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NORMAL AND NORMALLY OUTER AUTOMORPHISMS OF FREE METABELIAN NILPOTENT LIE ALGEBRAS*

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Abstract. Let $L_{m,c}$ be the free m-generated metabelian nilpotent of class c Lie algebra over a field of characteristic 0. An automorphism φ of $L_{m,c}$ is called normal if $\varphi(I) = I$ for every ideal I of the algebra $L_{m,c}$. Such automorphisms form a normal subgroup $\mathrm{N}(L_{m,c})$ of $\mathrm{Aut}(L_{m,c})$ containing the group of inner automorphisms. We describe the group of normal automorphisms of $L_{m,c}$ and the quotient group of $\mathrm{Aut}(L_{m,c})$ modulo $\mathrm{N}(L_{m,c})$.

Introduction. Let L_m be the free m-generated Lie algebra over a field K of characteristic 0, $m \geq 2$, and let $L_{m,c} = L_m/(L_m'' + L_m^{c+1})$ be the free m-generated metabelian nilpotent of class c Lie algebra. This is the relatively free algebra of rank m in the variety of Lie algebras $\mathfrak{A}^2 \cap \mathfrak{N}_c$, where $\mathfrak{A}^2$ is the

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metabelian (solvable of class 2) variety of Lie algebras and $\mathfrak{N}_c$ is the variety of all nilpotent Lie algebras of class at most c.

An automorphism of an algebra is called normal if it preserves every ideal of the algebra. Similarly, an automorphism of a group is normal if it preserves every normal subgroup of the group. Such automorphisms form a normal subgroup of the group of all automorphisms. The goal of our paper is to describe the group of normal automorphisms $\mathrm{N}(L_{m,c})$ and the quotient group $\mathrm{Aut}(L_{m,c})/\mathrm{N}(L_{m,c})$ of normally outer automorphisms of the Lie algebra $L_{m,c}$. The corresponding problem for the group of normal automorphisms of free metabelian nilpotent groups was studied by Endimioni [5, 6, 7]. He showed that the normal automorphisms θ of a free metabelian nilpotent group G are exactly the atomorphisms of the form

$$\theta(x) = x(x, u_1)^{k(1)} \dots (x, u_m)^{k(m)},$$

where $u_1, \dots, u_m$ are elements of G, the exponents $k(1), \dots, k(m)$ are integers. (As usual, the commutator (a, b) in the group case is defined by $(a, b) = a^{-1}b^{-1}ab$.). Endimioni also proved that the group of normal automorphisms of the free metabelian nilpotent group G is metabelian, generalizing a result of Gupta [8] for the group of IA-automorphisms in a two-generated metabelian group. Initially, automorphisms of the form $\theta(x) = x(x, u_1)^{k(1)} \dots (x, u_m)^{k(m)}$ were studied by Kuzmin [9].

The group of normal automorphisms of free groups has been studied by Lubotzky [10]. He showed that $\mathrm{N}(G) = \mathrm{Inn}(G)$, for any finitely generated free group G. Lue [11] gave a short proof of this fact using the Freiheitssatz for groups established by Magnus [12]. The Freiheitssatz for Lie algebras was proved by Shirshov [15]. Makar-Limanov [13] proved it for associative algebras over a field of characteristic zero. Following the idea of Lue [11] we show that the free Lie algebra L_m does not have nontrivial normal automorphisms for any $m \geq 2$ and over a field of any characteristic. For the proof we apply the Freiheitssatz for Lie algebras and use the Hopf property of free Lie algebras. The same result holds for free associative algebras over a field of characteristic 0. The key step of the proof was suggested to us by Ualbai Umirbev. If we replace L_m with a relatively free algebra in a proper subvariety of the variety of all Lie algebras it may happen that many normal automorphisms appear. In particular, this holds for the free metabelian nilpotent Lie algebra $L_{m,c}$. Since every inner automorphism of $L_{m,c}$ is normal, the algebra $L_{m,c}$ posseses nontrivial normal automorphisms.

Our first main result is similar to the result of Endimioni [5, 7] in the case of groups but there are some essential differences. We show that the group of normal automorphisms is included in the subgroup $\mathrm{IA}(L_{m,c})$ of the automorphisms which induce the identity map modulo the commutator ideal of $L_{m,c}$ when $m \geq 3, c \geq 2$ or $m = 2, c \geq 4$. In the exceptional cases, i.e. $(m,c) = (2,2)$ or $(m,c) = (2,3)$, every normal automorphism acts on the generators of $L_{m,c}$ as a nonzero scalar times an IA-automorphism. For the proof we define a special type of automorphisms called generalized inner automorphisms and describe the group of normal automorphisms in terms of them. We also show that the group of normal automorphisms $\mathrm{N}(L_{m,c})$ is an abelian group when $m \geq 3, c = 2$, is a nilpotent of class 2 group when $m \geq 3, c = 3$ and is a metabelian group when $m \geq 2, c \geq 4$ or $(m,c) = (2,2)$. Finally, $\mathrm{N}(L_{m,c})$ is a nilpotent of class two–by–abelian group when $(m,c) = (2,3)$ which is an analogue of the result of Gupta [8] and Endimioni [6].

A result of Shmel'kin [16] states that the free metabelian Lie algebra $F_m = L_m/L''_m$ can be embedded into the abelian wreath product $A_m \mathrm{wr} B_m$, where A_m and B_m are m-dimensional abelian Lie algebras with bases $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$, respectively. The elements of $A_m \mathrm{wr} B_m$ are of the form

$$\sum_{i=1}^{m} a_i f_i(t_1, \ldots, t_m) + \sum_{i=1}^{m} \beta_i b_i,$$

where the f_i's are polynomials in $K[t_1, \ldots, t_m]$ and $\beta_i \in K$. This allows to introduce partial derivatives in F_m with values in $K[t_1, \ldots, t_m]$ and the Jacobian matrix $J(\phi)$ of an endomorphism ϕ of F_m. Restricted on the semigroup $\mathrm{IE}(F_m)$ of endomorphisms of F_m which are identical modulo the commutator ideal F'_m, the map $J : \phi \to J(\phi)$ is a semigroup monomorphism of $\mathrm{IE}(F_m)$ into the multiplicative semigroup of the algebra $M_m(K[t_1, \ldots, t_m])$ of $m \times m$ matrices with entries from $K[t_1, \ldots, t_m]$. In the present work we consider the embedding of the free metabelian nilpotent Lie algebra $L_{m,c}$ into the wreath product $A_m \mathrm{wr} B_m$ modulo the ideal $(A_m \mathrm{wr} B_m)^{c+1}$. The automorphism group $\mathrm{Aut}(L_{m,c})$ is a semidirect product of the normal subgroup $\mathrm{IA}(L_{m,c})$ and the general linear group $\mathrm{GL}_m(K)$. Considering the group $\mathrm{IN}(L_{m,c})$ of normal IA-automorphisms, for the description of the factor group $\Gamma\mathrm{N}(L_{m,c}) = \mathrm{Aut}(L_{m,c})/\mathrm{N}(L_{m,c})$ it is sufficient to know only $\mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$. Drensky and Fındık [4] gave the explicit form of the Jacobian matrices of the coset representatives of the outer automorphisms in $\mathrm{IA}(L_{m,c})/\mathrm{Inn}(L_{m,c})$. Since $\mathrm{Inn}(L_{m,c})$ is included in the group of normal automorphisms, $\mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$ is a homomorphic image of $\mathrm{IA}(L_{m,c})/\mathrm{Inn}(L_{m,c})$ and we find explicitely coset representatives of $\mathrm{IN}(L_{m,c})$.

The paper is organized as follows. In the first section, we introduce normal and normally outer automorphisms and discuss the relations between $\mathrm{N}(L_{m,c})$ and the normal subgroup $\mathrm{IA}(L_{m,c})$. We also discuss the normal automorphisms of the free Lie algebra L_m. In the second section we define the group of generalized inner automorphisms and give necessary information about its group structure. In the third section we describe the group of normal automorphisms in terms of the group of generalized inner automorphisms. Finally we give the explicit form of the Jacobian matrices of the normal automorphisms and of the Jacobian matrices of the coset representatives of normally outer IA-automorphisms. We also give the explicit form of the Jacobian matrices of the coset representatives of the normal automorphisms modulo the group of inner automorphisms $\mathrm{Inn}(L_{m,c})$.

1. Preliminaries. Let L_m be the free Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 with free generators $y_1, \ldots, y_m$ and let $L_{m,c} = L_m/(L_m'' + L_m^{c+1})$ be the free metabelian nilpotent of class c Lie algebra freely generated by $x_1, \ldots, x_m$, where $x_i = y_i + (L_m'' + L_m^{c+1})$, $i = 1, \ldots, m$. We use the commutator notation for the Lie multiplication. Our commutators are left normed:

$$[u_1, \ldots, u_{n-1}, u_n] = [[u_1, \ldots, u_{n-1}], u_n], \quad n = 3, 4, \ldots.$$

In particular,

$$L_{m,c}^k = \underbrace{[L_{m,c}, \ldots, L_{m,c}]}_{k \text{ times}}.$$

For each $v \in L_{m,c}$, the linear operator $\mathrm{ad}v : L_{m,c} \to L_{m,c}$ defined by

$$u(\mathrm{ad}v) = [u, v], \quad u \in L_{m,c},$$

is a derivation of $L_{m,c}$ which is nilpotent and $\mathrm{ad}^c v = 0$ because $L_{m,c}^{c+1} = 0$. Hence the linear operator

$$\exp(\mathrm{ad}v) = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots + \frac{\mathrm{ad}^{c-1} v}{(c-1)!}$$

is well defined and is an inner automorphism of $L_{m,c}$. The set of all such automorphisms forms a normal subgroup $\mathrm{Inn}(L_{m,c})$ of the group of all automorphisms $\mathrm{Aut}(L_{m,c})$ of $L_{m,c}$.

Let φ be an automorphism of an algebra R such that $\varphi(I) = I$ for every ideal I of the algebra R. Such automorphisms are called *normal* automorphisms. Clearly they form a normal subgroup of the group of all automorphisms $\mathrm{Aut}(R)$ of R which we denote by $\mathrm{N}(R)$. The factor group $\mathrm{Aut}(R)/\mathrm{N}(R)$ is the group of *normally outer* (or N-outer) automorphisms and is denoted by $\Gamma\mathrm{N}(R)$.

The next lemma gives the form of normal automorphisms of $L_{m,c}$.

Lemma 1.1. *Let φ be a normal automorphism of the algebra $L_{m,c}$. Then φ is of the form*

$$\varphi : x_i \to \alpha x_i + \sum_{j=1}^{m} [x_i, x_j] f_{ij}(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m), \quad i = 1, \ldots, m, \quad \alpha \in K^*,$$

where $f_{ij}(t_1, \ldots, t_m) \in K[t_1, \ldots, t_m]$ and K^* is the set of nonzero elements of the field K.

Proof. Let φ be a normal automorphism of the algebra $L_{m,c}$. Hence φ induces a normal automorphism $\overline{\varphi}$ of the abelian algebra $\overline{L}_{m,c} = L_{m,c}/L'_{m,c}$. The automorphism group of $\overline{L}_{m,c}$ coincides with the general linear group $GL_m(K)$ and the normal automorphisms of $\overline{L}_{m,c}$ are the elements of $GL_m(K)$ which preserve the vector subspaces of $\overline{L}_{m,c}$. Applying to the vector subspace $K\overline{x}_i$ we obtain that $\overline{\varphi}(\overline{x}_i) = \alpha_i \overline{x}_i$, $\alpha_i \in K^*$. Similarly, for $i \neq j$,

$$\begin{aligned} \overline{\varphi}(\overline{x_i + x_j}) &= \alpha_i \overline{x}_i + \alpha_j \overline{x}_j \\ &= \alpha(\overline{x}_i + \overline{x}_j), \quad \alpha \in K^*. \end{aligned}$$

Thus $\alpha_i = \alpha_j = \alpha$. Hence φ has the form

$$\varphi : x_i \to \alpha x_i + u_i, \quad \alpha \in K^*,\ u_i \in L'_{m,c},\ i = 1, \ldots, m.$$

It is well known in a metabelian Lie algebra G, see e.g. [1], that

$$[v_1, v_2, v_{\sigma(3)}, \ldots, v_{\sigma(k)}] = [v_1, v_2, v_3, \ldots, v_k], \quad v_1, \ldots, v_k \in G,$$

where σ is an arbitrary permutation of $3, \ldots, k$, i.e. the operators $\mathrm{ad}v$, $v \in G$, commute when acting on G'. The vector space $L'_{m,c}$ has a basis consisting of all

$$[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_k}], \quad 1 \leq i_j \leq m,\ i_1 > i_2 \leq i_3 \leq \cdots \leq i_k,\ k \leq c,$$

and we may permute the elements $x_{i_3}, \ldots, x_{i_k}$. Reordering the elements $x_1, \ldots, x_m$ by

$$x_i < x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_m$$

we obtain that the subspace of $L'_{m,c}$ spanned by the commutators essentially depending on x_i, has a basis

$$[x_i, x_j, x_{i_3}, \ldots, x_{i_k}], \quad j \neq i,\ 1 \leq i_3 \leq \cdots \leq i_k,\ k \leq c.$$

Hence the normal automorphism φ of $L_{m,c}$ has the form

$$\varphi : x_i \to \alpha x_i + \sum_{j \neq i} [x_i, x_j] f_{ij}(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m) + g_i(\hat{x}_i),$$

where $\alpha \in K^*$, $f_{ij}(t_1, \ldots, t_m) \in K[t_1, \ldots, t_m]$ and $g_i(\hat{x}_i) \in L'_{m,c}$ does not depend on x_i.

For a fixed $i = 1, \ldots, m$ let us consider the ideal J_i of $L_{m,c}$ generated by the element x_i, Since φ is normal and $\varphi(x_i) \in J_i$ we obtain that

$$g_i(\hat{x}_i) \in J_i, \quad i = 1, \ldots, m,$$

and hence

$$g_i(\hat{x}_i) = 0, \quad i = 1, \ldots, m,$$

because every element in J_i depends on x_i. Thus we have

$$\varphi(x_i) = \alpha x_i + \sum_{j \neq i} [x_i, x_j] f_{ij}(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m),$$

which completes the proof. $\square$

A similar proof holds also for the free Lie algebra L_m. But we use the fact that, applying the anticommutativity and the Jacobian identity, linear combinations of commutators of L_m depending essentially on y_i can be rewritten as linear combinations of left normed commutators of the form

$$[y_i, y_{i_2}, \ldots, y_{i_k}], \quad y_{i_2}, \ldots, y_{i_k} \in \{y_1, \ldots, y_m\}.$$

Lemma 1.2. *Let φ be a normal automorphism of the algebra L_m. Then φ is of the form*

$$\varphi : y_i \to \alpha y_i + y_i f_i(\mathrm{ad}Y), \quad i = 1, \ldots, m, \quad \alpha \in K^*,$$

where $f_i(\mathrm{ad}Y) = f_i(\mathrm{ad}y_1, \ldots, \mathrm{ad}y_m)$ and every polynomial $f_i(t_1, \ldots, t_m)$, $i = 1, \ldots, m$, belongs to the free associative algebra $K\langle t_1, \ldots, t_m\rangle$.

Recall that an algebra R is *Hopfian*, if it cannot be mapped onto itself with nontrivial kernel. The following lemma is folklorely known.

Lemma 1.3. *Finitely generated free Lie algebras and free associative algebras over any field of arbitrary characteristic are Hopfian.*

For example this fact is stated for relatively free algebras of finite rank as Exercise 4.10.21, p. 137 in the book of Bahturin [1]. The proof is similar to the proof in the group case, see Section 4.1 of the book by Neumann [14], and repeats the steps of the proof of Theorem 9, p. 104 [1]. The proof of [1, Exercise 4.10.21] uses only the fact that over an infinite field relatively free algebras $F(\mathfrak{U})$ are graded and that $\cap_{m \geq 1} F^m(\mathfrak{U}) = 0$ which is obviously true for free Lie algebras and free associative algebras over any field.

The analogue of the Freiheitssatz in group theory [12] was proved by Shirshov [15] in the case of Lie algebras in any characteristic. For associative algebras it was obtained by Makar-Limanov [13] when characteristic of the base field is 0. The problem is still open for associative algebras over a field of positive characteristic (see e.g. the book by Bokut' and Kukin [3]). We shall state the result for free Lie algebras only.

Theorem 1.4 (Shirshov [15]). *Let $L(Y)$ be the free Lie algebra freely generated by $Y = \{y_1, \dots, y_m\}$. If $f(Y) \in L(Y)$ does not belong to the subalgebra generated by $y_1, \dots, y_{m-1}$, then $(f(Y)) \cap L(y_1, \dots, y_{m-1}) = 0$ where $(f(Y))$ is the ideal of $L(Y)$ generated by $f(Y)$.*

The idea to use the Freiheitssatz in the following proof was suggested to us by Ualbai Umirbaev.

Corollary 1.5. *If every monomial of $f(Y) \in L(Y)$ depends on y_m and $f(Y) \notin L(y_m) = Ky_m$, then $f(Y)$ is not an image of y_m under an automorphism of the algebra $L(Y)$, i.e. $f(Y)$ is not a coordinate.*

P r o o f. Let φ be an automorphism of $L(Y)$ and let $\varphi(y_m) = f(Y)$, i.e. $f = f(Y)$ be a coordinate. Clearly $f \in (y_m) \lhd L(Y)$ because every monomial of f depends on y_m. Let $y_m \notin (f) \lhd L(Y)$. This means that f depends also on the variables $y_1, \dots, y_{m-1}$. Since $\varphi : L(Y) \to L(Y)$ is an automorphism and $\varphi(y_m) = f$, then

$$L(Y)/(f) \cong L(\varphi(y_1), \dots, \varphi(y_{m-1})) \cong L(y_1, \dots, y_{m-1}).$$

On the other hand $L(Y)/(y_m) \cong L(y_1, \dots, y_{m-1})$. As a result

$$L(Y)/(f) \cong L(Y)/(y_m) \cong L(y_1, \dots, y_{m-1}).$$

Let us consider the natural homomorphism

$$\pi : L(y_1, \dots, y_{m-1}) \cong L(Y)/(f) \to L(Y)/(y_m) \cong L(y_1, \dots, y_{m-1}).$$

π is onto and $\ker \pi \neq 0$ because $y_m \notin (f) \subset (y_m)$. But $L(y_1, \dots, y_{m-1})$ has the *Hopf property* by Lemma 1.3. This is in contradiction with $\ker \pi \neq 0$. Then $(f) = (y_m) \lhd L(Y)$. But f depends also on the other variables, for example, without loss of generality, it depends also on y_1. Applying Theorem 1.4 (the Freiheitssatz) we get that $y_2, \dots, y_m$ generate a free algebra of rank $m - 1$ in $L(Y)/(f) = L(Y)/(y_m)$. But this is not true for $L(Y)/(y_m)$, because $\overline{y}_m \neq \overline{0}$ in $L(Y)/(f)$ while $\overline{y}_m = \overline{0}$ in $L(Y)/(y_m)$. $\square$

Lubotzky [10] showed that the group of normal automorphisms of a free group G is equal to the group of inner automorphisms of G, i.e. $\mathrm{N}(G) = \mathrm{Inn}(G)$

and Lue [11] gave an alternative proof of the statement. Our next theorem states that the free Lie algebra L_m does not have nontrivial normal automorphisms for any $m \geq 2$. The idea of the proof is similar to the idea of the proof of the paper by Lue [11] for free groups.

Theorem 1.6. *Let L_m be the free Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 with free generators $y_1, \ldots, y_m$. Then L_m does not have nontrivial normal automorphisms.*

Proof. Applying Lemma 2.1 and Corollary 1.5 we see that L_m does not have nontrivial normal IA-automorphisms, i.e. the normal automorphisms of L_m are of the form $y_i \to \alpha y_i$, $i = 1, \ldots, m$, $\alpha \in K^*$.

Let $m = 2$ and let us consider the ideal I of L_2 generated by $f = y_1 - [y_1, y_2]$. Let φ be a normal automorphism of L_2 of the form

$$y_1 \to \alpha y_1, \quad y_2 \to \alpha y_2, \quad \alpha \in K^*,$$

and assume that $\alpha \neq 1$. Since φ is normal

$$\varphi(f) = \alpha y_1 - \alpha^2 [y_1, y_2] \in I.$$

Hence we have the system

$$\begin{aligned} y_1 - [y_1, y_2] &\equiv 0 \pmod{I} \\ \alpha y_1 - \alpha^2 [y_1, y_2] &\equiv 0 \pmod{I} \end{aligned}$$

Since $\alpha \neq 0, 1$, then $y_1 \equiv 0 \pmod{I})$ and $[y_1, y_2] \equiv 0 \pmod{I}$ which means that $I = (y_1) \lhd L_2$. Now consider the ideal J of L_2 generated by all commutators $u \in L_2$ such that $\deg_{y_1}(u) \geq 2$. Then

$$\overline{L}_2 = L_2/J = \text{span}\{\overline{y}_2, [\overline{y}_1, \underbrace{\overline{y}_2, \ldots, \overline{y}_2}_{k}] \mid k \geq 0\}.$$

Recall that $f = y_1 - [y_1, y_2]$. Clearly $[\overline{f}, \overline{y}_1] = \overline{0}$ in $\overline{L}_2$. So the elements of $\overline{I}$ in $\overline{L}_2$ are linear combinations of

$$u_k = [\overline{f}, \underbrace{\overline{y}_2, \ldots, \overline{y}_2}_{k}], \quad k \geq 0.$$

Thus

$$\overline{I} = \left\{ \sum_{k \geq 0} \beta_k [\overline{y}_1, \underbrace{\overline{y}_2, \ldots, \overline{y}_2}_{k}] \mid \beta_k \in K, \sum_{k \geq 0} \beta_k = 0 \right\}.$$

This means that $\overline{y}_1 \in (\overline{y}_1)$ while $\overline{y}_1 \notin \overline{I}$, because the sum of the coefficients of $\overline{y}_1$ is 1. Thus $(\overline{y}_1) \neq \overline{I}$ which is in contradiction with $I = (y_1)$. Hence $\alpha = 1$.

Now let $m \geq 3$ and let φ be a normal automorphism of L_m of the form

$$\varphi(y_i) = \alpha y_i \quad i = 1, \dots, m, \quad \alpha \in K^*.$$

We consider the ideal I of L_m generated by the elements $y_1 - [y_1, y_2], y_3 \dots, y_m$. Since φ is normal then $\varphi(I) = I$ and it induces a normal automorphism of L_m/I which is isomorphic to $L_2/(y_1 - [y_1, y_2])$. But we already know that in this case $\alpha = 1$. □

Remark 1.7. An analogue of Theorem 1.6 holds for free associative algebras $K\langle Y\rangle = K\langle y_1, \dots, y_m\rangle$ over a field of characteristic 0. Repeating the main steps of the proof of Theorem 1.6 we obtain that the only possibility is that $f(Y) = f(y_m)$ depends on y_m only. We extend φ to an automorphism $\overline{\varphi}$ of the algebra $\overline{K}\langle y_1, \dots, y_m\rangle$ where $\overline{K}$ is the algebraic closure of K. If $\deg f(Y) = \deg f(y_m) = d > 1$, then

$$\varphi(y_m) = f(y_m) = a_0(y_m - \alpha_1) \cdots (y_m - \alpha_d), \quad a_0, \alpha_1, \dots, \alpha_d \in \overline{K},$$

is a product of several polynomials which is impossible:

Applying φ^{-1} we obtain that the degree of

$$y_m = a_0(\varphi^{-1}(y_m) - \alpha_1) \cdots (\varphi^{-1}(y_m) - \alpha_d)$$

is bigger than 1.

The situation in the case of free metabelian nilpotent Lie algebra $L_{m,c}$ is different. Applying Lemma 1.1 it is easy to see that $\mathrm{Inn}(L_{m,c}) \subset \mathrm{N}(L_{m,c})$. Hence $L_{m,c}$ posseses nontrivial normal automorphisms. The group $\mathrm{N}(L_{m,c})$ is not necessarily included in the normal subgroup $\mathrm{IA}(L_{m,c})$ of $\mathrm{Aut}(L_{m,c})$ of the automorphisms which induce the identity map modulo the commutator ideal of $L_{m,c}$. Our next lemma states that in some cases $\mathrm{N}(L_{m,c}) \subset \mathrm{IA}(L_{m,c})$.

Lemma 1.8. *(i) If $m \geq 3$ and $c = 2$, then $\mathrm{N}(L_{m,2}) \subset \mathrm{IA}(L_{m,2})$.*
(ii) If $m \geq 3$ and $c = 3$, then $\mathrm{N}(L_{m,3}) \subset \mathrm{IA}(L_{m,3})$.
(iii) If $m \geq 2$ and $c \geq 4$, then $\mathrm{N}(L_{m,c}) \subset \mathrm{IA}(L_{m,c})$.

Proof. *(i)* Let φ be a normal automorphism of $L_{m,2}$, $m \geq 3$. By Lemma 1.1, φ has the form

$$\varphi : x_i \to \alpha x_i + \sum_{j=1}^{m} \beta_{ij}[x_i, x_j], \quad i = 1, \dots, m, \quad \alpha \in K^*,$$

where $\beta_{ij} \in K$. Let us consider the ideal J generated by $u = x_1 + [x_2, x_3]$. It has a basis

$$x_1 + [x_2, x_3],\ [x_1, x_j],\ j = 2, \dots, m.$$

Since φ is normal

$$\varphi(u) = \alpha x_1 + \alpha^2[x_2, x_3] + \beta_{12}[x_1, x_2] + \cdots + \beta_{1m}[x_1, x_m] \in J.$$

Clearly the summand $\alpha x_1 + \alpha^2[x_2, x_3]$ is included in the vector space spanned by the element u. Thus $\alpha = \alpha^2$ or $\alpha = 1$.

(ii) Let φ be a normal automorphism of $L_{m,3}$, $m \geq 3$. By Lemma 1.1, φ has the form

$$\varphi : x_i \to \alpha x_i + \sum_{j=1}^{m}[x_i, x_j]f_{ij}, \quad i = 1, \dots, m, \quad \alpha \in K^*,$$

where $f_{ij} \in K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$. We can express φ as

$$\varphi : x_i \to \alpha x_i + \sum_{j=1}^{m}[x_i, x_j](f_{ij,0} + f_{ij,1}),$$

where $f_{ij,0} \in K$, $f_{ij,1} \in \omega/\omega^2$. Here ω states for the augmentation ideal of $K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$.

Let us consider the ideal J generated by $u = [x_1, x_2] + [x_1, x_3, x_3]$. J has a basis

$$[x_1, x_2] + [x_1, x_3, x_3],\ [x_1, x_2, x_j],\ j = 1, \dots, m.$$

Since φ is normal, $\varphi(u) \in J$. Easy calculations give that

$$\varphi(u) = \alpha^2[x_1, x_2] + \alpha^3[x_1, x_3, x_3] - \alpha\sum_{j=1}^{m}[x_2, x_j, x_1]f_{2j,0} + \alpha\sum_{j=1}^{m}[x_1, x_j, x_2]f_{1j,0}.$$

Clearly the summand $\alpha^2[x_1, x_2] + \alpha^3[x_1, x_3, x_3]$ is included in the vector space spanned by the element $[x_1, x_2] + [x_1, x_3, x_3]$. Thus $\alpha^2 = \alpha^3$ or $\alpha = 1$.

(iii) Let φ be a normal automorphism of $L_{m,c}$, $m \geq 2$, $c \geq 4$. By Lemma 1.1, φ has the form

$$\varphi : x_i \to \alpha x_i + \sum_{j=1}^{m}[x_i, x_j]f_{ij}, \quad i = 1, \dots, m, \quad \alpha \in K,$$

where $f_{ij} \in K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$.

Let us consider the ideal J generated by

$$v = \underbrace{[x_1, x_2, \dots, x_2]}_{c-1} + \underbrace{[x_1, x_2, x_1, \dots, x_1]}_{c}.$$

J has a basis consisting of v and the elements of the form

$$\underbrace{[x_1, x_2, \ldots, x_2, x_j]}_{c}, \quad j = 1, \ldots, m.$$

Since φ is normal $\varphi(v) \in J$. Similar steps as (ii) give that

$$\alpha^{c-1} \underbrace{[x_1, x_2, \ldots, x_2]}_{c-1} + \alpha^c \underbrace{[x_1, x_2, x_1, \ldots, x_1]}_{c}$$

is included in the vector space spanned by the element v. Thus $\alpha = 1$. □

Lemma 1.9. *In the cases* $(m, c) = (2, 2)$ *and* $(m, c) = (2, 3)$ *every normal automorphism of* $L_{m,c}$ *acts on the generators of* $L_{m,c}$ *as a nonzero scalar times a normal* IA-*automorphism.*

Proof. Firstly we analyze the structure of the ideals of the free metabelian nilpotent Lie algebra $L_{2,2}$. The monomials in $L_{2,2}$ are x_1, x_2 and $[x_1, x_2]$. Let $0 \neq u \in L_{2,2}$ posses the form

$$u = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 [x_1, x_2]$$

for some $\alpha_1, \alpha_2, \alpha_3 \in K$ and let I be the ideal of $L_{2,2}$ generated by u. If $\alpha_1 \neq 0$ then $[u, x_2] = \alpha_1 [x_1, x_2] \in I$ and I is a nonzero ideal containing the commutator ideal $L'_{2,2}$. Therefore all the nonzero ideals of $L_{2,2}$ are $L'_{2,2}$, $K(\alpha_1 x_1 + \alpha_2 x_2) + L'_{2,2}$ and $L_{2,2}$.

Now let $\alpha \in K^*$ and $\phi \in \mathrm{IN}(L_{2,2})$. Let us define id_α as follows:

$$\mathrm{id}_\alpha(x_1) = \alpha x_1,$$

$$\mathrm{id}_\alpha(x_2) = \alpha x_2.$$

We have to show that $\mathrm{id}_\alpha \phi$ preserves the ideals of $L_{2,2}$. Since ϕ is normal, it sufficies to show that id_α is normal. Clearly $\mathrm{id}_\alpha(L'_{2,2}) = L'_{2,2}$. If $w \in I = K(\alpha_1 x_1 + \alpha_2 x_2) + L'_{2,2}$ has the form

$$w = \beta_1(\alpha_1 x_1 + \alpha_2 x_2) + \beta_2 [x_1, x_2], \quad \beta_1, \beta_2 \in K,$$

then

$$\begin{aligned} \mathrm{id}_\alpha(w) &= \alpha\beta_1\alpha_1 x_1 + \alpha\beta_1\alpha_2 x_2 + \alpha^2\beta_2[x_1, x_2] \\ &= \alpha\beta_1(\alpha_1 x_1 + \alpha_2 x_2) + \alpha^2\beta_2[x_1, x_2] \in I. \end{aligned}$$

Now let us consider the free metabelian nilpotent Lie algebra $L_{2,3}$. The monomials in $L_{2,3}$ are x_1, x_2, $[x_1, x_2]$, $[x_1, x_2, x_1]$ and $[x_1, x_2, x_2]$. Let I be a nonzero ideal of $L_{2,3}$. If $I \subset L^3_{2,3}$ then I consists of homogeneous elements of degree three and if $w \in I$ then $\mathrm{id}_\alpha(w) = \alpha^3 w \in I$. If I is not included in $L^3_{2,3}$ then it is easy to see that I contains $L^3_{2,3}$ and $\overline{I} = I/L^3_{2,3} \lhd L_{2,3}/L^3_{2,3} \cong L_{2,2}$. But we already know that in this case id_α is normal. $\square$

Let $F_m = L_m/L''_m$ be the free metabelian Lie algebra of rank m. We shall denote the free generators of F_m with the same symbols $x_1, \ldots, x_m$ as the free generators of $L_{m,c}$, but now $x_i = y_i + L''_m$, $i = 1, \ldots, m$. Let $K[t_1, \ldots, t_m]$ be the (commutative) polynomial algebra over K freely generated by the variables $t_1, \ldots, t_m$ and let A_m and B_m be the abelian Lie algebras with bases $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$, respectively. Let C_m be the free right $K[t_1, \ldots, t_m]$-module with free generators $a_1, \ldots, a_m$. We give it the structure of a Lie algebra with trivial multiplication. The abelian wreath product $A_m \mathrm{wr} B_m$ is equal to the semidirect sum $C_m \leftthreetimes B_m$. The elements of $A_m \mathrm{wr} B_m$ are of the form

$$\sum_{i=1}^{m} a_i f_i(t_1, \ldots, t_m) + \sum_{i=1}^{m} \beta_i b_i,$$

where f_i are polynomials in $K[t_1, \ldots, t_m]$ and $\beta_i \in K$. The multiplication in $A_m \mathrm{wr} B_m$ is defined by

$$[C_m, C_m] = [B_m, B_m] = 0,$$

$$[a_i f_i(t_1, \ldots, t_m), b_j] = a_i f_i(t_1, \ldots, t_m) t_j, \quad i, j = 1, \ldots, m.$$

Hence $A_m \mathrm{wr} B_m$ is a metabelian Lie algebra and every mapping $\{x_1, \ldots, x_m\} \to A_m \mathrm{wr} B_m$ can be extended to a homomorphism $F_m \to A_m \mathrm{wr} B_m$. In particular, as a special case of the embedding theorem of Shmel'kin [16], the mapping $x_i \to a_i + b_i$, $i = 1, \ldots, m$, can be extended to an embedding of F_m into $A_m \mathrm{wr} B_m$.

Both F_m and $A_m \mathrm{wr} B_m$ are graded algebras. The monomials in $A_m \mathrm{wr} B_m$ of degree 1 are of the form a_i, b_j and of degree $n \geq 2$ have the form $a_i t_1 \ldots t_{n-1}$. Let us consider the ideal $(A_m \mathrm{wr} B_m)^{c+1}$ spanned by the elements of $A_m \mathrm{wr} B_m$ of length at least $c+1$. Then the quotient $(A_m \mathrm{wr} B_m)/(A_m \mathrm{wr} B_m)^{c+1}$ is metabelian and nilpotent of class c and the homomorphism

$$\varepsilon : L_{m,c} \to (A_m \mathrm{wr} B_m)/(A_m \mathrm{wr} B_m)^{c+1}$$

defined by $\varepsilon(x_i) = a_i + b_i$, $i = 1, \ldots, m$, is a monomorphism. If

$$f = \sum [x_i, x_j] f_{ij}(\mathrm{ad} x_1, \ldots, \mathrm{ad} x_m), \quad f_{ij}(t_1, \ldots, t_m) \in K[t_1, \ldots, t_m]/\Omega^c,$$

where Ω is the augmentation ideal of $K[t_1, \ldots, t_m]$, then

$$\varepsilon(f) = \sum (a_i t_j - a_j t_i) f_{ij}(t_1, \ldots, t_m).$$

The next lemma follows from [16], see also [2].

Lemma 1.10. *The element* $\sum_{i=1}^m a_i f_i(t_1,\dots,t_m)$ *of* C_m *belongs to* $\varepsilon(L'_{m,c})$ *if and only if*

$$\sum_{i=1}^m t_i f_i(t_1,\dots,t_m) \equiv 0 \pmod{\Omega)^{c+1}}.$$

The embedding of $L_{m,c}$ into $A_m \mathrm{wr} B_m/(A_m \mathrm{wr} B_m)^{c+1}$ allows to introduce partial derivatives in $L_{m,c}$ with values in $K[t_1,\dots,t_m]/\Omega^c$. If $f \in L_{m,c}$ and

$$\varepsilon(f) = \sum_{i=1}^m \beta_i b_i + \sum_{i=1}^m a_i f_i(t_1,\dots,t_m), \quad \beta_i \in K, f_i \in K[t_1,\dots,t_m]/\Omega^c,$$

then

$$\frac{\partial f}{\partial x_i} = f_i(t_1,\dots,t_m).$$

The Jacobian matrix $J(\phi)$ of an endomorphism ϕ of $L_{m,c}$ is defined as

$$J(\phi) = \left(\frac{\partial \phi(x_j)}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial \phi(x_1)}{\partial x_1} & \cdots & \frac{\partial \phi(x_m)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi(x_1)}{\partial x_m} & \cdots & \frac{\partial \phi(x_m)}{\partial x_m} \end{pmatrix} \in M_m(K[t_1,\dots,t_m]/\Omega^c),$$

where $M_m(K[t_1,\dots,t_m]/\Omega^c)$ is the associative algebra of $m\times m$ matrices with entries from $K[t_1,\dots,t_m]/\Omega^c$. Let $\mathrm{IE}(L_{m,c})$ be the multiplicative semigroup of all endomorphisms of $L_{m,c}$ which are identical modulo the commutator ideal $L'_{m,c}$. Let I_m be the identity $m\times m$ matrix and let S be the subspace of $M_m(K[t_1,\dots,t_m]/\Omega^c)$ defined by

$$S = \left\{ (f_{ij}) \in M_m(K[t_1,\dots,t_m]/\Omega^c) \mid \sum_{i=1}^m t_i f_{ij} \equiv 0 \pmod{\Omega)^{c+1}},\ j=1,\dots,m \right\}.$$

Clearly $I_m + S$ is a subsemigroup of the multiplicative group of $M_m(K[t_1,\dots,t_m]/\Omega^c)$. If $\phi \in \mathrm{IE}(L_{m,c})$, then $J(\phi) = I_m + (s_{ij})$, where $s_{ij} \in S$. It is easy to check that if $\phi,\psi \in \mathrm{IE}(L_{m,c})$ then $J(\phi\psi) = J(\phi)J(\psi)$. The following proposition is well known, see e.g. [2].

Proposition 1.11. *The map* $J : \mathrm{IE}(L_{m,c}) \to I_m + S$ *defined by* $\phi \to J(\phi)$ *is an isomorphism of the semigroups* $\mathrm{IE}(L_{m,c}) = \mathrm{IA}(L_{m,c})$ *and* $I_m + S$.

Now we know that the group of normal automorphisms $\mathrm{N}(L_{m,c})$ is included in the subgroup $\mathrm{IA}(L_{m,c})$ when $m \geq 3$, $c = 2$, $m \geq 3$, $c = 3$, or $m \geq 2$, $c \geq 4$ and in other cases $m = 2$, $c = 2$ and $m = 2$, $c = 3$ every normal automorphism is a nonzero scalar times an IA-automorphism. The automorphism group $\mathrm{Aut}(L_{m,c})$ is a semidirect product of the normal subgroup $\mathrm{IA}(L_{m,c})$ and the general linear group $\mathrm{GL}_m(K)$. Considering the group of normal IA-automorphisms $\mathrm{IN}(L_{m,c})$, for the description of the factor group $\Gamma\mathrm{N}(L_{m,c}) = \mathrm{Aut}(L_{m,c})/\mathrm{N}(L_{m,c})$ it is sufficient to know only $\mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$. Drensky and Fındık [4] gave the explicit form of the Jacobian matrices of the coset representatives of the outer automorphisms in $\mathrm{IA}(L_{m,c})/\mathrm{Inn}(L_{m,c})$. Since $\mathrm{Inn}(L_{m,c})$ is included in the group of normal automorphisms, $\mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$ is the homomorphic image of the factor group $\mathrm{IA}(L_{m,c})/\mathrm{Inn}(L_{m,c})$.

Lemma 1.12 (Drensky and Fındık [4])**.** *The automorphisms with the following Jacobian matrices are coset representatives of the subgroup* $\mathrm{Inn}(L_{m,c})$ *of the group* $\mathrm{IA}(L_{m,c})$*:*

$$J(\theta) = I_m + \begin{pmatrix} s(t_2,\ldots,t_m) & f_{12} & \cdots & f_{1m} \\ t_1q_2(t_2,t_3,\ldots,t_m) + r_2(t_2,\ldots,t_m) & f_{22} & \cdots & f_{2m} \\ t_1q_3(t_3,\ldots,t_m) + r_3(t_2,\ldots,t_m) & f_{32} & \cdots & f_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ t_1q_m(t_m) + r_m(t_2,\ldots,t_m) & f_{m2} & \cdots & f_{mm} \end{pmatrix},$$

where $s, q_i, r_i, f_{ij} \in \Omega/\Omega^c$*, i.e., are polynomials of degree* $\leq c-1$ *without constant terms. They satisfy the conditions*

$$s + \sum_{i=2}^{m} t_iq_i \equiv 0, \quad \sum_{i=2}^{m} t_ir_i \equiv 0, \quad \sum_{i=1}^{m} t_if_{ij} \equiv 0 \pmod{\Omega}^{c+1}, \quad j = 2,\ldots,m,$$

$r_i = r_i(t_2,\ldots,t_m)$*,* $i = 1,\ldots,m$*, does not depend on* t_1*,* $q_i(t_i,\ldots,t_m)$*,* $i = 2,\ldots,m$*, does not depend on* $t_1,\ldots,t_{i-1}$ *and* f_{12} *does not contain a summand* dt_2*,* $d \in K$*.*

2. Generalized inner automorphisms. In this section we introduce a special type of automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$. We shall use these automorphisms in order to describe the group of normal automorphisms $\mathrm{N}(L_{m,c})$ of $L_{m,c}$ in the next section.

Definition 2.1. *An automorphism* ψ *of the algebra* $L_{m,c}$ *is called* generalized inner automorphism *if* ψ *has the form*

$$\psi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \dots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$.

Every inner automorphism is a generalized inner automorphism. We give necessary information for the structure of generalized inner automorphisms in the next lemmas and theorems.

Lemma 2.2. *Let* ψ *and* ϕ *be generalized inner automorphisms of the form*

$$\psi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \dots, m,$$

$$\phi : x_i \to x_i + \sum_{t=1}^{m} [x_i, x_t] g_t, \quad i = 1, \dots, m,$$

where $f_j, g_t \in K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$. *Then the composition* $\psi\phi$ *is of the form*

$$\psi\phi : x_i \to x_i + \sum_{t=1}^{m} [x_i, x_t] g_t + \sum_{j=1}^{m} [x_i, x_j] f_j + \sum_{j,t=1}^{m} [x_i, x_t, x_j] g_t f_j, \quad i = 1, \dots, m.$$

P r o o f. Let ψ and ϕ be as above. Then

$$\begin{aligned}
\psi(\phi(x_i)) &= \psi(x_i) + \sum_{t=1}^{m} [\psi(x_i), \psi(x_t)] g_t \\
&= \psi(x_i) + \sum_{t=1}^{m} \left[x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, x_t + \sum_{j=1}^{m} [x_t, x_j] f_j \right] g_t \\
&= \psi(x_i) + \sum_{t=1}^{m} [x_i, x_t] g_t + \sum_{t,j=1}^{m} ([x_i, x_j, x_t] - [x_t, x_j, x_i]) f_j g_t \\
&= x_i + \sum_{t=1}^{m} [x_i, x_j] g_t + \sum_{j=1}^{m} [x_i, x_j] f_j + \sum_{j,t=1}^{m} [x_i, x_t, x_j] g_t f_j, \\
&\qquad i = 1, \dots, m. \qquad \square
\end{aligned}$$

Theorem 2.3. *Generalized inner automorphisms form a subgroup of the automorphism group* $\mathrm{Aut}(L_{m,c})$.

P r o o f. Let ψ and ϕ be generalized inner automorphisms of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]f_j, \quad i = 1, \dots, m,$$

$$\phi : x_i \to x_i + \sum_{t=1}^{m}[x_i, x_t]g_t, \quad i = 1, \dots, m,$$

where $f_j, g_t \in K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$. Applying Lemma 2.2 we have that

$$(\psi\phi)(x_i) = x_i + \sum_{t=1}^{m}[x_i, x_t]g_t + \sum_{j=1}^{m}[x_i, x_j]f_j + \sum_{j,t=1}^{m}[x_i, x_t, x_j]g_t f_j$$
$$= x_i + \sum_{j=1}^{m}[x_i, x_j](g_j + f_j) + \sum_{t=1}^{m}[x_i, x_t]g_t \sum_{j=1}^{m}\mathrm{ad}x_j f_j,$$

for every $i = 1, \dots, m$. Let us put $h_t = g_t \sum_{j=1}^{m} \mathrm{ad}x_j f_j$, $t = 1, \dots, m$. So we have

$$(\psi\phi)(x_i) = x_i + \sum_{j=1}^{m}[x_i, x_j](g_j + f_j) + \sum_{t=1}^{m}[x_i, x_t]h_t$$
$$= x_i + \sum_{j=1}^{m}[x_i, x_j](g_j + f_j + h_j)$$
$$= x_i + \sum_{j=1}^{m}[x_i, x_j]F_j,$$

where $F_j = g_j + f_j + h_j$, $j = 1, \dots, m$. Thus the composition $\psi\phi$ is a generalized inner automorphism. It remains to prove that for any inverse automorphism ψ^{-1} of a generalized inner automorphism ψ is also a generalized inner automorphism. For this purpose it suffices to construct for each integer $n \geq 2$ a generalized inner automorphism ψ_n such that $\psi\psi_n$ is of the form

$$\psi\psi_n : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]h_j, \quad i = 1, \dots, m,$$

with $h_j \in \omega^{n-1}$, where ω states for the augmentation ideal of $K[\mathrm{ad}x_1, \dots, \mathrm{ad}x_m]$, i.e. the length of the commutator $[x_i, x_j]h_j$ is at least $n+1$. Let ψ be of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j](f_{j0} + \dots + f_{j,c-2}),$$

where $f_{j0} \in K$, $f_{jk} \in \omega^k/\omega^{k+1}$, $k = 1, \ldots, c-2$. Let us consider the generalized inner automorphism

$$\psi_2 : x_i \to x_i - \sum_{j=1}^{m} f_{j0}[x_i, x_j], \quad f_{j0} \in K.$$

From Lemma 2.2 we obtain that

$$\psi\psi_2 : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j](g_{j1} + \cdots + g_{j,c-2}), \quad g_{jk} \in \omega^k/\omega^{k+1}.$$

Now consider the generalized inner automorphism

$$\psi_3 : x_i \to x_i - \sum_{j=1}^{m} [x_i, x_j] g_{j1}, \quad g_{j1} \in \omega.$$

Similarly we have that

$$\psi\psi_2\psi_3 : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j](h_{j2} + \cdots + h_{j,c-2}), \quad h_{jk} \in \omega^k/\omega^{k+1}.$$

Repeating this process we construct $\psi_2, \psi_3, \ldots, \psi_c$ and obtain that

$$\psi\psi_2\psi_3 \ldots \psi_c = 1.$$

□

Lemma 2.4 *Let ψ be a generalized inner automorphism of the form*

$$\psi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \ldots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. Then for every $u \in L_{m,c}$

$$\psi(u) = u + \sum_{j=1}^{m} [u, x_j] f_j.$$

P r o o f. By linearity it is sufficient to show that for every $k = 1, 2, \ldots$

$$\psi([x_{i_1}, \ldots, x_{i_k}]) = [x_{i_1}, \ldots, x_{i_k}] + \sum_{j=1}^{m} [[x_{i_1}, \ldots, x_{i_k}], x_j] f_j.$$

We make induction on the degree k of the commutators. The case $k = 1$ is trivial.

It is true for $k = 2$:

$$\begin{aligned}\psi[x_p, x_q] &= [\psi(x_p), \psi(x_q)] \\ &= \left[x_p + \sum_{j=1}^{m}[x_p, x_j]f_j, x_q + \sum_{j=1}^{m}[x_q, x_j]f_j\right] \\ &= [x_p, x_q] + \sum_{j=1}^{m}([x_p, x_j, x_q] - [x_q, x_j, x_p])f_j \\ &= [x_p, x_q] + \sum_{j=1}^{m}[[x_p, x_q], x_j]f_j.\end{aligned}$$

Now assume that the equality holds for $k - 1$. Then

$$\begin{aligned}\psi([x_{i_1}, \ldots, x_{i_k}]) &= [\psi([x_{i_1}, \ldots, x_{i_{k-1}}]), \psi(x_{i_k})] \\ &= \left[[x_{i_1}, \ldots, x_{i_{k-1}}] + \sum_{j=1}^{m}[[x_{i_1}, \ldots, x_{i_{k-1}}], x_j]f_j, x_{i_k} + \sum_{j=1}^{m}[x_{i_k}, x_j]f_j\right] \\ &= [x_{i_1}, \ldots, x_{i_k}] + \sum_{j=1}^{m}[[x_{i_1}, \ldots, x_{i_k}], x_j]f_j. \qquad \square\end{aligned}$$

Corollary 2.5. *The group of generalized inner automorphisms* $\mathrm{GInn}(L_{m,c})$ *is a subgroup of the group of normal automorphisms* $\mathrm{N}(L_{m,c})$.

P r o o f. Let ψ be a generalized inner automorphism of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]f_j, \quad i = 1, \ldots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. Let u be an element of an ideal J of the free metabelian nilpotent Lie algebra $L_{m,c}$. From Lemma 2.4 we know that

$$\psi(u) = u + \sum_{j=1}^{m}[u, x_j]f_j.$$

Hence $\psi(u) \in J$. $\square$

Now we describe the group structure of the group of generalized inner automorphisms $\mathrm{GInn}(L_{m,c})$.

Theorem 2.6. *(i) The group* $\mathrm{GInn}(L_{m,2})$ *is abelian;*
(ii) The group $\mathrm{GInn}(L_{m,3})$ *is nilpotent of class* 2*;*
(iii) The group $\mathrm{GInn}(L_{m,c})$, $c \geq 4$, *is metabelian.*

Proof. (*i*) Let $\psi, \phi \in \mathrm{GInn}(L_{m,2})$ be generalized inner automorphisms of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m} \alpha_j [x_i, x_j], \quad i = 1, \ldots, m,$$

$$\phi : x_i \to x_i + \sum_{j=1}^{m} \beta_j [x_i, x_j], \quad i = 1, \ldots, m,$$

where $\alpha_j, \beta_j \in K$ for $j = 1, \ldots, m$. Then the composition $\psi\phi$ is

$$\begin{aligned}\psi(\phi(x_i)) &= \psi\left(x_i + \sum_{j=1}^{m} \beta_j [x_i, x_j]\right) \\ &= x_i + \sum_{j=1}^{m} \alpha_j [x_i, x_j] + \sum_{t=1}^{m} \beta_j [x_i, x_j] \\ &= x_i + \sum_{j=1}^{m} (\alpha_j + \beta_j)[x_i, x_j].\end{aligned}$$

Thus $\psi\phi = \phi\psi$.

(*ii*) Let $\varphi, \phi, \gamma \in \mathrm{GInn}(L_{m,3})$ be generalized inner automorphisms of the form

$$\varphi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \ldots, m,$$

$$\phi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] g_j, \quad i = 1, \ldots, m,$$

$$\gamma : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] h_j, \quad i = 1, \ldots, m,$$

where $f_j, g_j, h_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$ and let

$$u = \sum_{j=1}^{m} \mathrm{ad}x_j f_j, \quad v = \sum_{j=1}^{m} \mathrm{ad}x_j g_j, \quad w = \sum_{j=1}^{m} \mathrm{ad}x_j h_j.$$

Using the arguments of Theorem 2.3 we have that

$$\varphi^{-1} = \varphi_2 \varphi_3; \quad \phi^{-1} = \phi_2 \phi_3; \quad \gamma^{-1} = \gamma_2 \gamma_3,$$

where

$$\varphi_2 : x_i \to x_i - \sum_{j=1}^{m}[x_i, x_j]f_j; \quad \varphi_3 : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]f_j u,$$

$$\phi_2 : x_i \to x_i - \sum_{j=1}^{m}[x_i, x_j]g_j; \quad \phi_3 : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]g_j v,$$

$$\gamma_2 : x_i \to x_i - \sum_{j=1}^{m}[x_i, x_j]h_j; \quad \gamma_3 : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]h_j w.$$

Using Lemma 2.2 direct calculations give that

$$(\varphi, \phi) = \varphi^{-1}\phi^{-1}\varphi\phi = \varphi_2\varphi_3\phi_2\phi_3\varphi\phi$$

has the form

$$(\varphi, \phi) : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j](g_j u - f_j v).$$

Let us define $t = \sum_{j=1}^{m} \mathrm{ad}x_j(g_j u - f_j v)$. Then we obtain that

$$(\varphi, \phi, \gamma) : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j](h_j t - (g_j u - f_j v)w).$$

Since the polynomials $h_j t, (g_j u - f_j v)w \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$ have no components of degree ≤ 1, we obtain that $[x_i, x_j](h_j t - (g_j u - f_j v)w) = 0$ in $L_{m,3}$ and $(\varphi, \phi, \gamma) = 1$.

(iii) Let $m \geq 2, c \geq 4$ and let $\psi, \phi \in \mathrm{GInn}(L_{m,c})$ be generalized inner automorphisms of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]f_j, \quad i = 1, \ldots, m,$$

$$\phi : x_i \to x_i + \sum_{t=1}^{m}[x_i, x_t]g_t, \quad i = 1, \ldots, m,$$

where $f_j, g_t \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. Then we know from Lemma 2.2 that the composition $\psi\phi$ is of the form

$$\psi\phi : x_i \to x_i + \sum_{t=1}^{m}[x_i, x_t]g_t + \sum_{j=1}^{m}[x_i, x_j]f_j + \sum_{j,t=1}^{m}[x_i, x_t, x_j]g_t f_j, \quad i = 1, \ldots, m.$$

Lemma 2.4 states that

$$\psi(u) = u + \sum_{j=1}^{m} [u, x_j] f_j, \quad \phi(u) = u + \sum_{t=1}^{m} [u, x_t] g_t,$$

for every $u \in L_{m,c}$. Hence

$$\psi\phi : u \to u + \sum_{t=1}^{m} [u, x_t] g_t + \sum_{j=1}^{m} [u, x_j] f_j + \sum_{j,t=1}^{m} [u, x_t, x_j] g_t f_j.$$

If u is an element of the derived algebra $L'_{m,c}$, then

$$\psi\phi : u \to u + \sum_{t=1}^{m} [u, x_t] g_t + \sum_{j=1}^{m} [u, x_j] f_j + \sum_{j,t=1}^{m} u(\mathrm{ad}x_t)(\mathrm{ad}x_j) g_t f_j,$$

where $\mathrm{ad}x_t, \mathrm{ad}x_j, g_t, f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. Hence

$$\psi\phi(u) = \phi\psi(u), \quad u \in L'_{m,c}.$$

This means that the commutator

$$(\psi, \phi) = \psi^{-1}\phi^{-1}\psi\phi \in (\mathrm{GInn}(L_{m,c}), \mathrm{GInn}(L_{m,c}))$$

of ψ and ϕ acts trivially on $L'_{m,c}$.

Now let us define the generalized normal automorphisms ρ and σ in the commutator subgroup $(\mathrm{GInn}(L_{m,c}), \mathrm{GInn}(L_{m,c}))$ and let $w_1(x_i) = \rho(x_i) - x_i$ and $w_2(x_i) = \sigma(x_i) - x_i$, $i = 1, \ldots, m$. Then clearly the elements $w_1(x_i)$ and $w_2(x_i)$ are in $L'_{m,c}$, i.e. ρ and σ act trivially on them. Thus

$$\rho\sigma(x_i) = \rho(x_i + w_2(x_i)) = \rho(x_i) + w_2(x_i) = x_i + w_1(x_i) + w_2(x_i)$$
$$\sigma\rho(x_i) = \sigma(x_i + w_1(x_i)) = \sigma(x_i) + w_2(x_i) = x_i + w_1(x_i) + w_2(x_i),$$

which means that $\rho\sigma = \sigma\rho$. Hence $(\mathrm{GInn}(L_{m,c}), \mathrm{GInn}(L_{m,c}))$ is abelian and so $\mathrm{GInn}(L_{m,c})$ is metabelian. □

Example 2.7. Now we give an explicit proof of the fact that $\mathrm{GInn}(L_{2,3})$ is nilpotent of class 2. Let $\psi \in \mathrm{GInn}(L_{2,3})$ be a generalized inner automorphism of the form

$$\psi(x_1) = x_1 + \alpha[x_1, x_2] + \alpha_1[x_1, x_2, x_1] + \alpha_2[x_1, x_2, x_2]$$
$$\psi(x_2) = x_2 + \beta[x_1, x_2] + \beta_1[x_1, x_2, x_1] + \beta_2[x_1, x_2, x_2],$$

where $\alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \in K$. Easy calculations give that the inverse function ψ^{-1} has the form

$$\psi^{-1}(x_1) = x_1 - \alpha[x_1,x_2] - (\alpha\beta+\alpha_1)[x_1,x_2,x_1] + (\alpha^2-\alpha_2)[x_1,x_2,x_2]$$
$$\psi^{-1}(x_2) = x_2 - \beta[x_1,x_2] - (\beta^2+\beta_1)[x_1,x_2,x_1] + (\alpha\beta-\beta_2)[x_1,x_2,x_2].$$

If $\phi \in \mathrm{GInn}(L_{2,3})$ is another generalized inner automorphism,

$$\phi(x_1) = x_1 + p[x_1,x_2] + p_1[x_1,x_2,x_1] + p_2[x_1,x_2,x_2]$$
$$\phi(x_2) = x_2 + q[x_1,x_2] + q_1[x_1,x_2,x_1] + q_2[x_1,x_2,x_2],$$

where $p,p_1,p_2,q,q_1,q_2 \in K$ with inverse

$$\phi^{-1}(x_1) = x_1 - p[x_1,x_2] - (pq+p_1)[x_1,x_2,x_1] + (p^2-p_2)[x_1,x_2,x_2]$$
$$\phi^{-1}(x_2) = x_2 - q[x_1,x_2] - (q^2+q_1)[x_1,x_2,x_1] + (pq-q_2)[x_1,x_2,x_2],$$

calculating the composition $\psi\phi$ we have that

$$\psi\phi(x_1) = x_1 + (\alpha+p)[x_1,x_2] + (\alpha_1+p_1-p\beta)[x_1,x_2,x_1] + (\alpha_2+p_2+p\alpha)[x_1,x_2,x_2]$$
$$\psi\phi(x_2) = x_2 + (\beta+q)[x_1,x_2] + (\beta_1+q_1-q\beta)[x_1,x_2,x_1] + (\beta_2+q_2+q\alpha)[x_1,x_2,x_2],$$

and the composition $\phi^{-1}\psi\phi$ is of the form

$$\phi^{-1}\psi\phi(x_1) = x_1 + \alpha[x_1,x_2] + (\alpha_1-p\beta+q\alpha)[x_1,x_2,x_1] + \alpha_2[x_1,x_2,x_2]$$
$$\phi^{-1}\psi\phi(x_2) = x_2 + \beta[x_1,x_2] + \beta_1[x_1,x_2,x_1] + (\beta_2-p\beta+q\alpha)[x_1,x_2,x_2].$$

Finally we obtain that $(\psi,\phi) = \psi^{-1}\phi^{-1}\psi\phi$ has the form

$$(\psi,\phi)(x_1) = x_1 + (\alpha q-\beta p)[x_1,x_2,x_1]$$
$$(\psi,\phi)(x_2) = x_2 + (\alpha q-\beta p)[x_1,x_2,x_2].$$

Now let $\theta \in \mathrm{GInn}(L_{2,3})$ be a generalized inner automorphism of the form

$$\theta(x_1) = x_1 + a[x_1,x_2] + a_1[x_1,x_2,x_1] + a_2[x_1,x_2,x_2]$$
$$\theta(x_2) = x_2 + b[x_1,x_2] + b_1[x_1,x_2,x_1] + b_2[x_1,x_2,x_2],$$

where $a,a_1,a_2,b,b_1,b_2 \in K$. Direct calculations give that

$$((\psi,\phi),\theta)(x_1) = x_1 + (0.b-0.a)[x_1,x_2,x_1]$$
$$((\psi,\phi),\theta)(x_2) = x_2 + (0.b-0.a)[x_1,x_2,x_2],$$

which means that

$$((\psi,\phi),\theta) = (\psi,\phi)^{-1}\theta^{-1}(\phi,\psi)\theta = 1.$$

3. Main results. In this section we describe the group of normal automorphisms in terms of generalized inner automorphisms. We give the explicit form of the Jacobian matrices of the normal automorphisms and of the coset representatives of normally outer IA-automorphisms.

Lemma 3.1. *Let φ be a normal* IA*-automorphism of $L_{m,2}$. Then φ is a generalized inner automorphism of $L_{m,2}$. Furthermore φ is an inner automorphism of $L_{m,2}$.*

Proof. Clearly, $L'_{m,2}$ has a basis $[x_i, x_j]$, $1 \leq i < j \leq m$. Let φ be a normal automorphism in $\mathrm{IA}(L_{m,2})$. If $m = 2$, then $\mathrm{IA}(L_{2,2}) = \mathrm{Inn}(L_{2,2})$. Since $\mathrm{Inn}(L_{2,2}) \subset \mathrm{N}(L_{2,2})$ then φ is an inner automorphism. In particular φ is a generalized inner automorphism. Let $m \geq 3$ and φ be of the form

$$\begin{aligned}\varphi(x_1) =& x_1 + [x_1, c_{11}x_1 + c_{12}x_2 + \cdots + c_{1m}x_m]\\ \varphi(x_2) =& x_2 + [x_2, c_{21}x_1 + c_{22}x_2 + \cdots + c_{2m}x_m]\\ &\vdots\\ \varphi(x_m) =& x_m + [x_m, c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mm}x_m],\end{aligned}$$

where $c_{ij} \in K$ for every $i, j = 1, \ldots, m$. Now consider the ideal J_{12} of $L_{m,2}$ generated by $x_1 + x_2$. As a vector space J_{12} posseses a basis

$$x_1 + x_2,\ [x_1, x_2],\ [x_1 + x_2, x_j], \quad j = 3, \ldots, m.$$

Since φ is normal $\varphi(x_1 + x_2) \in J_{12}$. But

$$\varphi(x_1+x_2) = x_1+x_2+(c_{12}-c_{21})[x_1, x_2]+\sum_{j=3}^{m} c_{1j}[x_1+x_2, x_j]+\sum_{j=3}^{m}(c_{2j}-c_{1j})[x_2, x_j],$$

which means that $\sum_{j=3}^{m}(c_{2j} - c_{1j})[x_2, x_j] \in J_{12} \cap L'_{m,c}$. Then

$$\sum_{j=3}^{m} d_j[x_2, x_j] = p[x_1, x_2] + \sum_{j=3}^{m} q_j[x_1 + x_2, x_j],$$

for some $p, q_j \in K$, $d_j = c_{2j} - c_{1j}$, $j = 3, \ldots, m$, which means that $d_j = 0$. Hence $c_{2j} = c_{1j}$, $j = 3, \ldots, m$. Similarly, considering the ideals J_{1k} of $L_{m,2}$ generated by $x_1 + x_k$ for every $k = 3, \ldots, m$, we obtain that

$$c_{k2} = c_{12}, \ldots, c_{k,k-1} = c_{1,k-1}, c_{k,k+1} = c_{1,k+1}, \ldots, c_{km} = c_{1m}.$$

Finally, considering the ideals J_{2k} of $L_{m,2}$ generated by $x_2 + x_k$, $k = 3, \ldots, m$, similar arguments give that

$$c_{k1} = c_{21}, \quad k = 3, \ldots, m.$$

Thus

$$\varphi = \exp(\mathrm{ad}u), \quad u = c_{21}x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1m}x_m,$$

i.e. φ is an inner automorphism. $\square$

Lemma 3.2. *Every normal* IA-*automorphism of* $L_{2,3}$ *is generalized inner.*

P r o o f. Let φ be a normal IA-automorphism of $L_{2,3}$ such that

$$\begin{aligned}\varphi(x_1) =& x_1 + \alpha[x_1, x_2] + \alpha_1[x_1, x_2, x_1] + \alpha_2[x_1, x_2, x_2]\\ \varphi(x_2) =& x_2 + \beta[x_2, x_1] + \beta_1[x_2, x_1, x_1] + \beta_2[x_2, x_1, x_2],\end{aligned}$$

where $\alpha, \alpha_1, \alpha_2, \beta, \beta_1\beta_2 \in K$. Let us define $f_1 = \beta + \beta_1 \mathrm{ad}x_1 + \beta_2 \mathrm{ad}x_2$ and $f_2 = \alpha + \alpha_1 \mathrm{ad}x_1 + \alpha_2 \mathrm{ad}x_2$. Then we can rewrite φ in the following way.

$$\begin{aligned}\varphi(x_1) =& x_1 + \sum_{j=1}^{2}[x_1, x_j]f_j,\\ \varphi(x_2) =& x_2 + \sum_{j=1}^{2}[x_2, x_j]f_j,\end{aligned}$$

which completes the proof. $\square$

We know that $\mathrm{Inn}(L_{m,c}) \subset \mathrm{N}(L_{m,c})$. If $c = 2$, then $\mathrm{Inn}(L_{m,2}) = \mathrm{IN}(L_{m,2})$ by Lemma 3.1. But the elements φ of $\mathrm{IN}(L_{m,c})$ are not necessarily inner automorphisms when $c \geq 3$. For example it follows from Lemma 3.2 that

$$\begin{aligned}\varphi(x_1) =& x_1 + [x_1, x_2, x_2]\\ \varphi(x_2) =& x_2\end{aligned}$$

is a normal automorphism which is not an inner automorphism.

Lemma 3.3. *Let* φ *be a normal* IA-*automorphism of* $L_{m,c}$ *acting trivially on* $L_{m,c}/L_{m,c}^c$. *Then* φ *is a generalized inner automorphism.*

P r o o f. If $m = 2$ then φ is of the form

$$\begin{aligned}\varphi(x_1) =& x_1 + [x_1, x_2]f_{12}\\ \varphi(x_2) =& x_2 + [x_2, x_1]f_{21},\end{aligned}$$

where $[x_1, x_2]f_{12}, [x_2, x_1]f_{21} \in L_{2,c}^c$. This means that

$$\begin{aligned}\varphi(x_1) =& x_1 + [x_1, x_1]f_{21} + [x_1, x_2]f_{12}\\ \varphi(x_2) =& x_2 + [x_2, x_1]f_{21} + [x_2, x_2]f_{12}.\end{aligned}$$

Thus φ is a generalized inner automorphism. In the case $c = 2$, $m \geq 2$, we know from Lemma 3.1 that such automorphisms are generalized inner. Hence we can assume that $c \geq 3$, $m \geq 3$. Let φ be a normal automorphism of $L_{m,c}$ acting trivially on $L_{m,c}/L_{m,c}^c$. Since $c \geq 3$ and $m \geq 3$ then we can assume from Lemma 1.8 that φ is of the form

$$\varphi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_{ij},$$

where $[x_i, x_j] f_{ij}(\mathrm{ad}x_1, \dots, \mathrm{ad}x_m)$ is in the the center $L_{m,c}^c$ of the free metabelian nilpotent Lie algebra $L_{m,c}$ for every $i, j = 1, \dots, m$. Such automorphisms form an abelian subgroup of $\mathrm{Aut}L_{m,c}$. Let us define the generalized inner automorphism

$$\varphi_1 : x_i \to x_i + \sum_{j=2}^{m} [x_i, x_j] f_{1j}.$$

Then the composition $\varphi\varphi_1^{-1}$ has the form

$$\begin{aligned} \varphi\varphi_1^{-1}(x_1) =& x_1 \\ \varphi\varphi_1^{-1}(x_k) =& x_k + [x_k, x_1] f_{k1} + \sum_{j \neq 1,k}^{m} [x_k, x_j](f_{kj} - f_{1j}), \quad k \neq 1. \end{aligned}$$

Now consider the generalized inner automorphism $\varphi_2 : x_i \to x_i + [x_i, x_1] f_{21}$. Then

$$\begin{aligned} \varphi\varphi_1^{-1}\varphi_2^{-1}(x_1) =& x_1 \\ \varphi\varphi_1^{-1}\varphi_2^{-1}(x_2) =& x_2 + \sum_{j \neq 1,2} [x_2, x_j] g_{2j} \\ \varphi\varphi_1^{-1}\varphi_2^{-1}(x_3) =& x_3 + [x_3, x_1] g_{31} + \sum_{j \neq 1,3} [x_3, x_j] g_{3j} \\ &\vdots \\ \varphi\varphi_1^{-1}\varphi_2^{-1}(x_m) =& x_m + [x_m, x_1] g_{m1} + \sum_{j \neq 1,m} [x_m, x_j] g_{mj} \end{aligned}$$

where $g_{k1} = f_{k1} - f_{21}$ for $k \geq 3$ and $g_{kj} = f_{kj} - f_{1j}$ for $k \geq 2$, $j \geq 2$. Thus it suffices to show that $\phi = \varphi\varphi_1^{-1}\varphi_2^{-1}$ is a generalized inner automorphism. Let α be a nonzero constant and let us consider the ideal $J_{\alpha 12}$ of $L_{m,c}$ generated by $\alpha x_1 + x_2$. The vector space $J_{\alpha 12}$ has a basis modulo $L_{m,c}^3$:

$$\alpha x_1 + x_2,\ [x_1, x_2],\ [\alpha x_1 + x_2, x_j], \quad j = 3, \dots, m.$$

Since ϕ is normal, $\phi(\alpha x_1 + x_2) \in J_{\alpha 12}$,

$$\phi(\alpha x_1 + x_2) = \alpha x_1 + x_2 + \sum_{j=3}^{m} [x_2, x_j] g_{2j},$$

which means that

$$\sum_{j=3}^{m} [x_2, x_j] g_{2j} \in J_{\alpha 12} \cap L'_{m,c}.$$

Then

$$\sum_{j=3}^{m} [x_2, x_j] g_{2j} = [x_1, x_2] P + \sum_{j=3}^{m} [\alpha x_1 + x_2, x_j] Q_j,$$

for some $P, Q_j \in \omega^{c-2}/\omega^{c-1}$, $j = 3, \ldots, m$. Using the embedding $L_{m,c}$ into the wreath product we have that

$$\begin{aligned} & a_2 \sum_{j=3}^{m} t_j g_{2j} - \sum_{j=3}^{m} a_j t_2 g_{2j} \\ & \quad = a_1 \left(t_2 P + \alpha \sum_{j=3}^{m} t_j Q_j \right) + a_2 \left(-t_1 P + \sum_{j=3}^{m} t_j Q_j \right) - \sum_{j=3}^{m} a_j (\alpha t_1 + t_2) Q_j. \end{aligned}$$

we have that Since $a_1, \ldots, a_m$ are free generators of a free $K[t_1, \ldots, t_m]/\Omega^{c-1}$-module, for every $j = 3, \ldots, m$ we have that

$$t_2 g_{2j} = (\alpha t_1 + t_2) Q_j.$$

Thus $\alpha t_1 + t_2$ divides g_{2j}, $j = 3, \ldots, m$, for every $\alpha \in K^*$. Since characteristic of the field K is 0 we can choose more than $c - 2$ distinct scalars $\alpha \in K^*$. Then by nilpotency the function $g_{2j}(t_1, \ldots, t_m)$ is 0, $j = 3, \ldots, m$. Hence

$$g_{23} = \cdots = g_{2m} = 0.$$

Considering the ideals $J_{\alpha 1k}$, $k = 3, \ldots, m$ of $L_{m,c}$ generated by $\alpha x_1 + x_k$ the same argument gives that $g_{kj} = 0$, $j \neq 1, k$, $k = 3, \ldots, m$.

Now let us consider the ideal $J_{\alpha 23}$ of $L_{m,c}$ generated by $\alpha x_2 + x_3$. It has a basis

$$\alpha x_2 + x_3,\ [x_2, x_3],\ [\alpha x_2 + x_3, x_j], \quad j \neq 2, 3,$$

modulo $L^3_{m,c}$. Since ϕ is normal, $\phi(\alpha x_2 + x_3) \in J_{\alpha 23}$.

$$\phi(\alpha x_2 + x_3) = \alpha x_2 + x_3 + \sum_{j \neq 3} [x_3, x_j] g_{3j}.$$

This means that $[x_3,x_1]g_{31}\in J_{\alpha 23}\cap L'_{m,c}$ because we know that $g_{3k}=0$, $k\neq 1,3$. Then

$$[x_3,x_1]g_{31}=[x_2,x_3]P+\sum_{j\neq 2,3}[\alpha x_2+x_3,x_j]Q_j,$$

for some $P,Q_j\in\omega^{c-2}/\omega^{c-1}$, $j\neq 2,3$. Using the embedding $L_{m,c}$ into the wreath product, considering only the coefficient of a_1 we have that

$$t_3g_{31}=(\alpha x_2+x_3)Q_1.$$

Thus αt_2+t_3 divides g_{31} for every $\alpha\in K^*$. Hence the function $g_{31}(t_1,\dots,t_m)$ is 0 and $g_{31}=0$.

Finally, considering the ideals $J_{\alpha 2k}$, $k=4,\dots,m$, of $L_{m,c}$ generated by αx_2+x_k the same argument gives that $g_{k1}=0$, $k=4,\dots,m$. Hence $\phi=\varphi\varphi_1^{-1}\varphi_2^{-1}=1$, i.e. $\varphi=\varphi_2\varphi_1$ which means that φ is a generalized inner automorphism. $\square$

Theorem 3.4. *Let φ be a normal* IA-*automorphism of $L_{m,c}$. Then φ is a generalized inner automorphism.*

P r o o f. We argue by induction on the nilpotency class c of $L_{m,c}$. If $c=2$, the result follows from Lemma 3.1. (In this case, each normal IA-automorphism is inner.) Now consider a normal IA-automorphism φ of $L_{m,c}$, $c>2$, of the form

$$\varphi: x_i\to x_i+\sum_{j=1}^m[x_i,x_j](f_{ij,0}+\dots+f_{ij,c-2}),$$

where $f_{ij,0}\in K$, $f_{ij,k}\in\omega^k/\omega^{k+1}$, $k=1,\dots,c-2$. Then φ induces a normal IA-automorphism on $L_{m,c}/L^c_{m,c}$. By induction, since this quotient is isomorphic to $L_{m,c-1}$, there exists a generalized inner automorphism $\psi: L_{m,c}\to L_{m,c}$ such that

$$\varphi(x_i)=\psi(x_i)+\sum_{j=1}^m[x_i,x_j]f_{ij,c-2},\quad i=1,\dots,m.$$

It follows for $i=1,\dots,m$ that

$$\begin{aligned}\psi^{-1}\varphi(x_i)&=x_i+\sum_{j=1}^m\psi^{-1}([x_i,x_j]f_{ij,c-2})\\&=x_i+\sum_{j=1}^m[x_i,x_j]f_{ij,c-2}.\end{aligned}$$

Thus $\phi = \psi^{-1}\varphi$ is a normal IA-automorphism of $L_{m,c}$ acting trivially on $L_{m,c}/L_{m,c}^c$. By Lemma 3.3, ϕ is a generalized inner automorphism, and so is $\varphi = \psi\phi$. □

Now we give one of the main results which is obtained as a direct consequence of Lemma 1.1, Lemma 1.9, Lemma 3.1, Lemma 3.2 and Theorem 3.4.

Corollary 3.5. *Let K^* denote the set of invertible elements of the field K. Then*

(*i*) $\mathrm{N}(L_{m,1}) \cong K^*$;

(*ii*) $\mathrm{N}(L_{2,2}) \cong K^* \leftthreetimes \mathrm{Inn}(L_{2,2})$;

(*iii*) $\mathrm{N}(L_{2,3}) \cong K^* \leftthreetimes \mathrm{GInn}(L_{2,3})$;

(*iv*) $\mathrm{N}(L_{m,c}) = \mathrm{GInn}(L_{m,c}), \quad m \geq 3,\ c \geq 2$ or $m = 2,\ c \geq 4$,

where $\leftthreetimes$ stands for the semi-direct product of the groups.

Now we describe the group structure of the group of normal automorphisms $\mathrm{N}(L_{m,c})$.

Theorem 3.6. (*i*) *The group* $\mathrm{N}(L_{m,2})$, $m \geq 3$, *is abelian;*
(*ii*) *The group* $\mathrm{N}(L_{m,3})$, $m \geq 3$, *is nilpotent of class* 2*;*
(*iii*) *The group* $\mathrm{N}(L_{m,c})$, $m \geq 2, c \geq 4$ *or* $(m,c) = (2,2)$, *is metabelian;*
(*iv*) *The group* $\mathrm{N}(L_{2,3})$, *is nilpotent of class two–by–abelian.*

P r o o f. Let $\psi_\alpha, \phi_\beta \in \mathrm{N}(L_{2,2}) \cong K^* \leftthreetimes \mathrm{Inn}(L_{2,2})$ be normal automorphisms of the form

$$\begin{aligned}\psi_\alpha(x_1) =& \alpha x_1 + \alpha\alpha_2[x_1, x_2]\\ \psi_\alpha(x_2) =& \alpha x_2 + \alpha\alpha_1[x_2, x_1],\end{aligned}$$

$$\begin{aligned}\psi_\beta(x_1) =& \beta x_1 + \beta\beta_2[x_1, x_2]\\ \psi_\beta(x_2) =& \beta x_2 + \beta\beta_1[x_2, x_1],\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K$ and $\alpha, \beta \in K^*$. Easy calculations give that

$$\begin{aligned}\psi_\alpha^{-1}(x_1) =& \alpha^{-1}x_1 - \alpha^{-2}\alpha_2[x_1, x_2]\\ \psi_\alpha^{-1}(x_2) =& \alpha^{-1}x_2 - \alpha^{-2}\alpha_1[x_2, x_1],\end{aligned}$$

$$\begin{aligned}\psi_\beta^{-1}(x_1) =& \beta^{-1}x_1 - \beta^{-2}\beta_2[x_1, x_2]\\ \psi_\beta^{-1}(x_2) =& \beta^{-1}x_2 - \beta^{-2}\beta_1[x_2, x_1].\end{aligned}$$

By direct calculations we obtain that the commutator $(\psi_\alpha, \phi_\beta) = \psi_\alpha^{-1}\phi_\beta^{-1}\psi_\alpha\phi_\beta$ has the form

$$(\psi_\alpha, \phi_\beta) : x_i \to x_i + \sum_{j=1}^{m}(\alpha^{-1}\alpha_j(\beta^{-1} - 1) + \beta^{-1}\beta_j(1 - \alpha^{-1}))[x_i, x_j], \quad i = 1, 2.$$

This means that $(\psi_\alpha, \phi_\beta) \in \mathrm{GInn}(L_{2,2}) = \mathrm{Inn}(L_{2,2})$ which is abelian from Theorem 2.6. Hence $\mathrm{N}(L_{2,2})$ is metabelian.

We know from Corollary 3.5 that if $m \geq 3$ or $m = 2, c \geq 4$ then $\mathrm{N}(L_{m,c}) = \mathrm{GInn}(L_{m,c})$. Applying Theorem 2.6 we get that $\mathrm{N}(L_{m,2})$ is abelian when $m \geq 3$, $\mathrm{N}(L_{m,3})$ is nilpotent of class 2 when $m \geq 3$ and that $\mathrm{N}(L_{m,c})$ is metabelian when $m \geq 2$, $c \geq 4$. Thus it remains to show that $\mathrm{N}(L_{2,3})$ is a nilpotent of class two–by–abelian group.

Now let ψ_α, ϕ_β in $\mathrm{N}(L_{2,3})$ be normal automorphisms of the form

$$\psi_\alpha : x_i \to \alpha x_i + f_i, \quad i = 1, \ldots, m,$$

$$\phi_\beta : x_i \to \beta x_i + g_i, \quad i = 1, \ldots, m,$$

where $f_i, g_i \in L'_{m,c}$ and $\alpha, \beta \in K^*$. Clearly the inverse functions are of the form

$$\psi_\alpha^{-1} : x_i \to \alpha^{-1} x_i + f'_i, \quad i = 1, \ldots, m,$$

$$\phi_\beta^{-1} : x_i \to \beta^{-1} x_i + g'_i, \quad i = 1, \ldots, m,$$

where $f'_i, g'_i \in L'_{m,c}$. Easy calculations give that the commutator $(\psi_\alpha, \phi_\beta)$ of ψ_α and ϕ_β is included in $\mathrm{GInn}(L_{2,3})$ which is nilpotent of class 2 by Theorem 2.6. Thus $\mathrm{N}(L_{2,3})$ is a nilpotent of class two–by–abelian group. $\square$

Now we have collected the necessary information for the description of the group of normally outer automorphisms $\Gamma\mathrm{N}(L_{m,c})$. We shall find the coset representatives of the normal subgroup $\mathrm{IN}(L_{m,c})$ of the group $\mathrm{IA}(L_{m,c})$ of IA-automorphisms $L_{m,c}$, i.e., we shall find a set of IA-automorphisms θ of $L_{m,c}$ such that the factor group $\mathrm{I\Gamma N}(L_{m,c}) = \mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$ of the outer IA-automorphisms of $L_{m,c}$ is presented as the disjoint union of the cosets $\mathrm{IN}(L_{m,c})\theta$.

Lemma 3.7. *Let $m = 2$, then the group of normally outer* IA-*automorphisms* $\mathrm{I\Gamma N}(L_{2,c})$ *is trivial.*

Proof. Let φ be an IA-automorphism of $L_{2,c}$. Then φ has the form

$$\varphi(x_1) \to x_1 + [x_1, x_2]f$$
$$\varphi(x_2) \to x_2 + [x_1, x_2]g,$$

where $f, g \in K[\mathrm{ad}x_1, \mathrm{ad}x_2]$. Then clearly

$$\varphi(x_1) = x_1 + [x_1, x_1]f_1 + [x_1, x_2]f_2$$
$$\varphi(x_2) = x_2 + [x_2, x_1]f_1 + [x_2, x_2]f_2,$$

where $f_1 = g$, $f_2 = f$, i.e. φ is a generalized inner automorphism or from Theorem 3.4 φ is a normal IA-automorphism. Thus $\mathrm{IA}(L_{2,c}) = \mathrm{IN}(L_{2,c})$. $\square$

Theorem 3.8. *(i) Let φ be a normal* IA-*automorphism of the form*

$$\varphi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j]f_j, \quad i = 1, \ldots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. Then the Jacobian matrix of φ is

$$J(\varphi) = I_m + \begin{pmatrix} t_2f_2 + \cdots + t_mf_m & -t_2f_1 & \cdots & -t_mf_1 \\ -t_1f_2 & \sum_{j\neq 2} t_jf_j & \cdots & -t_mf_2 \\ -t_1f_3 & -t_2f_3 & \cdots & -t_mf_3 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1f_m & -t_2f_m & \cdots & \sum_{j\neq m} t_jf_j \end{pmatrix},$$

(ii) Let Θ be the set of automorphisms θ of $L_{m,c}$ with Jacobian matrix of the form

$$J(\theta) = I_m + \begin{pmatrix} 0 & f_{12}(\hat{t}_2) & \cdots & f_{1m} \\ p_2(\hat{t}_1) & f_{22} & \cdots & f_{2m} \\ p_3(\hat{t}_1) & f_{32} & \cdots & f_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\hat{t}_1) & f_{m2} & \cdots & f_{mm} \end{pmatrix},$$

where p_i, f_{ij}, are polynomials of degree $\leq c-1$ without constant terms with the following conditions

$$\sum_{i=2}^{m} t_ip_i \equiv 0, \quad \sum_{i=1}^{m} t_if_{ij} \equiv 0 \pmod{\Omega}^{c+1}, \quad j = 2, \ldots, m,$$

$p_i = p_i(\hat{t}_1)$, $i = 1, \ldots, m$, does not depend on t_1, and $f_{12} = f_{12}(\hat{t}_2)$ does not depend on t_2.

Then Θ consists of coset representatives of the subgroup $\mathrm{IN}(L_{m,c})$ *of the group* $\mathrm{IA}(L_{m,c})$ *and* $\mathrm{I\Gamma N}(L_{m,c})$ *is a disjoint union of the cosets* $\mathrm{IN}(L_{m,c})\theta$, $\theta \in \Theta$.

(iii) Let Ψ be the set of normal IA*-automorphisms ψ of $L_{m,c}$ with Jacobian matrix of the form*

$$J(\psi) = I_m + \begin{pmatrix} \sum_{j\neq 1} t_j q_j(T_j) & -t_2 q_1(T_1) & \cdots & -t_m q_1(T_1) \\ -t_1 q_2(T_2) & \sum_{j\neq 2} t_j q_j(T_j) & \cdots & -t_m q_2(T_2) \\ -t_1 q_3(T_3) & -t_2 q_3(T_3) & \cdots & -t_m q_3(T_3) \\ \vdots & \vdots & \ddots & \vdots \\ -t_1 q_m(T_m) & -t_2 q_m(T_m) & \cdots & \sum_{j\neq m} t_j q_j(T_j) \end{pmatrix},$$

where $q_j(T_j)$, $j = 1, \ldots, m$, are polynomials of degree $\leq c-1$ in Ω^2 with the following conditions

$$\sum_{i=2}^{m} q_j(T_j) \equiv 0 \pmod{\Omega}^{c+1},$$

and $q_j(T_j)$ depends on $t_j, \ldots, t_m$ only, $j = 1, \ldots, m$.

Then Ψ consists of coset representatives of the subgroup $\mathrm{Inn}(L_{m,c})$ *of the group* $\mathrm{IN}(L_{m,c})$ *and* $\mathrm{IN}(L_{m,c})/\mathrm{Inn}(L_{m,c})$ *is a disjoint union of the cosets* $\mathrm{Inn}(L_{m,c})\psi$, $\psi \in \Psi$.

Proof. (*i*) Let φ be a normal IA-automorphism of the form

$$\psi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \ldots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$. The Jacobian matrix of φ is

$$J(\varphi) = \left(\frac{\partial \varphi(x_j)}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial \varphi(x_1)}{\partial x_1} & \cdots & \frac{\partial \varphi(x_m)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi(x_1)}{\partial x_m} & \cdots & \frac{\partial \varphi(x_m)}{\partial x_m} \end{pmatrix} \in M_m(K[t_1, \ldots, t_m]/\Omega^c).$$

Easy calculations give

$$\frac{\partial \varphi(x_j)}{\partial x_i} = \delta_{ij} + \begin{cases} \sum_{r\neq j} t_r f_r & i = j, \\ -t_j f_i & i \neq j, \end{cases}$$

where δ_{ij} is Kronecker symbol. Thus we obtain the desired form of the matrix $J(\varphi)$.

(*ii*) When $m = 2$ then from Lemma 3.7 the factor group $\mathrm{IA}(L_{2,c})/\mathrm{IN}(L_{2,c})$ is trivial which satisfies the conditions. Let $m \geq 3$. Since $\mathrm{Inn}(L_{m,c})$ is included in the group of normal automorphisms, the factor group $\mathrm{IA}(L_{m,c})/\mathrm{IN}(L_{m,c})$ is the homomorphic image of $\mathrm{IA}(L_{m,c})/\mathrm{Inn}(L_{m,c})$. Then from Lemma 1.12 we can consider the Jacobian matrix of the IA-automorphism ψ of the form

$$J(\psi) = I_m + \begin{pmatrix} s(t_2,\ldots,t_m) & f_{12} & \cdots & f_{1m} \\ t_1q_2(t_2,t_3,\ldots,t_m)+r_2(t_2,\ldots,t_m) & f_{22} & \cdots & f_{2m} \\ t_1q_3(t_3,\ldots,t_m)+r_3(t_2,\ldots,t_m) & f_{32} & \cdots & f_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ t_1q_m(t_m)+r_m(t_2,\ldots,t_m) & f_{m2} & \cdots & f_{mm} \end{pmatrix},$$

where s, q_i, r_i, f_{ij} are polynomials of degree $\leq c-1$ without constant terms with the conditions

$$s + \sum_{i=2}^{m} t_iq_i \equiv 0, \quad \sum_{i=2}^{m} t_ir_i \equiv 0, \quad \sum_{i=1}^{m} t_if_{ij} \equiv 0 \pmod{\Omega}^{c+1}, \quad j = 2,\ldots,m,$$

$s = s(t_2,\ldots,t_m)$, $r_i = r_i(t_2,\ldots,t_m)$, $i = 1,\ldots,m$, does not depend on t_1, $q_i(t_i,\ldots,t_m)$, $i = 2,\ldots,m$, does not depend on $t_1,\ldots,t_{i-1}$ and f_{12} does not contain a summand dt_2, $d \in K$.

Let

$$f_1 = 0, \quad f_k = q_k, \quad k = 2,\ldots,m,$$

and let us define the normal automorphism

$$\varphi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1,\ldots,m.$$

Then from (i) the Jacobian matrix of φ is of the form

$$J(\varphi) = I_m + \begin{pmatrix} -s & 0 & \cdots & 0 \\ -t_1q_2 & -s-t_2q_2 & \cdots & -t_mq_2 \\ -t_1q_3 & -t_2q_3 & \cdots & -t_mq_3 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1q_m & -t_2q_m & \cdots & -s-t_mq_m \end{pmatrix}.$$

Let us denote the $m \times 2$ matrix consisting of the first two columns of $J(\varphi\psi)$ and I_m by $J(\varphi\psi)_2$ and I_{m2}, respectively. Direct calculations give that $J(\varphi\psi)_2$ is of the form

$$J(\varphi\psi)_2 = I_{m2} + \begin{pmatrix} -s^2 & -sf_{12} + f_{12} \\ -s(t_1q_2 + r_2) + r_2 & * \\ -s(t_1q_3 + r_3) + r_3 & * \\ \vdots & \vdots \\ -s(t_1q_m + r_m) + r_m & * \end{pmatrix},$$

where we have denoted by $*$ the corresponding entries of the second column of the Jacobian matrix of $\varphi\psi$.

Now let

$$g_1 = 0, \quad g_k = -sq_k, \quad k = 2, \dots, m,$$

and let us define the normal automorphism

$$\phi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] g_j, \quad i = 1, \dots, m.$$

The Jacobian matrix of ϕ is of the form

$$J(\phi) = I_m + \begin{pmatrix} s^2 & 0 & \cdots & 0 \\ st_1q_2 & s(s + t_2q_2) & \cdots & st_mq_2 \\ st_1q_3 & st_2q_3 & \cdots & st_mq_3 \\ \vdots & \vdots & \ddots & \vdots \\ st_1q_m & st_2q_m & \cdots & s(s + t_mq_m) \end{pmatrix}.$$

Calculating $J(\phi\varphi\psi)$ we have that

$$J(\phi\varphi\psi)_2 = I_{m2} + \begin{pmatrix} -s^4 & f_{12}(1 - s + s^2 - s^3) \\ -s^3t_1q_2 + r_2(1 - s + s^2 - s^3) & * \\ -s^3t_1q_3 + r_3(1 - s + s^2 - s^3) & * \\ \vdots & \vdots \\ -s^3t_1q_m + r_m(1 - s + s^2 - s^3) & * \end{pmatrix}.$$

Repeating this process sufficiently many times, we get that the $(1,1)$-th entry and the coefficients of the elements t_1q_j, $j = 2, \dots, m$, are zero, because $L_{m,c}$ is nilpotent. So we have the form

$$J(\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & g_{12} \\ p_2(\hat{t}_1) & * \\ p_3(\hat{t}_1) & * \\ \vdots & \vdots \\ p_m(\hat{t}_1) & * \end{pmatrix},$$

where $p_i = p_i(\hat{t}_1)$, $i = 2, \ldots, m$, does not depend on t_1, g_{12} does not contain a summand dt_2, $d \in K$. Let us express g_{12} as

$$g_{12} = t_2 f + \hat{f}_2,$$

where $\hat{f}_2$ does not depend on t_2 and $f \in \Omega$ because g_{12} does not contain a summand dt_2, $d \in K$. Let us consider the normal automorphism

$$\phi_1 : x_i \to x_i + [x_i, x_1]f, \quad i = 1, \ldots, m.$$

The Jacobian matrix of ϕ_1 is of the form

$$J(\phi_1) = I_m + \begin{pmatrix} 0 & -t_2 f & \cdots & -t_m f \\ 0 & t_1 f & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_1 f \end{pmatrix},$$

Calculating $J(\phi_1\gamma)$ we have that

$$J(\phi_1\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & t_1 f(t_2 f + \hat{f}_2) + t_2 f + \hat{f}_2 - t_2 f \\ t_1 f p_2 + p_2 & * \\ t_1 f p_3 + p_3 & * \\ \vdots & \vdots \\ t_1 f p_m + p_m & * \end{pmatrix}.$$

Now let

$$g_1 = 0, \quad g_k = f p_k, \quad k = 2, \ldots, m,$$

and let us define the normal automorphism

$$\phi_2 : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] g_j, \quad i = 1, \ldots, m.$$

Calculating $J(\phi_2\phi_1\gamma)$ we see that the summands $-t_1 f p_j$ in the first column disappears:

$$J(\phi_2\phi_1\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & t_1 t_2 f^2 + t_1 f \hat{f}_2 + \hat{f}_2 \\ p_2 & * \\ p_3 & * \\ \vdots & \vdots \\ p_m & * \end{pmatrix}.$$

Let us consider the $(1,2)$-th entry $t_1t_2f^2+t_1f\hat{f}_2+\hat{f}_2$ of the matrix $J(\phi_2\phi_1\gamma)$ and express the element f as

$$f = t_2F + \hat{F}_2,$$

where $\hat{F}_2$ does not depend on t_2. Now we have that

$$\begin{aligned} t_1t_2f^2 + t_1f\hat{f}_2 + \hat{f}_2 &= t_1t_2(f^2 + F\hat{f}_2) + (t_1\hat{F}_2 + 1)\hat{f}_2 \\ &= t_1t_2h + \hat{h}_2, \end{aligned}$$

where $\hat{h}_2 = (t_1\hat{F}_2 + 1)\hat{f}_2$ does not depend on t_2 and $h = f^2 + F\hat{f}_2$. Note that the minimal degree of the monomials of the summand which depend on t_2 (in this step this is t_1t_2h) is bigger than of the minimal degree of the corresponding summand t_2f of the previous step which means that the degree increases.

We repeat the process one more step and consider the normal automorphism

$$\varphi_1 : x_i \to x_i + [x_i, x_1](\mathrm{ad}x_1h), \quad i = 1, \dots, m.$$

Calculating $J(\varphi_1\phi_2\phi_1\gamma)$ we have that

$$J(\varphi_1\phi_2\phi_1\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & t_1^2h(t_1t_2h + \hat{h}_2) + t_1t_2h + \hat{h}_2 - t_2t_1h \\ t_1^2hp_2 + p_2 & * \\ t_1^2hp_3 + p_3 & * \\ \vdots & \vdots \\ t_1^2hp_m + p_m & * \end{pmatrix}.$$

Now let

$$g_1 = 0, \quad g_k = t_1hp_k, \quad k = 2, \dots, m,$$

and let us define the normal automorphism

$$\varphi_2 : x_i \to x_i + \sum_{j=1}^{m}[x_i, x_j]g_j, \quad i = 1, \dots, m.$$

Then

$$J(\varphi_2\varphi_1\phi_2\phi_1\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & t_1^3t_2h^2 + t_1^2h\hat{h}_2 + \hat{h}_2 \\ p_2 & * \\ p_3 & * \\ \vdots & \vdots \\ p_m & * \end{pmatrix}.$$

Let us consider the $(1,2)$-th entry $t_1^3t_2h^2 + t_1^2h\hat{h}_2 + \hat{h}_2$ of the matrix $J(\varphi_2\varphi_1\phi_2\phi_1\gamma)_2$ and express the element h as

$$h = t_2H + \hat{H}_2,$$

where $\hat{H}_2$ does not depend on t_2. Now we have that

$$\begin{aligned} t_1^3t_2h^2 + t_1^2h\hat{h}_2 + \hat{h}_2 &= t_1^2t_2(t_1h^2 + H\hat{h}_2) + (t_1^2\hat{H}_2 + 1)\hat{h}_2 \\ &= t_1^2t_2Q(t_1,\ldots,t_m) + \hat{Q}(\hat{t}_2), \end{aligned}$$

Again, the length of the summands in $t_1^2t_2Q(t_1,\ldots,t_m)$ which depend on t_2 increases step by step. Repeating this argument sufficiently many times, by nilpotency, we get finally that

$$J(\gamma)_2 = I_{m2} + \begin{pmatrix} 0 & q(\hat{t}_2) \\ p_2(\hat{t}_1) & * \\ p_3(\hat{t}_1) & * \\ \vdots & \vdots \\ p_m(\hat{t}_1) & * \end{pmatrix}.$$

Hence, starting from an arbitrary coset of IA-automorphisms $\mathrm{IN}(L_{m,c})\psi$, we have found that it contains an automorphism $\theta \in \Theta$ with Jacobian matrix prescribed in the theorem. Now, let θ_1 and θ_2 be two different automorphisms in Θ with $\mathrm{IN}(L_{m,c})\theta_1 = \mathrm{IN}(L_{m,c})\theta_2$. Hence, there exists a nontrivial automorphism φ in $\mathrm{IN}(L_{m,c})$ such that $\theta_1 = \varphi\theta_2$. Direct calculations show that this is in contradiction with the form of $J(\theta_1)$.

(iii) Let φ be a normal IA-automorphism of $L_{m,c}$. From *(i)*, the Jacobian matrix of φ is

$$J(\varphi) = I_m + \begin{pmatrix} t_2f_2 + \cdots + t_mf_m & -t_2f_1 & \cdots & -t_mf_1 \\ -t_1f_2 & \sum_{j\neq 2} t_jf_j & \cdots & -t_mf_2 \\ -t_1f_3 & -t_2f_3 & \cdots & -t_mf_3 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1f_m & -t_2f_m & \cdots & \sum_{j\neq m} t_jf_j \end{pmatrix},$$

where $f_j(t_1,\ldots,t_m) \in K[t_1,\ldots,t_m]$, $j = 1,\ldots,m$. When $c = 2$ then from Lemma 3.1, $\mathrm{IN}(L_{m,2}) = \mathrm{Inn}(L_{m,2})$. As a result we may consider that $f_j(t_1,\ldots,t_m) \in \Omega$.

Let us express the polynomials $f_j(t_1,\ldots,t_m)$, $j = 2,\ldots,m$, in the following way:

$$f_j(t_1,\ldots,t_m) = t_1\overline{f}_j(t_1,\ldots,t_m) + h_j(T_2).$$

Now let $u \in L_{m,c}$ be of the form

$$u = -\sum_{i>1} [x_i, x_1]\overline{f}_j(\mathrm{ad}x_1, \dots, \mathrm{ad}x_m), \quad g_{i1}(t_1, \dots, t_m) \in \Omega,$$

and let consider the inner automorphism $\phi_1 = \exp(\mathrm{ad}u)$. Then the Jacobian matrix of ϕ_1 has the form

$$J(\phi_1) = I_m + \begin{pmatrix} -t_1 G_1 & -t_2 G_1 & \cdots & -t_m G_1 \\ -t_1 G_2 & -t_2 G_2 & \cdots & -t_m G_2 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1 G_m & -t_2 G_m & \cdots & -t_m G_m \end{pmatrix},$$

where

$$\begin{aligned} G_1 =& t_2\overline{f}_2 - t_3\overline{f}_3 - \cdots - t_m\overline{f}_m, \\ G_2 =& - t_1\overline{f}_2,\ G_3 = -t_1\overline{f}_3, \dots, G_m = -t_1\overline{f}_m. \end{aligned}$$

The element u belongs to the commutator ideal of $L_{m,c}$ and the linear operator $\mathrm{ad}u$ acts trivially on $L'_{m,c}$. Hence $\exp(\mathrm{ad}u)$ is the identity map restricted on $L'_{m,c}$. Since the automorphism φ is IA, we obtain that

$$J(\phi_1\varphi)_2 = I_{m2} + \begin{pmatrix} t_2h_2(T_2), + \cdots + t_mh_m(T_2), & -t_2F_1 \\ -t_1h_2(T_2), & t_1F_1 + \sum_{j=3}^{m} t_jh_j(T_2), \\ -t_1h_3(T_2), & -t_2h_3(T_2) \\ \vdots & \vdots \\ -t_1h_m(T_2), & -t_2h_m(T_2) \end{pmatrix},$$

Now we write $h_i(T_2)$ in the form

$$h_i(T_2) = t_2h'_i(T_2) + h''_i(T_3), \quad i = 3, \dots, m,$$

and define

$$\phi_2 = \exp(\mathrm{ad}u_2), \quad u_2 = \sum_{i=3}^{m} [x_i, x_2]h'_i(\mathrm{ad}x_2, \dots, \mathrm{ad}x_m).$$

Then we obtain that

$$J(\phi_2\phi_1\phi_0\psi)_2 = \begin{pmatrix} 1 + t_2H_2(T_2) + \cdots + t_mh''_m(T_m) & -t_2F'_1 \\ -t_1H_2(T_2) & * \\ -t_1h''_3(T_3) & * \\ \vdots & \vdots \\ -t_1h''_m(T_3) & * \end{pmatrix},$$

$$H_2(T_2) = h_2(T_2) - \sum_{i=3}^{m} t_i h_i'(T_2).$$

Repeating this process we construct inner automorphisms $\phi_3, \ldots, \phi_{m-1}$ such that

$$\psi = \phi_{m-1} \cdots \phi_2 \phi_1 \varphi,$$

$$J(\phi_{m-1} \cdots \phi_2 \phi_1 \varphi)_2 = \begin{pmatrix} 1 + t_2 H_2(T_2) + \cdots + t_m H_m(T_m) & -t_2 H_1(T_1) \\ -t_1 H_2(T_2) & * \\ -t_1 H_3(T_3) & * \\ \vdots & \vdots \\ -t_1 H_m(T_m) & * \end{pmatrix},$$

Hence, starting from an arbitrary coset of normal IA-automorphisms $\mathrm{Inn}(L_{m,c})\varphi$, we found that it contains an automorphism $\psi \in \Psi$ with Jacobian matrix prescribed in the theorem. Now, let ψ_1 and ψ_2 be two different automorphisms in Ψ with $\mathrm{Inn}(L_{m,c})\psi_1 = \mathrm{Inn}(L_{m,c})\psi_2$. Hence, there exists a nonzero element $u \in L_{m,c}$ such that $\psi_1 = \exp(\mathrm{ad} u)\psi_2$. Direct calculations show that this is in contradiction with the form of $J(\psi_1)$. □

Example 3.9. When $m = 3$ the results of Theorem 3.8 have the following simple form. If φ is a normal automorphism of the form

$$\begin{aligned} \varphi : & x_1 \to x_1 + [x_1, x_2] f_2 + [x_1, x_3] f_3 \\ & x_2 \to x_2 + [x_2, x_1] f_1 + [x_2, x_3] f_3 \\ & x_3 \to x_3 + [x_3, x_1] f_1 + [x_3, x_2] f_2, \end{aligned}$$

where $f_1, f_2, f_3 \in K[\mathrm{ad}x_1, \mathrm{ad}x_2, \mathrm{ad}x_3]$ then the Jacobian matrix of φ is

$$J(\varphi) = \begin{pmatrix} 1 + t_2 f_2 + t_3 f_3 & -t_2 f_1 & -t_3 f_1 \\ -t_1 f_2 & 1 + t_1 f_1 + t_3 f_3 & -t_3 f_2 \\ -t_1 f_3 & -t_2 f_3 & 1 + t_1 f_1 + t_2 f_2 \end{pmatrix}.$$

The Jacobian matrix of the normally outer automorphism θ is

$$J(\theta) = \begin{pmatrix} 1 & f_{12}(t_1, t_3) & f_{13} \\ t_3 p(t_2, t_3) & 1 + f_{22} & f_{23} \\ -t_2 p(t_2, t_3) & f_{32} & 1 + f_{33} \end{pmatrix},$$

where $p(t_2, t_3)$, f_{ij}, are polynomials of degree $\leq c-1$ without constant terms with the following conditions

$$t_1 f_{1j} + t_2 f_{2j} + t_3 f_{3j} \equiv 0 \pmod{\Omega}^{c+1}, \quad j = 2, 3,$$

$p(t_2, t_3)$ does not depend on t_1 and $f_{12} = f_{12}(t_1, t_3)$ does depend on t_2.

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REFERENCES

[1] Yu. A. Bahturin. Identical Relations in Lie Algebras, Nauka, Moscow, 1985 (in Russian). Translation: VNU Science Press, Utrecht, 1987.

[2] R. M. Bryant, V. Drensky. Dense subgroups of the automorphism groups of free algebras. *Canad. J. Math.* **45** (1993), 1135–1154.

[3] L. A. Bokut', G. P. Kukin. Algorithmic and Combinatorial Algebra. Mathematics and its Applications, vol **255**, Kluwer Academic Publishers Group, Dordrecht, 1994.

[4] V. Drensky, Ş. Findik. Inner and outer automorphisms of free metabelian nilpotent Lie algebras, `http://lanl.arxiv.org/abs/1003.0350`.

[5] G. Endimioni. Normal automorphisms of a free metabelian nilpotent group. *Glasgow Math. J.* **52** (2010), 169–177.

[6] G. Endimioni. On the polynomial automorphisms of a free group. *Acta Sci. Math. (Szeged)* **73** (2007), 61–69.

[7] G. Endimioni. Pointwise inner automorphisms in a free nilpotent group. *Quart. J. Math.* **53** (2002), 397–402.

[8] C. K. Gupta. IA-automorphisms of two-generator metabelian groups. *Arch. Math.* **37** (1981), 106–112.

[9] J. V. Kuzmin. Inner endomorphisms of metabelian groups. *Sibirsk. Mat. Zh.* **16** (1975), 736–744 (in Russian); Translation in *Siberian Math. J.* **16** (1976), 563–568.

[10] A. Lubotzky. Normal automorphisms of free groups. *J. Algebra* **63** (1980), 494–498.

[11] A. Lue. Normal automorphisms of free groups. *J. Algebra* **64** (1980), 52–53.

[12] W. Magnus. Über diskontinuierliche Gruppen mit einer definierenden Relation (der Freiheitssatz). *J. Reine Angew. Math.* **163** (1930), 141–165.

[13] L. Makar-Limanov. Algebraically closed skew fields. *J. Algebra* **93**, *1* (1985), 117–135.

[14] H. Neumann. Varieties of Groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **37**, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[15] A. I. Shirshov. Some algorithmical problems for Lie algebras. *Sib. Mat. Zh.* **3** (1962), 292–296 (in Russian).

[16] A. L. Shmel'kin. Wreath products of Lie algebras and their application in the theory of groups. *Trudy Moskov. Mat. Obshch.* **29** (1973), 247–260 (in Russian); Translation: *Trans. Moscow Math. Soc.* **29** (1973), 239–252.

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