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SOME FIXED POINT THEOREMS FOR KANNAN MAPPINGS

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Abstract. Some results on the existence and uniqueness of fixed points for Kannan mappings on admissible subsets of bounded metric spaces and on bounded closed convex subsets of complete convex metric spaces having uniform normal structure are proved in this paper. These results extend and generalize some results of Ismat Beg and Akbar Azam [*Ind. J. Pure Appl. Math.* **18** (1987), 594–596], A. A. Gillespie and B. B. Williams [*J. Math. Anal. Appl.* **74** (1980), 382–387] and of Yoichi Kijima and Wataru Takahashi [*Kodai Math Sem. Rep.* **21** (1969), 326–330].

Kannan mappings have inspired a branch of fixed point theory devoted exclusively to the study of generalizations of contraction type conditions. These mappings have been used by Subrahmanyam [14] to characterize the metric completeness of the underlying spaces. Many results on the existence and uniqueness

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of fixed points for Kannan mappings have been proved in Banach spaces by Gillespie and Williams [4] and by Kannan [5, 6, 7, 8, 9]. Some of these results were extended to convex metric spaces by Beg and Azam [1, 2]. In this paper, we also prove some results on the existence and uniqueness of fixed points for Kannan mappings on admissible subsets of bounded metric spaces and on bounded closed convex subsets of complete convex metric spaces having uniform normal structure. Our results extend and generalize some results of [2, 4] and [11]. The proofs given here are modifications of those given in these papers.

We start with a few definitions and observations.

Definition 1. *A mapping T of a metric space (X, d) into itself is said to be a Kannan mapping on a subset K of X if*

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in K$.

The following examples show that Kannan mappings need not be continuous, these mappings need not be non-expansive and non-expansive mappings need not be Kannan.

Example 1. Consider $T : [0, 1] \to [0, 1]$ defined as

$$T(x) = \begin{cases} 1 - x & x \in \left[0, \frac{1}{3}\right) \\ \dfrac{x+1}{3} & x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

For any $x, y \in [0, 1]$, $d(x, y) = |x - y|$.

If $x, y \in \left[0, \frac{1}{3}\right)$, then $d(Tx, Ty) = |Tx - Ty| = |x - y|$ and $d(x, Tx) = |x - Tx| = |2x - 1|$ for all x. $\frac{1}{2}(d(x, Tx) + d(y, Ty)) = \frac{1}{2}(|2x - 1| + |2y - 1|) = \left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right| \geq |x - y| = |(1 - y) - (1 - x)| = d(Ty, Tx)$. This implies $d(Tx, Ty) \leq \dfrac{d(x, Tx) + d(y, Ty)}{2}$ for all $x, y \in \left[0, \frac{1}{3}\right)$. Similarly we can prove this inequality for other x, y's. Thus T is a Kannan mapping.

This mapping is not non-expansive because it is not continuous at $x = \frac{1}{3}$.

Example 2. Define $T : \mathbb{R}^+ \to \mathbb{R}^+$ by $T(x) = x + 1$ and $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}^+$.

Clearly $d(Tx, Ty) = d(x, y)$ for all $x, y \in \mathbb{R}^+$ i.e. T is non-expansive. But T is not Kannan: Take $x, y \in \mathbb{R}^+$ with $|x - y| > 1$. $d(x, Tx) = 1$ for all $x \in \mathbb{R}^+$.

$$\frac{d(x,Tx) + d(y,Ty)}{2} = \frac{1+1}{2} = 1 < |x - y| = |(1 + x) - (1 + y)| = d(Tx, Ty).$$

Therefore, $d(Tx, Ty) > \dfrac{d(x,Tx) + d(y,Ty)}{2}$ for all $x, y \in \mathbb{R}^+$ with $|x - y| > 1$.

Definition 2. *For a metric space (X, d) and the closed interval $I = [0, 1]$, a mapping $W : X \times X \times I \to X$ is said to be a* convex structure *on X if for all $x, y \in X, \lambda \in I$*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a convex metric space [15].

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [15]*).*

Definition 3. *A subset K of a convex metric space (X, d) is said to be* convex *[15] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.*

If A is a subset of a convex metric space (X, d) then the intersection of all convex sets in X containing A is called the convex hull *of A and is denoted by conv A.*

Definition 4. *If A is a subset of a metric space (X, d) then the diameter of A, denoted by $\delta(A)$, is defined as*

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

Definition 5. *The set A is said to be* admissible (*cf.* [3, p.459]) *if it can be written as the intersection of a family of closed balls in X.*

Admissible sets have played a central role in proving fixed point theorems (see [10]).

If A is a subset of a bounded metric space (X, d) then *conv* A is an admissible set in X (see [10, p. 36]). Every bounded closed interval on the real line is an admissible set. For more examples of admissible sets one may refer to [10].

Definition 6. *A convex metric space* (X, d) *is said to have* uniform normal structure [4] *if we can find a real number* $h \in (0, 1)$ *such that if* C *is a closed bounded convex subset of* X*, then there exists some* $x_0 \in C$ *such that*

$$\sup\{d(x_0, y) : y \in C\} \leq h\delta(C).$$

This notion was initially investigated by A. A. Gillespie and B. B. Williams in 1979 *and has been widely used in fixed point theory* (*see e.g.* [4], [10] and [13].)

Example 3. For $f \in l^2$, define $\|f\|_{2,1} = \|f^+\|_2 + \|f^-\|_2$. Then $\|\cdot\|_{2,1}$ is a norm on l^2 and is equivalent to the usual norm. If l^2 equipped with $\|\cdot\|_{2,1}$ is denoted by $l_{2,1}$ then $l_{2,1}$ has uniform normal structure (see [13]).

Every uniformly convex Banach space has uniform normal structure (see [10]).

For more examples of spaces with uniform normal structure one may refer to [10] and [13]. Throughout this paper, $B[x, r]$ denotes a closed ball with center x and radius r.

Theorem 1 proved below guarantees the existence and uniqueness of fixed point for Kannan mappings on admissible sets:

Theorem 1. *Let* T *be a Kannan mapping of a non-empty subset* K *of a bounded metric space* (X, d) *into itself. Suppose*

$$\sup\{d(y, Ty) : y \in F\} < \delta(F)$$

for every non-empty admissible subset F *of* K *which has non-zero diameter and is invariant under* T*. Then* T *has unique fixed point in* K *if there exists a minimal* T*-invariant admissible subset* K^* *of* K*.*

Proof. If $\delta(K^*) = 0$ then the point in K^* is a fixed point of T. Suppose $\delta(K^*) > 0$. For any $x, y \in K^*$, we have

$$d(Ty, Tx) \leq \frac{1}{2}\{d(y, Ty) + d(x, Tx)\} \leq \sup_{s \in K^*} d(s, Ts).$$

Therefore $T(K^*) \subseteq C \equiv B\left[Tx, \sup_{s \in K^*} d(s, Ts)\right]$ for any $x \in K^*$. Moreover, $K^* \cap C$ is T-invariant, for if $u \in K^* \cap C$ then $Tu \in C$ as $T(K^*) \subseteq C$. Since

$Tu \in K^*$, $Tu \in K^* \cap C$. Therefore, by the minimality of K^*, it follows that $K^* \subseteq C$. Hence for any arbitrary but fixed $x \in K^*$ and for every $y \in K^*$, we have $d(y, Tx) \leq \sup_{s \in K^*} d(s, Ts)$ implying thereby $\sup_{y \in K^*} d(y, Tx) \leq \sup_{s \in K^*} d(s, Ts)$.

Let $K' = \left\{ z \in K^* : \sup_{y \in K^*} d(y, z) \leq \sup_{s \in K^*} d(s, Ts) \right\}$. Obviously, K' is non-empty as for any $x \in K^*$, $Tx \in K'$. The previous argument shows that $T(K^*) \subset K'$. Since $K' \subset K^*$, it follows that K' is invariant under T. Also K' is admissible as

$$K' = K^* \cap \left\{ \cap B \left[y,\ \sup_{s \in K^*} d(s, Ts) + \frac{1}{n} \right] : y \in K^*,\ n = 1, 2, \dots \right\}.$$

Further, $\delta(K') \leq \sup_{y \in K^*} d(y, Ty) < \delta(K^*)$ by the hypothesis. Hence K' is a proper invariant admissible subset of K^*, contradicting the minimality of K^*. Thus $\delta(K^*) = 0$ proving thereby that T has a fixed point in K.

For uniqueness, suppose x and y are fixed points of T then $d(x, y) = d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\} = 0$ and so $x = y$. $\square$

Corollary 1. *Let T be a Kannan mapping of a non-empty admissible subset K of a bounded metric space (X, d) into itself satisfying:*

(1) *if a family of closed balls has finite intersection property (f.i.p.) then the intersection of the family is non-empty,*

(2) $\sup\{d(y, Ty) : y \in F\} < \delta(F)$ *for every non-empty admissible subset F of K which has non-zero diameter and is invariant under T,*

then T has unique fixed point in K.

Proof. Let Φ be the family of all non-empty admissible subsets of X which are invariant under T. Since $K \in \Phi$, Φ is non-empty. Φ is partially ordered with respect to set inclusion. Let $\{A_i : i \in I\}$ be totally ordered subfamily of Φ. We show that $A = \cap\{A_i : i \in I\}$ is an element of Φ. A is admissible as each $A_i = \cap\{B[x_j, r_j] : j \in J_i\}$ so $A = \cap\{B[x_j, r_j] : j \in J = \cup_{i \in I} J_i\}$. Also A is invariant under T. Consider the family $\{B[x_j, r_j] : j \in J\}$ and take arbitrary finite elements $B[x_{j_1}, r_{j_1}], B[x_{j_2}, r_{j_2}], \dots, B[x_{j_n}, r_{j_n}]$ from it. Every $B[x_j, r_j]$, $j \in J$ contains some A_i, $i \in I$ and so $\cap_{k=1}^n A_{i_k} \subset \cap_{k=1}^n B[x_{j_k}, r_{j_k}]$. Since the family $\{A_i : i \in I\}$ is totally ordered, $\cap_{k=1}^n A_{i_k}$ is non-empty and so is $\cap_{k=1}^n B[x_{j_k}, r_{j_k}]$. This shows that the family $\{B[x_j, r_j] : j \in J\}$ has f.i.p. and so A is non-empty by (1). Obviously, A is lower bound of the family $\{A_i : i \in I\}$ and therefore by Zorn's lemma, Φ has a minimal element, say K^*. The result now follows by Theorem 1. $\square$

Note 1. For non-expansive mappings, the following similar result was proved by Kijima and Takahashi [11]:

Suppose a bounded metric space (X, d) satisfies

(1) if a family of closed balls has f.i.p, then the intersection of the family is nonempty,

(2) each admissible subset which contains more than one point contains a non diametral point,

then every non-expansive mapping T of X into X has a fixed point.

Note 2. For nonempty bounded closed starshaped subset K of a convex metric space (X, d) the following result was proved by Beg and Azam [2]:

Let T be a Kannan mapping of a nonempty bounded closed starshaped subset K of a convex metric space X into itself. Suppose $\sup_{y \in F} d(y, Ty) < \frac{1}{2}\delta(F)$ for every closed T-invariant starshaped subset F of K with non-zero diameter. Then T has a unique fixed point if there exists a minimal closed T-invariant starshaped subset K^* of K.

Gillespie and Williams [4] proved the existence and uniqueness of fixed point of Kannan mappings for closed bounded convex subsets of Banach spaces having uniform normal structure. We extend the result to convex metric spaces.

Theorem 2. *If C is a bounded closed convex subset of a complete convex metric space (X, d), $T : C \to C$ a Kannan mapping and X has uniform normal structure, then T has unique fixed point.*

Proof. Since X has uniform normal structure, there exists $h \in (0, 1)$ such that $P = \{x \in C : d(x, Tx) \leq h\delta(C)\} \neq \emptyset$. If $x \in P$, then in view of

$$d(Tx, T^2x) \leq \frac{1}{2}(d(x, Tx) + d(Tx, T^2x)),$$

we have $d(Tx, T^2x) \leq h\delta(C)$ which implies that $Tx \in P$ for all $x \in P$ and hence $T(P) \subseteq P$. Let $C_1 = cl\ conv[T(P)] \equiv$ closed convex hull of $T(P)$. We claim that $C_1 \subseteq P$.

If $z \in C_1$, then the following three cases may arise:

(1) $z \in T(P)$. In this case, since $T(P) \subset P,\ z \in P$.

(2) $z \in conv[T(P)] = \cup_{i \in N} A_i$ (Proposition 1 [11]), where $A_1 = W(T(P) \times T(P) \times [0,1])$, $A_2 = W(A_1 \times A_1 \times [0,1]), \ldots, A_{n+1} = W(A_n \times A_n \times [0,1]), \ldots$. Then $z \in A_m$ for some m. Applying Principle of Mathematical Induction, we get

$$d(z, Ty) \leq \frac{h\delta(C)}{2} + \frac{d(Ty, y)}{2}$$

for all $z \in A_m,\ m \geq 1$ and $y \in C$ as shown below:

Suppose $z \in A_1$ and $y \in C$. $z \in A_1 \Rightarrow z \in W(T(P) \times T(P) \times [0,1])$. This implies $z = W(T(u), T(v), \lambda)$ for some $\lambda \in [0,1], u, v \in P$. Consider

$$\begin{aligned} d(z, Ty) &= d(W(Tu, Tv, \lambda), Ty) \\ &\leq \lambda d(Tu, Ty) + (1-\lambda) d(Tv, Ty) \\ &\leq \lambda \frac{d(u, Tu) + d(y, Ty)}{2} + \\ &\quad (1-\lambda) \frac{d(v, Tv) + d(y, Ty)}{2} \\ &\leq \lambda \left[\frac{h\delta(C)}{2} + \frac{d(y, Ty)}{2} \right] + \\ &\quad (1-\lambda) \left[\frac{h\delta(C)}{2} + \frac{d(y, Ty)}{2} \right] \\ &= \frac{h\delta(C)}{2} + \frac{d(y, Ty)}{2} \end{aligned}$$

Suppose $z \in A_2$ and $y \in C$. $z \in A_2 \Rightarrow z \in W(A_1 \times A_1 \times [0,1])$. This implies $z = W(p, q, \kappa), p, q \in A_1$. Now $p \in A_1 \Rightarrow d(p, Ty) \leq \dfrac{h\delta(C)}{2} + \dfrac{d(Ty, y)}{2}$ and $q \in A_1 \Rightarrow d(q, Ty) \leq \dfrac{h\delta(C)}{2} + \dfrac{d(Ty, y)}{2}$.

Consider

$$\begin{aligned} d(z,Ty) &= d(W(p,q,\kappa),Ty) \\ &\leq \kappa d(p,Ty)+(1-\kappa)d(q,Ty) \\ &\leq \kappa\left[\frac{h\delta(C)}{2}+\frac{d(Ty,y)}{2}\right]+ \\ &\quad (1-\kappa)\left[\frac{h\delta(C)}{2}+\frac{d(Ty,y)}{2}\right] \\ &= \frac{h\delta(C)}{2}+\frac{d(Ty,y)}{2} \end{aligned}$$

Proceeding so on, we get $d(z,Tz) \leq \frac{h\delta(C)}{2}+\frac{d(Tz,z)}{2}$ for every $z \in A_m$, for all $m \geq 1$ and $y \in C$. As a consequence, we have $d(z,Tz) \leq h\delta(C)$ and therefore, $z \in P$.

(3) z is a limit point of $conv[T(P)]$. Then there exists a sequence $\langle z_n \rangle$ in $conv[T(P)]$ such that $\langle z_n \rangle \to z$. As $z_n \in conv[T(P)]$,

$$d(z_n,Ty) \leq \frac{h\delta(C)}{2}+\frac{d(Ty,y)}{2}$$

for all $y \in C$. In particular, $d(z_n,Tz) \leq \frac{h\delta(C)}{2}+\frac{d(Tz,z)}{2}$. Thus

$$\begin{aligned} d(z,Tz) &\leq d(z,z_n)+d(z_n,Tz) \\ &\leq d(z,z_n)+\frac{h\delta(C)}{2}+\frac{d(Tz,z)}{2}. \end{aligned}$$

Taking limit as $n \to \infty$, $d(z,Tz) \leq \frac{h\delta(C)}{2}+\frac{d(Tz,z)}{2}$ i.e. $d(z,Tz) \leq h\delta(C)$ and so $z \in P$.

Thus in all the three cases $C_1 \subseteq P$. Also C_1 is invariant under T for if, $x \in C_1 \subseteq P$ then $Tx \in T(P) \subseteq C_1$. Now $C_1 = cl\ conv[T(P)] \Rightarrow \delta(C_1) = \delta(conv[T(P)]) = \delta(T(P))(see[12]) \leq h\delta(C)$ as $Tx, Ty \in T(P) \Rightarrow d(Tx,Ty) \leq \frac{1}{2}\{d(x,Tx)+d(y,Ty)\} \leq h\delta(C)$.

Taking $C_n = cl\ conv[T(P_{n-1})]$, where for C_{n-1} we consider the set P_{n-1} defined in the similar way as P for C and noting that $T(P_1) \subset P_1$. So,we can find a decreasing sequence $\langle C_n \rangle$ of non-empty closed convex T invariant subsets of C such that $\delta(C_n) \leq h^n \delta(C)$ for each n. Taking $n \to \infty$, we get $\lim \delta(C_n) = 0$. Therefore Cantor's intersection Theorem gives a fixed point for T. Uniqueness of fixed point follows as in Theorem 1. □

Corollary ([4]). *If C is a bounded closed convex subset of a Banach space X. $T : C \to C$ is a Kannan mapping and X has uniform normal structure then T has unique fixed point.*

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