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Serdica Math. J. **36** (2010), 255–264

Serdica
Mathematical Journal

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OSCILLATION CRITERIA OF SECOND-ORDER QUASI-LINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Communicated by I. D. Iliev

Abstract. The oscillatory and nonoscillatory behaviour of solutions of the second order quasi linear neutral delay difference equation

$$\Delta(a_n|\Delta(x_n + p_n x_{n-\tau})|^{\alpha-1}\Delta(x_n + p_n x_{n-\tau}) + q_n f(x_{n-\sigma})g(\Delta x_n) = 0$$

where $n \in N(n_0)$, $\alpha > 0$, τ, σ are fixed non negative integers, $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ are real sequences and f and g real valued continuous functions are studied. Our results generalize and improve some known results of neutral delay difference equations.

1. Introduction. In this paper, we consider the second order quasi linear neutral delay difference equation of the form

$$(1) \qquad \Delta(a_n|\Delta(x_n + p_n x_{n-\tau})|^{\alpha-1}\Delta(x_n + p_n x_{n-\tau}) + q_n f(x_{n-\sigma})g(\Delta x_n) = 0$$

2010 *Mathematics Subject Classification:* 39A10.
Key words: Oscillation, quasi-linear, neutral type, delay difference equations.

where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$ n_0 a non negative integer, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\alpha > 0$, τ, σ are fixed non negative integers.

Throughout this paper we assume that the following conditions hold:

(C_1) $\{a_n\}$ is a positive real sequence and $\{q_n\}$ is a non negative real sequence with q_n is not identically zero for large n,

(C_2) $\{p_n\}$ is a real sequence,

(C_3) $g : \mathbb{R} \to \mathbb{R}$ such that $g(u) \geq c > 0$ for $u \neq 0$,

(C_4) $f : R \to R$ is continuous and $uf(u) > 0$ for $u \neq 0$ and $f(u) - f(v) = h(u, v)(u - v)$ for all $u \neq 0$ and h is a non negative function.

Let $m = \max\{\tau, \sigma\}$. By a solution of equation (1) we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0 - m$ and satisfies (1) for large $n \geq n_0$. A solution $\{x_n\}$ of (1) is said to be nonoscillatory if all the terms x_n are eventually of fixed sign, otherwise the solution $\{x_n\}$ is called oscillatory. A nonoscillatory solution $\{x_n\}$ of (1) is said to be weakly oscillatory if $\{\Delta x_n\}$ changes sign for arbitrarily large values of n.

In this paper, we investigate oscillatory and asymptotic behaviour of non oscillatory solution of equation (1), when q_n is either non negative or changing sign for large n.

Let S denote the set of all nontrivial solutions of (1). With respect to their asymptotic nature, the nonoscillatory solutions of equation (1) may be a priori divided into the following classes:

$$\begin{aligned}
M^+ &= \{\{x_n\} \in S : \text{there exists an integer } N \text{ such that} \\
&\qquad x_n \Delta x_n \geq 0,\ \forall n \geq N\} \\
M^- &= \{\{x_n\} \in S : \text{there exists an integer } N \text{ such that} \\
&\qquad x_n \Delta x_n \leq 0,\ \forall n \geq N\} \\
OS &= \{\{x_n\} \in S : \text{there exists an integer } N \text{ such that} \\
&\qquad x_n x_{n+1} \leq 0,\ \forall n \geq N\} \\
WOS &= \{\{x_n\} \in S : \{x_n\} \text{ is nonoscillatory for every } N\ \exists\, n \geq N \\
&\qquad \text{such that } \Delta x_n \Delta x_{n+1} \leq 0\}
\end{aligned}$$

In [1] and [3] the authors studied the oscillatory and asymptotic behaviour of nonoscillatory solution of equation (1) when $g(u) \equiv 1$, $\alpha = 1$ and p_n either identically zero or $p_n = p$ via the above said classification. Hence the results obtained in this paper generalize that in [3].

2. Main results. Define

$$z_n = x_n + p_n x_{n-\tau} \tag{2}$$

First we examine the non-existence of solutions of equation (1) in the class M^+.

Theorem 2.1. *With respect to difference equation* (1)*, assume that*

$$-1 < -h \leq p_n \tag{3}$$

$$q_n \text{ is non negative and } \lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} q_s = \infty \tag{4}$$

$$\text{and} \qquad \sum_{s=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty \tag{5}$$

hold. Then for equation (1) *we have* $M^+ = \phi$.

Proof. Suppose that equation (1) has a solution $\{x_n\} \in M^+$. Without loss of generality we can assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $\Delta x_n \geq 0$, $x_{n-m} > 0$, $\Delta x_{n-m} \geq 0$ for all $n \geq n_1 = n_0 + m$ (the proof is similar if $x_n < 0$, $\Delta x_n \leq 0$ for all large n). If $p_n \geq 0$, we have $z_n \geq x_n > 0$. If $-1 < -h \leq p_n < 0$ we claim that $z_n > 0$, for all $n \geq n_1$. Otherwise, there is a $n_2 \geq n_1$, such that $z_{n_2} \leq 0$, then

$$x_{n_2} \leq h x_{n_2-\tau}$$

and therefore

$$x_{n_2+\tau} \leq h x_{n_2}$$

by induction

$$x_{n_2+2\tau} \leq h x_{n_2+\tau} \leq h^2 x_{n_2}$$

we obtain

$$x_{n_2+j\tau} \leq h^j x_{n_2}$$

implying that $x_{n_2+j\tau} \leq 0$ for large n, which contradicts the fact that $x_n > 0$, $\Delta x_n \geq 0$ for $n \geq n_1$.
Hence $z_n > 0$ for all $n \geq n_1$.

Now from the equation (1), it follows that

$$\Delta(a_n|\Delta z_n|^{\alpha-1}\Delta z_n) = -q_n f(x_{n-\sigma})g(\Delta x_n) \leq 0 \quad n \geq n_1 \tag{6}$$

we claim that $\Delta z_n \geq 0$ for $n \geq n_1$.

Otherwise, there exists an integer $n_3 \geq n_1$ such that $\Delta z_{n_3} < 0$.
It follows from (6) that

$$z_n \leq z_{n_3} - (-a_{n_3}|\Delta z_{n_3}|^{\alpha-1}\Delta z_{n_3})^{1/\alpha} \sum_{s=n_3}^{n-1} 1/a_n^{1/\alpha} \quad n \geq n_3$$

By using (5), we have $\lim_{n\to\infty} z_n = -\infty$ which contradicts the fact that $z_n > 0$ for $n \geq n_1$. So

$$\Delta z_n \geq 0 \quad \text{for} \quad n \geq n_1 \tag{7}$$

Summing equation (6) and using (C_1) - (C_4)

$$\begin{aligned} \frac{a_n(\Delta z_n)^\alpha}{f(x_{n-\sigma})} &\leq \frac{a_{n_1}(\Delta z_{n_1})^\alpha}{f(x_{n_1-\sigma}} - \sum_{s=n_1}^{n-1} \frac{a_n(\Delta z_n)^\alpha h(x_{s+1-\sigma}, x_{s-\sigma})\Delta x_{s-\sigma}}{f(x_{s+1-\sigma}f(x_{s-\sigma})} - c\sum_{s=n_1}^{n-1} q_s. \\ &\leq \frac{a_{n_1}(\Delta z_{n_1})^\alpha}{f(x_{n_1-\sigma})} - c\sum_{s=n_1}^{n-1} q_s \qquad n \geq n_1 \end{aligned}$$

From (4) we obtain

$$\lim_{n\to\infty} \inf \frac{a_n(\Delta z_n)^\alpha}{f(x_{n-\sigma})} = -\infty$$

which contradicts (7). The proof is complete. □

Theorem 2.2. *With respect to the difference equation* (1)*, assume that*

$$\{p_n\} \textit{ is non negative and nondecreasing for all } n \in \mathbb{N}(n_0) \tag{8}$$

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} q_s = \infty \tag{9}$$

hold. Then for equation (1) *we have* $M^+ = \phi$.

Proof. Suppose that equation (1) has a solution $\{x_n\} \in M^+$. There is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x_n > 0$, $\Delta x_n \geq 0$, $x_{n-m} > 0$, $\Delta x_{n-m} \geq 0$ for all $n \geq n_1 = n_0 + m$. The proof is similar if $x_n < 0$, $\Delta x_n \leq 0$ for all large n.

By condition (8) we see that

$$z_n > 0, \Delta z_n \geq 0, \quad n \geq n_1. \tag{10}$$

Similar to the proof of Theorem 2.1, we obtain

$$\lim_{n\to\infty} \inf \frac{a_n(\Delta z_n)^\alpha}{f(x_{n-\sigma})} = -\infty$$

which contradicts (10). The proof is complete. □

Now we examine existence of solutions of equation (1) in the class M^-.

Theorem 2.3. *Assume that* $\tau \leq \sigma$. *If the function* $\frac{1}{(f(u))^{1/\alpha}}$ *is locally integrable on* $(0, \alpha)$ *and* $(-\alpha, 0)$ *for all* $\alpha > 0$ *and*

$$\int_0^\alpha \frac{du}{(f(u))^{1/\alpha}} < \infty, \qquad \int_{-\alpha}^0 \frac{du}{(f(u))^{1/\alpha}} > -\infty \tag{11}$$

(12) f *is sub multiplicative;*

(13) $\{p_n\}$ *is non negative and nonincreasing for all* $n \in N(n_0)$

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \frac{1}{(a_s f(1+p_s))^{1/\alpha}} \left(\sum_{t=N}^{s-1} (q_\tau)^{1/\alpha} \right) = \infty \quad N \in \mathbb{N}(n_0) \tag{14}$$

hold, then for equation (1) *we have* $M^- = \phi$.

P r o o f. Suppose that equation (1) has a solution $\{x_n\} \in M^-$. Then there is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x_n > 0$, $\Delta x_n \leq 0$ $x_{n-m} > 0$, $\Delta x_{n-m} \leq 0$ for all $n \geq n_1$. The proof is similar if $x_n < 0$ $\Delta x_n \geq 0$ for all large n. Then from (2) by using (13) we see that

$$z_n > 0, \quad \Delta z_n \leq 0 \quad n \geq n_1.$$

Summing (6), using summation by parts from n_1 to $n-1$ and by (C_3) and (C_4)

$$\sum_{s=n_1}^{n-1} \frac{\Delta[a_n(\Delta z_n)^\alpha]}{f(x_{s-\sigma})} \leq -c \sum_{s=n_1}^{n-1} q_s \quad n \geq n_1$$

$$\frac{a_n(\Delta z_n)^\alpha}{f(x_{n-\sigma})} - \frac{a_{n_1}(\Delta z_{n_1})^\alpha}{f(x_{n_1-\sigma})} + \sum_{s=n_1}^{n-1} \frac{a_s(\Delta z_s)^\alpha h(x_{s-\sigma}, x_{s+1-\sigma})\Delta x_{s-\sigma}}{f(x_{s+1-\sigma})f(x_{s-\sigma})} \leq -c \sum_{s=n_1}^{n-1} q_s.$$

$$\begin{aligned} \frac{a_n(\Delta z_n)^\alpha}{f(x_{n-\sigma})} &\leq \frac{a_{n_1}(\Delta z_{n_1})^\alpha}{f(x_{n_1-\sigma})} - \sum_{s=n_1}^{n-1} \frac{a_s(\Delta z_s)^\alpha h(x_{s-\sigma}, x_{s+1-\sigma}\Delta x_{s-\sigma}}{f(x_{s+1-\sigma})f(x_{s-\sigma})} - c \sum_{s=n_1}^{n-1} q_s. \\ &\leq -c \sum_{s=n_1}^{n-1} q_s \quad n \geq n_1. \end{aligned}$$

By (C_1)

$$-\frac{(\Delta z_n)^\alpha}{f(x_{n-\sigma})} \geq c/a_n \sum_{s=n_1}^{n-1} q_s \quad \text{for } n \geq n_1 \tag{15}$$

Since $\{x_n\}$ is non increasing and $\tau \leq \sigma$ we have $z_n \leq (1+p_n)x_{n-\sigma}$ and hence by using (12)

$$f(z_n) \leq f(1+p_n)f(x_{n-\sigma}) \tag{16}$$

Combining (15) and (16)

$$-\frac{(\Delta z_n)^\alpha}{f(z_n)} \geq \frac{c}{a_n f(1+p_n)} \sum_{s=n_1}^{n-1} q_s \quad n \geq n_1.$$

Then we have

$$-\frac{(\Delta z_n)}{(f(z_n))^{1/\alpha}} \geq c^{1/\alpha} \left(\frac{\sum_{s=n_1}^{n-1} q_s}{a_n f(1+p_n)} \right)^{1/\alpha}, \quad n \geq n_1.$$

Using (by parts) summing the last inequality from n_1 to $n-1$

$$\sum_{s=n_1}^{n-1} -\frac{(\Delta z_s)^\alpha}{(f(z_s))^{1/\alpha}} \geq c^{1/\alpha} \sum_{s=n_1}^{n-1} \frac{1}{(a_s f(1+p_s))^{1/\alpha}} \left(\sum_{t=n_1}^{s-1} q_t \right)^{1/\alpha} \qquad n \geq n_1 \tag{17}$$

For $t+1 \leq z_n \leq t$

$$\int_{t+1}^{t} \frac{dt}{f(z_t)^{1/\alpha}} \geq -\frac{\Delta z_s}{f(z_s)^{1/\alpha}}$$

hence

$$\int_0^{z_{n_1}} \frac{dt}{f(z_t)^{1/\alpha}} \geq \sum_{s=n_1}^{n-1} -\frac{\Delta z_s}{f(z_s)^{1/\alpha}} \tag{18}$$

Combining (17) and (18) and taking limit sup we get a contradiction to (11) and (14).

The proof is complete. □

Next we establish sufficient conditions under which equation (1) has no weakly oscillatory solution.

Theorem 2.4. *Let $q_n \geq 0$ for all $n \geq n_0$. If*

$$p_n \equiv p \geq 0 \quad \textit{for } n \in N(n_0). \tag{19}$$

Then for equation (1), $WOS = \phi$.

Proof. Let $\{x_n\}$ be a weakly oscillatory solution of (1). Without loss of generality we assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-m} > 0$ for $n \geq n_1$.

(The proof is similar if $x_n < 0$ for all large n)

Using (2) and (19), $z_n > 0$

$$\begin{aligned}
\Delta z_n &= \Delta x_n + p\Delta x_{n-\tau} \\
\Delta z_{n+1} &= \Delta x_{n+1} + p\Delta x_{n-\tau+1} \\
\Delta z_n \Delta z_{n+1} &= \Delta x_n \Delta x_{n+1} + p(\Delta x_n \Delta x_{n-\tau+1} + \Delta x_{n+1}\Delta x_{n-\tau}) \\
&\quad + p^2 \Delta x_{n-\tau}\Delta x_{n-\tau+1} \\
&\leq 0.
\end{aligned}$$

Hence $z_n > 0$ and weakly oscillatory. In equation (1) putting $F_n = a_n|\Delta z_n|^{\alpha-1}\Delta z_n$ for $n \geq n_0$ we get $\Delta F_n = -q_n f(x_{n-\sigma})g(\Delta x_n) \leq 0$ which implies $\{F_n\}$ is non-increasing hence F_n is eventually of one sign which gives a contradiction, since $\{F_n\}$ an oscillatory sequence. □

Theorem 2.5. *Assume conditions* (5), (9), (19) *hold. Then every solution of equation* (1) *is either oscillatory or weakly oscillatory.*

Proof. From Theorem 2.2 it follows that for equation (1) $M^+ = \phi$. In order to complete the proof it suffices to show that for (1) $M^- = \phi$.

Suppose that $\{x_n\} \in M^-$. Then as earlier we can assume that $x_n > 0$, $\Delta x_n \leq 0$, $x_{n-m} > 0$, $\Delta x_{n-m} \leq 0$ for all $n \geq n_1$ the proof is similar if $x_n < 0$, $\Delta x_n \geq 0$ for large n.

Then by using (2) and (19) we see that

$$z_n > 0 \quad \Delta z_n \leq 0 \qquad n \geq n_1$$

Let $w_n = a_n(\Delta z_n)^\alpha$, so that $w_n \leq 0$ for $n \geq n_1$. From (1)

$$\begin{aligned} \Delta w_n &\leq -cq_n f(x_{n-\sigma}) \\ w_n &\leq w_{n_1} - c\sum_{s=n_1}^{n-1} q_s f(x_{s-\sigma}) \end{aligned}$$

using Abel's transformation. (1, p. 35)

$$w_n \leq w_{n_1} - cf(x_{n-\sigma})\sum_{s=n_1}^{n-1} q_s - \sum_{s=n_1}^{n-1} \Delta f(x_{s-\sigma})\left(\sum_{t=n_1}^{s} q_t\right)$$

From the above relation

$$\begin{aligned} w_n &\leq w_{n_1} \\ (\Delta z_n)^\alpha &\leq \frac{w_{n_1}}{a_n} < 0 \qquad \text{for } n \geq n_1 \\ z_n - z_{n_1} &\leq w_{n_1}^{1/\alpha} \sum_{s=n_1}^{n-1} \frac{1}{a_s^{1/\alpha}} \to -\infty \qquad \text{as } n \to \infty \end{aligned}$$

which contradicts $z_n > 0$. The proof is complete. □

From Theorems 2.4 and 2.5 we can easily get the following theorem.

Theorem 2.6. *Let* $q_n \geq 0$ *for all* $n \geq n_0$ *and conditions* (5), (9), (19) *hold. Then every solution of equation* (1) *is oscillatory.*

Now we study the asymptotic behaviour of the eventually monotone solution of equation (1).

Theorem 2.7. *Assume conditions* (12), (13), (14) *are satisfied. Then for every solution* $x_n \in M^-$ *we have* $\lim_{n\to\infty} x_n = 0$.

Proof. The assertion follows from the same argument as given in the proof of Theorem 2.3. Taking into account (18) which implies $\lim_{n\to\infty} z_n = 0$, together with $z_n \geq x_n$ for all $n \geq M$ we have $\lim_{n\to\infty} x_n = 0$.

This completes the proof. □

Example 2.1. *Consider the quasi linear neutral delay difference equation*

$$\begin{aligned} & \Delta\left[\frac{1}{n^2}\left|\Delta x_n + 2x_{n-1}\right|^{\alpha-1}\Delta(x_n + 2x_{n-1})\right] \\ (E_1) \qquad & + \frac{n}{n-2}x_{n-2}(1+(\Delta x_n)^2) = 0 \quad n \geq 3. \end{aligned}$$

$\tau = 1 \quad \sigma = 2 \quad f(y) = y \quad g(y) = 1 + y^2 \geq 1$
$p_n = 2 > 0 \quad a_n = 1/n^2 > 0 \quad q_n = n/n-2$
All conditions of Theorem 2.6 *are satisfied and hence* (E_1) *is oscillatory by Theorem* 2.6.

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Received August 28, 2010