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Growth Theorem and the Radius of Starlikeness of Close-to-Spirallike Functions

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Let A be the class of all analytic functions in the open unit disc $\mathbb{D} = \{z| \quad |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $g(z)$ be an element of A and satisfy the condition $Re(e^{i\alpha}\frac{g'(z)}{g(z)}) > 0$ for some α, $|\alpha| < \frac{\pi}{2}$. Then $g(z)$ is said to be α-spirallike. Such functions are known to be univalent in $\mathbb{D}$. It was shown by L. Spacek [11], that the α-spirallike functions are univalent in $|z| < 1$.

Let S^*_α denote the class of all functions $g(z)$ satisfying the above condition for a given α. A function $f(z) \in A$ is called close-to-α spirallike, if there exists a function $g(z)$ in S^*_α such that $Re(\frac{f(z)}{g(z)}) > 0$. The class of such functions is denoted by $S^*_\alpha K$.

The aim of this paper is to give a growth theorem and the radius of starlikeness of the class $S^*_\alpha K$.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the condition $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. The family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ analytic in $\mathbb{D}$, and satisfying the conditions $p(0) = 1$, $Rep(z) > 0$ is denoted by $\mathcal{P}$ such that $p(z)$ in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some $\phi(z) \in \Omega$, and every $z \in \mathbb{D}$. Then we say that $p(z) \in P$ is the Caratheodory function, see [1].

Next, let A be the family of functions which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$. Let $g(z)$ be an element of A and satisfying the condition,

$$Re(e^{i\alpha}\frac{g'(z)}{g(z)}) > 0 \tag{1.2}$$

for some α, $|\alpha| < \frac{1}{2}$, then $g(z)$ is said to be α-spirallike. The class of such functions is denoted by S^*_α, see [2]. A function $f(z) \in A$ is close-to-α-spirallike if there exists a function $g(z)$ in S^*_α such that

$$Re(\frac{f(z)}{g(z)}) > 0 \tag{1.3}$$

for all $z \in \mathbb{D}$. The class of such functions is denoted by $S^*_\alpha K$. By taking $\alpha = 0$, we see that every close-to-α-spirallike functions reduces to a close-to-star function, see [6]. Close-to-star functions are not always univalent in $\mathbb{D}$; consequently, close-to-α-spirallike functions need not to be univalent in $\mathbb{D}$. We also note that if $g(z) \in S^*_\alpha$, the introduction of appropriate normalizing factors enables us to write

$$\sec\alpha[e^{i\alpha}z\frac{g'(z)}{g(z)} - i\sin\alpha]_{z=0} = 1. \tag{1.4}$$

This leads to a useful representation formula for being member of S^*_α in terms of functions $\mathcal{P}$. The function $g(z)$ is α-spirallike function if and only if there exists a function $p(z)$ in P,

$$e^{i\alpha}z\frac{g'(z)}{g(z)} = \cos\alpha p(z) + i\sin\alpha. \tag{1.5}$$

Many geometric and analytic properties of α-spirallike functions can be obtained from (1.5). Using (1.5), D. Pashkouleva [8] found the radius of spiral convexity of a close-to-spirallike functions.

Finally, we need to give a description. Let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$. The following lemma is due to I. S. Jack, see [2] and plays very important role in our proof of Theorem 2.2.

Lemma 1.1. *Let $w(z)$ be regular in the unit disc with $w(0) = 0$. Then if $|w(z)|$ obtains its maximum value on the circle $|z| = r$ at the point z_1, one has. $z_1 w'(z_1) = kw(z_1)$, for some $k \geq 1$.*

2. Main results

Lemma 2.1. ([7]) *Let $g(z)$ be an element of S^*_α, then*

$$\frac{r}{(1-r)^{\cos^2\alpha-\cos\alpha}(1+r)^{\cos^2\alpha+\cos\alpha}} \leq$$

$$|g(z)| \leq \frac{r}{(1-r)^{\cos^2\alpha+\cos\alpha}(1+r)^{\cos^2\alpha-\cos\alpha}}\,. \tag{2.1}$$

This inequality is sharp because the extremal function is

$$f(z) = z(1-z)^{2\cos\alpha e^{-i\alpha}}$$

with

$$\zeta = \frac{r(r-e^{i\alpha})}{1-re^{i\alpha}},$$

then

$$\zeta\frac{g'(z)}{g(z)} = \frac{1-2\cos\alpha.r+e^{-2i\alpha}r^2}{1-r^2}. \tag{2.2}$$

P r o o f. Let $g(z) \in S^*_\alpha$. If there exists a function p(z) in $\mathcal{P}$ such that

$$e^{i\alpha}z\frac{g'(z)}{g(z)} = \cos\alpha p(z) + i\sin\alpha. \tag{2.3}$$

By using $p(z)$ is subordinate to $(\frac{1+z}{1-z})$, we obtain ([1]).

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}. \tag{2.4}$$

Equations (2.3) and (2.4) yield

$$\left|z\frac{g'(z)}{g(z)} - \frac{1+e^{-2i\alpha}r^2}{1-r^2}\right| \leq \frac{2r\cos\alpha}{1-r^2}. \tag{2.5}$$

which gives:

$$\frac{1-(2\cos\alpha)r+(\cos 2\alpha)r^2}{1-r^2} \leq Re(z\frac{g'(z)}{g(z)}) \leq \frac{1+(2\cos\alpha)r+(\cos 2\alpha)r^2}{1-r^2}. \tag{2.6}$$

On the other hand we have

$$Re(z\frac{g'(z)}{g(z)}) = r\frac{\partial}{\partial r}\log|g(z)|\,. \tag{2.7}$$

Using (2.7) and after the simple calculations we get,

$$\frac{1-(2\cos\alpha)r+(\cos 2\alpha)r^2}{r(1-r)(1+r)} \leq \frac{\partial}{\partial r}\log|g(z)| \leq \frac{1+(2\cos\alpha)r+(\cos 2\alpha)r^2}{r(1-r)(1+r)}. \quad (2.8)$$

then after integration we obtain (2.1). ■

Theorem 2.2.

$$g(z) \in S_\alpha^* \Leftrightarrow (z\frac{g'(z)}{g(z)}-1) \prec \frac{2(e^{-i\alpha}\cos\alpha)z}{1-z} = F(z). \quad (2.9)$$

Proof. Let $g(z)$ be an element of S_α^*, then we define the function $\phi(z)$ by

$$\frac{g(z)}{z} = (1-\phi(z))^{-2\cos\alpha e^{-i\alpha}}, \quad (2.10)$$

where $(1-\phi(z))^{-2\cos\alpha e^{-i\alpha}}$ has the value 1 at $z=0$, then $\phi(z)$ is analytic and $\phi(0)=0$. If we take logarithmic derivative from (2.10), we get

$$(z\frac{g'(z)}{g(z)}-1) = \frac{(2\cos\alpha e^{-i\alpha})z\phi'(z)}{1-\phi(z)}. \quad (2.11)$$

Now it is easy to realize that the subordination is equivalent $|\phi(z)|<1$ for all $z \in \mathbb{D}$. Indeed, assume to the contrary: Then there exists $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. So from I. S. Jack Lemma, $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$ and for such $z_1 \in \mathbb{D}$, we have

$$(z_1\frac{g'(z_1)}{g(z_1)}-1) = \frac{(2\cos\alpha e^{-i\alpha})k\phi(z_1)}{1-\phi(z_1)} = F(\phi(z_1)) \notin F(\mathbb{D}).$$

But this contradicts (2.11); so our assumption is wrong, i.e., $|\phi(z)|<1$ for all $z \in \mathbb{D}$. This shows that

$$(z\frac{g'(z)}{g(z)}-1) \prec \frac{(2e^{-i\alpha}\cos\alpha)z}{1-z}.$$

Conversely,

$$(z\frac{g'(z)}{g(z)}-1) \prec \frac{e^{-i\alpha}(2\cos\alpha)z}{1-z} \Rightarrow$$

$$z\frac{g'(z)}{g(z)}-1 = e^{-i\alpha}\frac{(2\cos\alpha)\phi(z)}{1-\phi(z)} \Rightarrow$$

$$e^{i\alpha}z\frac{g'(z)}{g(z)} = \cos\alpha\frac{1+\phi(z)}{1-\phi(z)} + i\sin\alpha.$$

This shows that $g(z) \in S^*_\alpha$. ■

Corollary 2.3. *Let $g(z)$ be an element of S^*_α, then*

$$\left|(\frac{z}{g(z)})^{\frac{e^{i\alpha}}{2\cos\alpha}} - 1\right| < 1. \tag{2.12}$$

P r o o f. Since

$$\frac{g(z)}{z} = (1-\phi(z))^{-2\cos\alpha e^{-i\alpha}}. \tag{2.13}$$

is analytic and $(1-\phi(z))^{-2\cos\alpha e^{-i\alpha}}$ has the value 1 at $z=0$, then after simple calculations from (2.12) with $|\phi(z)| < 1$ we get (2.13). We also note that the inequality (2.12) is the Marx-Strohhacker inequality for α-spirallike functions. If we take $\alpha = 0$ we obtain,

$$\left|(\frac{z}{g(z)})^{\frac{1}{2}} - 1\right| < 1. \tag{2.14}$$

The inequality (2.14) is the Marx-Strohhacker inequality for starlike functions, see [5]. ■

Theorem 2.4. *Let $f(z)$ be an element of $S^*_\alpha K$, then*

$$r(1+r)^{\cos^2\alpha+\cos\alpha-1}(1-r)^{\cos^2\alpha-\cos\alpha+1} \le |f(z)|$$

$$\le r(1+r)^{\cos^2\alpha-\cos\alpha+1}(1-r)^{\cos^2\alpha+\cos\alpha-1}.$$

This inequality is sharp because the extremal function is

$$f(z) = z(1+z)(1-z)^{-2\cos\alpha e^{-i\alpha}-1}$$

with

$$\zeta = \frac{r(r-e^{i\alpha})}{1-re^{i\alpha}}.$$

P r o o f. Using the definition of close-to-α-spirallike function, we can write;

$$Re(\frac{f(z)}{g(z)}) > 0 \Rightarrow \frac{f(z)}{g(z)} = p(z) \Rightarrow \left|\frac{f(z)}{g(z)}\right| = |p(z)|. \tag{2.15}$$

On the other hand, since $p(z) \in \mathcal{P}$, then we have;

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}. \tag{2.16}$$

Considering the inequalities (2.15) and (2.16) together we get the result. ■

Lemma 2.5. *Let $f(z)$ be an element of $S^*_\alpha K$, then*

$$Re\left[(1-z)^{2\cos\alpha e^{-i\alpha}}\frac{f(z)}{z}\right] > 0. \tag{2.17}$$

P r o o f. Since

$$g(z) = \frac{z}{1-z)^{2\cos\alpha e^{-i\alpha}}}$$

is α-spirallike function, then using the definition of close-to-α-spirallike function we get (2.17). ■

Theorem 2.6. ([7]) *The radius of starlikeness of the class of $S^*_\alpha K$ is*

$$r = \frac{1}{(1+\cos\alpha) + \sqrt{(1+\cos\alpha)^2 - \cos 2\alpha}}. \tag{2.18}$$

This radius is sharp because the extremal function is

$$f(z) = \frac{z(1+z)}{(1-z)^{2\cos\alpha e^{-i\alpha}-1}}$$

with

$$\zeta = \frac{r(r-e^{i\alpha})}{1-re^{i\alpha}}.$$

P r o o f. Using Lemma 2.5 and after straightforward calculations, we get

$$z\frac{f'(z)}{f(z)} = z\frac{p'(z)}{p(z)} + \frac{1+(2\cos\alpha e^{-i\alpha}-1)z}{1-z}. \tag{2.19}$$

On the other hand if we take

$$w = \frac{1+(2\cos\alpha e^{-i\alpha}-1)z}{1-z},$$

then we obtain

$$Rew \geq \frac{1-2r\cos\alpha r + \cos 2\alpha r^2}{1-r^2}. \tag{2.20}$$

Therefore, we have

$$Re(z\frac{f'(z)}{f(z)}) = Rez\frac{p'(z)}{p(z)} + Rew \geq \frac{-2r}{1-r^2} + \frac{1-2\cos\alpha r + \cos 2\alpha r^2}{1-r^2} \Rightarrow$$

$$Re(z\frac{f'(z)}{f(z)}) \geq \frac{1-2(1+\cos\alpha)r + \cos 2\alpha r^2}{1-r^2}. \tag{2.21}$$

From (2.21) we conclude that the radius of starlikeness of $S^*_\alpha K$ is the smallest positive root of

$$1-2(1+\cos\alpha)r+\cos 2\alpha r^2,$$

which is the result. ∎

Corollary 2.7. *If we take $\alpha = 0$, then we obtain*

$$r = \frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}. \tag{2.22}$$

This is the radius of starlikeness of the class of close-to-star functions which was obtained by Sakaguchi, see [10].

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