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FRACTIONAL KOROVKIN THEORY BASED ON STATISTICAL CONVERGENCE

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Abstract. In this paper, we obtain some statistical Korovkin-type approximation theorems including fractional derivatives of functions. We also show that our new results are more applicable than the classical ones.

1. Introduction.

In [11], Gadjiev and Orhan improved the classical Korovkin theory via the concept of statistical convergence, which is known as "*Statistical Korovkin Theory*" in the literature (see also [4, 6, 7, 17]). In the very recent paper [2], Anastassiou studied the Korovkin theory by considering the fractional derivatives of functions, the so-called "*Fractional Korovkin Theory*". Here, we refer to readers [1, 13] for the Korovkin theory; [3, 14, 16] for the fractional calculus, and also [5, 8, 9, 10, 15] for the statistical convergence. The aim of the present paper is to obtain some fractional Korovkin-type results based

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on statistical convergence. We also show that our new results are more applicable than the classical ones. More precisely, we display a sequence of positive linear operators which obeys our all statistical approximation conditions but not ones in the fractional Korovkin theory.

We first recall some basic definitions and notations used in the paper. Let $A := [a_{jn}]$, $j, n = 1, 2, \ldots$, be an infinite summability matrix and assume that, for a given sequence (x_n), the series $\sum_{n=1}^{\infty} a_{jn} x_n$ converges for every $j \in \mathbb{N}$. Then, by the A-transform of x, we mean the sequence $((Ax)_j)$ such that, for every $j \in \mathbb{N}$, $(Ax)_j := \sum_{n=1}^{\infty} a_{jn} x_n$. A summability matrix A is said to be regular (see [12]) if for every (x_n) for which $\lim_n x_n = L$ we get $\lim_j (Ax)_j = L$. For a given non-negative regular summability matrix A, we say that a sequence (x_n) is A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_{j \to \infty} \sum_{n : |x_n - L| \geq \varepsilon} a_{nj} = 0$, which is denoted by $st_A - \lim_n x_n = L$. (see [10]). Observe now that if $A = C_1 = [c_{jn}]$, the Cesáro matrix defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [8]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$ (see the last section for this situation). Notice that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, however, the converse is not always true. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded of an A-statistically convergent sequence.

2. Fractional Derivatives and Positive Linear Operators. In this section we first recall the Caputo fractional derivatives. Let r be a positive real number and $m = \lceil r \rceil$, where $\lceil \cdot \rceil$ is the ceiling of the number. As usual, by $AC([a, b])$ we denote the space of all real-valued absolutely continuous functions on $[a, b]$. Also, consider the space

$$AC^m([a, b]) := \left\{ f : [a, b] \to \mathbb{R} : f^{(m-1)} \in AC([a, b]) \right\}.$$

Then, the left Caputo fractional derivative of a function f belonging to $AC^m[a,b]$ is defined by

$$D_{*a}^{r}f(x) := \frac{1}{\Gamma(m-r)} \int_{a}^{x} (x-t)^{m-r-1} f^{(m)}(t)dt \quad \text{for} \quad x \in [a,b], \tag{2.1}$$

where Γ is the usual Gamma function. Also, the right Caputo fractional derivative of a function f belonging to $AC^m([a,b])$ is defined to be

$$D_{b-}^{r}f(x) := \frac{(-1)^m}{\Gamma(m-r)} \int_{x}^{b} (\zeta - x)^{m-r-1} f^{(m)}(\zeta)d\zeta \quad \text{for} \quad x \in [a,b]. \tag{2.2}$$

In (2.1) and (2.2), we set $D_{*a}^{0}f = f$ and $D_{b-}^{0}f = f$ on $[a,b]$. Throughout the paper we consider the following assumptions:

$$D_{*a}^{r}f(y) = 0 \text{ for every } y < a$$

and

$$D_{b-}^{r}f(y) = 0 \text{ for every } y > b.$$

Then we know the following facts (see, e.g., [2, 3]):

(**a**) If $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$, $f \in C^{m-1}([a,b])$ and $f^{(m)} \in L_{\infty}([a,b])$, then we have $D_{*a}^{r}f(a) = 0$ and $D_{b-}^{r}f(b) = 0$.

(**b**) Let $y \in [a,b]$ be fixed. For $r > 0$, $m = \lceil r \rceil$, $f \in C^{m-1}([a,b])$ with $f^{(m)} \in L_{\infty}[a,b]$, consider the following Caputo fractional derivatives:

$$U_f(x,y) := D_{*x}^{r}f(y) = \frac{1}{\Gamma(m-r)} \int_{x}^{y} (y-t)^{m-r-1} f^{(m)}(t)dt \quad \text{for} \quad y \in [x,b] \tag{2.3}$$

and

$$V_f(x,y) := D_{x-}^{r}f(y) = \frac{(-1)^m}{\Gamma(m-r)} \int_{y}^{x} (\zeta - y)^{m-r-1} f^{(m)}(\zeta)d\zeta \quad \text{for} \quad y \in [a,x]. \tag{2.4}$$

Then, by [2], for each fixed $x \in [a,b]$, $U_f(x,\cdot)$ is continuous on the interval $[x,b]$, and also $V_f(x,\cdot)$ is continuous on $[a,x]$. In addition, if $f \in C^m([a,b])$, then, $U_f(\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ are continuous on the set $[a,b] \times [a,b]$.

(**c**) Let $\omega(f,\delta)$, $\delta>0$, denote the usual modulus of continuity of a function f on $[a,b]$. If $g\in C([a,b]\times[a,b])$, then, for any $\delta>0$, both the functions $s(x):=\omega\left(g\left(\cdot,x\right),\delta\right)_{[a,x]}$ and $t(x):=\omega\left(g\left(\cdot,x\right),\delta\right)_{[x,b]}$ are continuous at the point $x\in[a,b]$.

(**d**) If $f\in C^{m-1}([a,b])$ with $f^{(m)}\in L_\infty[a,b]$, then we get from [2] that

$$\sup_{x\in[a,b]}\omega\left(U_f\left(x,\cdot\right),\delta\right)_{[x,b]}<\infty \tag{2.5}$$

and

$$\sup_{x\in[a,b]}\omega\left(V_f\left(x,\cdot\right),\delta\right)_{[a,x]}<\infty. \tag{2.6}$$

(**e**) Now let $\Psi(y):=\Psi_x(y)=y-x$ and $e_0(y):=1$ on the interval $[a,b]$. Following the paper by Anastassiou (see [2]) if $L_n:C([a,b])\to C([a,b])$ is a sequence of positive linear operators and if $r>0$, $r\notin\mathbb{N}$, $m=\lceil r\rceil$, $f\in AC^m([a,b])$ with $f^{(m)}\in L_\infty([a,b])$, then we obtain that ($\|\cdot\|$ is the supremum norm)

$$\begin{aligned}\|L_n(f)-f\| \le\ & \|f\|\,\|L_n(e_0)-e_0\|+\sum_{k=1}^{m-1}\frac{\left\|f^{(k)}\right\|}{k!}\left\|L_n\left(|\Psi|^k\right)\right\| \\ &+\left(\frac{r+2}{\Gamma(r+2)}+\frac{1}{\Gamma(r+1)}\|L_n(e_0)-e_0\|^{\frac{1}{r+1}}\right) \\ &\times\left\|L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{r}{r+1}}\left\{\sup_{x\in[a,b]}\omega\left(U_f(x,\cdot),\left\|L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{1}{r+1}}\right)_{[x,b]}\right. \\ &\left.+\sup_{x\in[a,b]}\omega\left(V_f(x,\cdot),\left\|L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{1}{r+1}}\right)_{[a,x]}\right\}.\end{aligned}$$

Then setting

$$\delta_{n,r}:=\left\|L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{1}{r+1}}, \tag{2.7}$$

and also using (2.5), (2.6) we may write that

$$
\begin{aligned}
\|L_n(f) - f\| \leq K_{m,r} \Bigg\{ & \|L_n(e_0) - e_0\| + \sum_{k=1}^{m-1} \left\| L_n\left(|\Psi|^k\right)\right\| \\
& + \delta_{n,r}^r \left(\sup_{x\in[a,b]} \omega\left(U_f\left(x,\cdot\right), \delta_{n,r}\right)_{[x,b]} \right) \\
& + \delta_{n,r}^r \left(\sup_{x\in[a,b]} \omega\left(V_f\left(x,\cdot\right), \delta_{n,r}\right)_{[a,x]} \right) \\
& + \delta_{n,r}^r \left\|L_n\left(e_0\right) - e_0\right\|^{\frac{1}{r+1}} \left(\sup_{x\in[a,b]} \omega\left(U_f\left(x,\cdot\right), \delta_{n,r}\right)_{[x,b]} \right) \\
& + \delta_{n,r}^r \left\|L_n\left(e_0\right) - e_0\right\|^{\frac{1}{r+1}} \left(\sup_{x\in[a,b]} \omega\left(V_f\left(x,\cdot\right), \delta_{n,r}\right)_{[a,x]} \right) \Bigg\},
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
K_{r,m} := \max\Bigg\{ & \frac{1}{\Gamma\left(r+1\right)}, \frac{r+2}{\Gamma\left(r+2\right)}, \|f\|, \left\|f'\right\|, \\
& \frac{\|f''\|}{2!}, \frac{\|f'''\|}{3!}, \ldots, \frac{\left\|f^{(m-1)}\right\|}{(m-1)!} \Bigg\}
\end{aligned} \tag{2.9}
$$

We should note that the sum in the right hand-side of (2.8) collapses when $r \in (0,1)$.

Therefore, the next theorem is a fractional Korovkin-type approximation result for a sequence of positive linear operators.

Theorem A (see [2]). *Let $L_n : C([a,b]) \to C([a,b])$ be a sequence of positive linear operators, and let $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$. If the sequence $\{\delta_{n,r}\}_{n\in\mathbb{N}}$ given by (2.7) is convergent to zero as n tends to infinity and $\{L_n(e_0)\}_{n\in\mathbb{N}}$ converges uniformly to e_0 on $[a,b]$, then, for every $f \in AC^m([a,b])$ with $f^{(m)} \in L_\infty([a,b])$, the sequence $\{L_n(f)\}_{n\in\mathbb{N}}$ converges uniformly to f on the interval $[a,b]$. Furthermore, this uniform convergence is still valid on $[a,b]$ when $f \in C^m([a,b])$.*

3. Fractional Korovkin Results Based on Statistical Convergence. In this section, we mainly obtain the statistical version of Theorem A. We first need the following lemma.

Lemma 3.1. *Let* $A := [a_{jn}]$ *be a non-negative regular summability matrix, and let* $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$. *Assume that* $L_n : C([a,b]) \to C([a,b])$ *is a sequence of positive linear operators. If*

$$st_A - \lim_n \|L_n(e_0) - e_0\| = 0 \tag{3.1}$$

and

$$st_A - \lim_n \delta_{n,r} = 0, \tag{3.2}$$

where $\delta_{n,r}$ *is the same as in* (2.7)*, then we have, for every* $k = 1, 2, \ldots, m-1$,

$$st_A - \lim_n \left\| L_n\left(|\Psi|^k\right)\right\| = 0.$$

P r o o f. Let $k \in \{1, 2, \ldots, m-1\}$ be fixed. Then, using Hölder's inequality for positive linear operators with $p = \dfrac{r+1}{k}$, $q = \dfrac{r+1}{r+1-k}$ $\left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$, we obtain that

$$\left\| L_n\left(|\Psi|^k\right)\right\| \le \left\| L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{k}{r+1}} \|L_n(e_0)\|^{\frac{r+1-k}{r+1}},$$

which gives

$$\left\| L_n\left(|\Psi|^k\right)\right\| \le \left\| L_n\left(|\Psi|^{r+1}\right)\right\|^{\frac{k}{r+1}} \left\{ \|L_n(e_0) - e_0\|^{\frac{r+1-k}{r+1}} + 1 \right\}.$$

Hence, for each $k = 1, 2, \ldots, m-1$, we get the following inequality

$$\left\| L_n\left(|\Psi|^k\right)\right\| \le \delta_{n,r}^k \|L_n(e_0) - e_0\|^{\frac{r+1-k}{r+1}} + \delta_{n,r}^k. \tag{3.3}$$

Then, for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} A &:= \left\{ n \in \mathbb{N} : \left\| L_n\left(|\Psi|^k\right)\right\| \ge \varepsilon \right\}, \\ A_1 &:= \left\{ n \in \mathbb{N} : \delta_{n,r}^k \|L_n(e_0) - e_0\|^{\frac{r+1-k}{r+1}} \ge \frac{\varepsilon}{2} \right\} \\ A_2 &:= \left\{ n \in \mathbb{N} : \delta_{n,r} \ge \left(\frac{\varepsilon}{2}\right)^{\frac{1}{k}} \right\}. \end{aligned}$$

Then, it follows from (3.3) that $A \subseteq A_1 \cup A_2$. Also, defining

$$\begin{aligned} A_1' &:= \left\{ n \in \mathbb{N} : \delta_{n,r} \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{2k}} \right\}, \\ A_1'' &:= \left\{ n \in \mathbb{N} : \| L_n(e_0) - e_0 \| \geq \left(\frac{\varepsilon}{2}\right)^{\frac{r+1}{2(r+1-k)}} \right\}, \end{aligned}$$

we observe that $A_1 \subseteq A_1' \cup A_2''$, which implies that

$$A \subseteq A_1' \cup A_1'' \cup A_2.$$

Hence, for every $j \in \mathbb{N}$, we get

$$\sum_{n \in A} a_{jn} \leq \sum_{n \in A_1'} a_{jn} + \sum_{n \in A_1''} a_{jn} + \sum_{n \in A_2} a_{jn}.$$

Letting $j \to \infty$ in the last inequality and also using the hypotheses (3.1) and (3.2) we immediately see that

$$\lim_j \sum_{n \in A} a_{jn} = 0.$$

Hence, we conclude that, for each $k = 1, 2, \ldots, m-1$,

$$st_A - \lim_n \left\| L_n \left(|\Psi|^k \right) \right\| = 0,$$

whence the result. □

Now we are ready to give our first fractional approximation result based on statistical convergence.

Theorem 3.2. *Let $A := [a_{jn}]$ be a non-negative regular summability matrix, and let $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$. Assume that $L_n : C([a,b]) \to C([a,b])$ is a sequence of positive linear operators. If* (3.1) *and* (3.2) *hold, then, for every $f \in AC^m([a,b])$ with $f^{(m)} \in L_\infty([a,b])$, we have*

$$st_A - \lim_n \| L_n(f) - f \| = 0. \tag{3.4}$$

P r o o f. Let $f \in AC^m([a,b])$ with $f^{(m)} \in L_\infty([a,b])$. Then, using (2.5), (2.6) and (2.8), we get

$$\begin{aligned} \|L_n(f)-f\| &\leq M_{m,r}\left\{\|L_n(e_0)-e_0\| + 2\delta_{n,r}^r \right. \\ &\left. \quad +2\delta_{n,r}^r \left\|L_n(e_0)-e_0\right\|^{\frac{1}{r+1}} + \sum_{k=1}^{m-1}\left\|L_n\left(|\Psi|^k\right)\right\|\right\}, \end{aligned} \tag{3.5}$$

where

$$M_{m,r} := \max\left\{K_{m,r}, \sup_{x\in[a,b]} \omega\left(U_f(x,\cdot), \delta_{n,r}\right)_{[x,b]}, \sup_{x\in[a,b]} \omega\left(V_f(x,\cdot), \delta_{n,r}\right)_{[a,x]}\right\}$$

and $K_{m,r}$ is given by (2.9). Now, for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} B &: = \left\{n \in \mathbb{N} : \|L_n(f)-f\| \geq \varepsilon\right\}, \\ B_k &: = \left\{n \in \mathbb{N} : \left\|L_n\left(|\Psi|^k\right)\right\| \geq \frac{\varepsilon}{(m+2)M_{m,r}}\right\},\ k = 1,2,\ldots,m-1. \\ B_m &: = \left\{n \in \mathbb{N} : \|L_n(e_0)-e_0\| \geq \frac{\varepsilon}{(m+2)\,M_{m,r}}\right\} \\ B_{m+1} &: = \left\{n \in \mathbb{N} : \delta_{n,r} \geq \left(\frac{\varepsilon}{2(m+2)M_{m,r}}\right)^{\frac{1}{r}}\right\}, \\ B_{m+2} &: = \left\{n \in \mathbb{N} : \delta_{n,r}^r \left\|L_n(e_0)-e_0\right\|^{\frac{1}{r+1}} \geq \frac{\varepsilon}{2(m+2)M_{m,r}}\right\}. \end{aligned}$$

Then, it follows from (3.5) that $B \subseteq \bigcup_{i=1}^{m+2} B_i$. Also defining

$$\begin{aligned} B_{m+3} &: = \left\{n \in \mathbb{N} : \|L_n(e_0)-e_0\| \geq \left(\frac{\varepsilon}{2(m+2)M_{m,r}}\right)^{\frac{r+1}{2}}\right\}, \\ B_{m+4} &: = \left\{n \in \mathbb{N} : \delta_{n,r} \geq \left(\frac{\varepsilon}{2(m+2)M_{m,r}}\right)^{\frac{1}{2r}}\right\} \end{aligned}$$

we see that

$$B_{m+2} \subseteq B_{m+3} \cup B_{m+4},$$

which implies for i not $m+2$:

$$B \subseteq \bigcup_{i=1}^{m+4} B_i.$$

Hence, for every $j \in \mathbb{N}$, we have for i not $m+2$:

$$\sum_{n \in B} a_{jn} \leq \sum_{i=1}^{m+4} \sum_{n \in B_i} a_{jn}. \tag{3.6}$$

Taking limit as $n \to \infty$ in the both sides of (3.6) and also using (3.1), (3.2), and also considering Lemma 3.1 we conclude that

$$\lim_j \sum_{n \in B} a_{jn} = 0,$$

which gives (3.4). □

If we use the space $C^m([a,b])$ instead of $AC^m([a,b])$, then we can get a slight modification of Theorem 3.2. To see this we need the next lemma.

Lemma 3.3. *Let $A := [a_{jn}]$ be a non-negative regular summability matrix, and let $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$. Assume that $L_n : C([a,b]) \to C([a,b])$ is a sequence of positive linear operators. If (3.2) holds, then, for every $f \in C^m([a,b])$, we have:*

$$(i)\ \ st_A - \lim_n \left(\sup_{x \in [a,b]} \omega \left(U_f(x, \cdot), \delta_{n,r} \right)_{[x,b]} \right) = 0,$$

$$(ii)\ \ st_A - \lim_n \left(\sup_{x \in [a,b]} \omega \left(V_f(x, \cdot), \delta_{n,r} \right)_{[a,x]} \right) = 0,$$

where $\delta_{n,r}$ is the same as in (2.7); $U_f(\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ are given respectively by (2.3) and (2.4).

Proof. We know from (**b**) that if $f \in C^m([a,b])$, then both $U_f(\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ belong to $C([a,b] \times [a,b])$. Then, by (**c**), the functions $\omega \left(U_f(x,\cdot), \delta_{n,r} \right)_{[x,b]}$ and $\omega \left(V_f(x,\cdot), \delta_{n,r} \right)_{[a,x]}$ are continuous at the point $x \in [a,b]$. Hence, there exist the points $x_0, x_1 \in [a,b]$ such that

$$\sup_{x \in [a,b]} \omega \left(U_f(x,\cdot), \delta_{n,r} \right)_{[x,b]} = \omega \left(U_f(x_0,\cdot), \delta_{n,r} \right)_{[x_0,b]} =: g(\delta_{n,r})$$

and

$$\sup_{x\in[a,b]} \omega\left(V_f\left(x,\cdot\right),\delta_{n,r}\right)_{[a,x]} = \omega\left(V_f\left(x_1,\cdot\right),\delta_{n,r}\right)_{[a,x_1]} =: h\left(\delta_{n,r}\right).$$

Since $U_f(x_0,\cdot)$ and $V_f(x_1,\cdot)$ are continuous on $[a,b]$, the functions g and h are right continuous at the origin. By (3.2), we get, for any $\delta > 0$, that

$$\lim_j \sum_{n:\delta_{n,r}\geq\delta} a_{jn} = 0. \tag{3.7}$$

Now, by the right continuity of g and h at zero, for a given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $g(\delta_{n,r}) < \varepsilon$ whenever $\delta_{n,r} < \delta_1$ and that $h(\delta_{n,r}) < \varepsilon$ whenever $\delta_{n,r} < \delta_2$. Then, we may write that $g(\delta_{n,r}) \geq \varepsilon$ implies $\delta_{n,r} \geq \delta_1$, and also that $h(\delta_{n,r}) \geq \varepsilon$ implies $\delta_{n,r} \geq \delta_2$. Hence, we see that

$$\{n \in \mathbb{N} : g(\delta_{n,r}) \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : \delta_{n,r} \geq \delta_1\} \tag{3.8}$$

and

$$\{n \in \mathbb{N} : h(\delta_{n,r}) \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : \delta_{n,r} \geq \delta_2\} \tag{3.9}$$

So, it follows from (3.8) and (3.9) that, for each $j \in \mathbb{N}$,

$$\sum_{n:g(\delta_{n,r})\geq\varepsilon} a_{jn} \leq \sum_{n:\delta_{n,r}\geq\delta_1} a_{jn} \tag{3.10}$$

and

$$\sum_{n:h(\delta_{n,r})\geq\varepsilon} a_{jn} \leq \sum_{n:\delta_{n,r}\geq\delta_2} a_{jn} \tag{3.11}$$

Then, taking limit as $j \to \infty$ on the both sides of the inequalities (3.10), (3.11); and also using (3.7) we immediately get, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:g(\delta_{n,r})\geq\varepsilon} a_{jn} = \lim_j \sum_{n:h(\delta_{n,r})\geq\varepsilon} a_{jn} = 0,$$

which means that

$$st_A - \lim_n \left(\sup_{x\in[a,b]} \omega\left(U_f\left(x,\cdot\right),\delta_{n,r}\right)_{[x,b]} \right) = 0$$

and

$$st_A - \lim_n \left(\sup_{x\in[a,b]} \omega \left(V_f\left(x,\cdot\right), \delta_{n,r}\right)_{[a,x]} \right) = 0.$$

Therefore, the proof of Lemma is completed. □

Then, we get the following result.

Theorem 3.4. *Let $A := [a_{jn}]$ be a non-negative regular summability matrix, and let $r > 0$, $r \notin \mathbb{N}$, $m = \lceil r \rceil$. Assume that $L_n : C([a,b]) \to C([a,b])$ is a sequence of positive linear operators. If* (3.1) *and* (3.2) *hold, then, for every $f \in C^m([a,b])$, we have* (3.4).

Proof. By (2.8), we get

$$\begin{aligned} \|L_n(f) - f\| \le \ & K_{m,r} \left\{ \|L_n(e_0) - e_0\| + \sum_{k=1}^{m-1} \left\| L_n\left(|\Psi|^k\right) \right\| \right. \\ & + \delta_{n,r}^r g\left(\delta_{n,r}\right) + \delta_{n,r}^r h\left(\delta_{n,r}\right) \\ & + \delta_{n,r}^r g\left(\delta_{n,r}\right) \|L_n\left(e_0\right) - e_0\|^{\frac{1}{r+1}} \\ & \left. + \delta_{n,r}^r h\left(\delta_{n,r}\right) \|L_n\left(e_0\right) - e_0\|^{\frac{1}{r+1}} \right\}, \end{aligned} \tag{3.12}$$

where $g(\delta_{n,r})$ and $h(\delta_{n,r})$ are the same as in the proof of Lemma 3.3. Now, for a given $\varepsilon > 0$, consider the following sets

$$C \ : \ = \left\{ n \in \mathbb{N} : \|L_n(f) - f\| \ge \varepsilon \right\},$$

$$C_k \ : \ = \left\{ n \in \mathbb{N} : \left\| L_n\left(|\Psi|^k\right) \right\| \ge \frac{\varepsilon}{(m+4)K_{m,r}} \right\}, \ k = 1, 2, \ldots, m-1.$$

$$C_m \ : \ = \left\{ n \in \mathbb{N} : \|L_n(e_0) - e_0\| \ge \frac{\varepsilon}{(m+4)\,K_{m,r}} \right\}$$

$$C_{m+1} \ : \ = \left\{ n \in \mathbb{N} : \delta_{n,r}^r g\left(\delta_{n,r}\right) \ge \frac{\varepsilon}{(m+4)\,K_{m,r}} \right\},$$

$$C_{m+2} \quad : \quad = \left\{ n \in \mathbb{N} : \delta_{n,r}^{r} h\left(\delta_{n,r}\right) \geq \frac{\varepsilon}{(m+4) K_{m,r}} \right\},$$

$$C_{m+3} \quad : \quad = \left\{ n \in \mathbb{N} : \delta_{n,r}^{r} g\left(\delta_{n,r}\right) \left\| L_n\left(e_0\right) - e_0 \right\|^{\frac{1}{r+1}} \geq \frac{\varepsilon}{(m+4) K_{m,r}} \right\}$$

$$C_{m+4} \quad : \quad = \left\{ n \in \mathbb{N} : \delta_{n,r}^{r} h\left(\delta_{n,r}\right) \left\| L_n\left(e_0\right) - e_0 \right\|^{\frac{1}{r+1}} \geq \frac{\varepsilon}{(m+4) K_{m,r}} \right\}.$$

Then, by (3.12), we have

$$C \subseteq \bigcup_{i=1}^{m+4} C_i.$$

So, for every $j \in \mathbb{N}$, we get

$$\sum_{n \in C} a_{jn} \leq \sum_{i=1}^{m+4} \left(\sum_{n \in C_i} a_{jn} \right). \tag{3.13}$$

On the other hand, by (3.1), (3.2) and Lemmas 3.1, 3.3, we see that

$$\begin{aligned}
st_A - \lim_n \left\| L_n\left(|\Psi|^k\right) \right\| &= 0, \ (k = 1, .., m-1), \\
st_A - \lim_n \delta_{n,r}^{r} g\left(\delta_{n,r}\right) &= 0, \\
st_A - \lim_n \delta_{n,r}^{r} h\left(\delta_{n,r}\right) &= 0, \\
st_A - \lim_n \delta_{n,r}^{r} g\left(\delta_{n,r}\right) \left\| L_n\left(e_0\right) - e_0 \right\|^{\frac{1}{r+1}} &= 0, \\
st_A - \lim_n \delta_{n,r}^{r} h\left(\delta_{n,r}\right) \left\| L_n\left(e_0\right) - e_0 \right\|^{\frac{1}{r+1}} &= 0.
\end{aligned}$$

Hence, we observe that, for every $i = 1, 2, \ldots, m+4$,

$$\lim_j \sum_{n \in C_i} a_{jn} = 0. \tag{3.14}$$

Now, taking limit as $j \to \infty$ in the both sides of (3.13) and using (3.14) we obtain that

$$\lim_j \sum_{n \in C} a_{jn} = 0.$$

The last equality implies that

$$st_A - \lim_n \|L_n(f) - f\| = 0,$$

which completes the proof. □

4. Concluding Remarks. In this section we introduce a sequence of positive linear operators which satisfies all conditions of Theorem 3.2 but not Theorem A.

Now take $A = C_1 = [c_{jn}]$, the Cesáro matrix, and define the sequences (u_n) and (v_n) by

$$u_n := \begin{cases} \sqrt{n}, & \text{if } n = m^2 \ (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

and

$$v_n := \begin{cases} 1/2, & \text{if } n = m^2 \ (m \in \mathbb{N}), \\ 1, & \text{otherwise.} \end{cases}$$

Then observe that

$$st - \lim_n u_n = 0 \quad \text{and} \quad st - \lim_n v_n = 1.$$

Let $r = \dfrac{1}{2}$. Then we get $m = \left\lceil \dfrac{1}{2} \right\rceil = 1$. Now consider the following Bernstein-like positive linear operators:

(3.15)

$$L_n(f;x) := (1 + u_n) \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} v_n^k x^k (1 - v_n x)^{n-k}, \quad x \in [0,1], \quad n \in \mathbb{N},$$

where $f \in AC([0,1])$ with $f' \in L_\infty([0,1])$. Since

$$L_n(e_0) = 1 + u_n,$$

we easily get,

$$st-\lim_n \|L_n(e_0)-e_0\| = st-\lim_n u_n = 0,$$

which gives (3.1). Also, by Hölder's inequality with $p=\frac{4}{3}$ and $q=4$, since

$$\begin{aligned} L_n\left(|\Psi|^{\frac{3}{2}};x\right) &= (1+u_n)\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^{3/2}\binom{n}{k}v_n^k x^k(1-v_nx)^{n-k} \\ &\leq (1+u_n)\left(\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\binom{n}{k}v_n^k x^k(1-v_nx)^{n-k}\right)^{3/4} \\ &= (1+u_n)\left(x^2(1-v_n)^2+\frac{v_nx-v_n^2x^2}{n}\right)^{3/4} \end{aligned}$$

we have

$$\delta_{n,\frac{1}{2}}^{3/2} = \left\|L_n\left(|\Psi|^{\frac{3}{2}}\right)\right\| \leq (1+u_n)\left((1-v_n)^2+\frac{1}{4n}\right)^{3/4}.$$

Using the fact that

$$st-\lim_n u_n = 0 \quad \text{and} \quad st-\lim_n v_n = 1,$$

we get

$$st-\lim_n (1+u_n)\left((1-v_n)^2+\frac{1}{4n}\right)^{3/4} = 0.$$

Hence, we obtain that

$$st-\lim_n \delta_{n,\frac{1}{2}} = 0,$$

which verifies (3.2). Therefore, by Theorem 3.2, for every $f\in AC([0,1])$ with $f'\in L_\infty([0,1])$, we have

$$st_A-\lim_n \|L_n(f)-f\| = 0.$$

However, since neither (u_n) nor (v_n) converges to zero (in the usual sense), it is impossible to approximate f by the sequence $\{L_n(f)\}$ for every $f\in AC([0,1])$

with $f' \in L_\infty([0,1])$. This example clearly shows that our statistical result in Theorem 3.2 is more applicable than Theorem A.

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