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Complex Oscillations and Limit Cycles in Autonomous Two-Component Incommensurate Fractional Dynamical Systems

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In the paper, long-time behavior of solutions of autonomous two-component incommensurate fractional dynamical systems with derivatives in the Caputo sense is investigated. It is shown that both the characteristic times of the systems and the orders of fractional derivatives play an important role for the instability conditions and system dynamics. For these systems, stationary solutions can be unstable for wider range of parameters compared to ones in the systems with integer order derivatives. As an example, the incommensurate fractional FitzHugh-Nagumo model is considered. For this model, different kinds of limit cycles are obtained by the method of computer simulation. A common picture of non-linear dynamics in fractional dynamical systems with positive and negative feedbacks is presented.

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Key Words: fractional dynamical system, linear stability analysis, limit cycles, fractional FitzHugh-Nagumo model

1. Introduction and problem formulation

Autonomous non-linear systems of ordinary differential equations are important mathematical models of dynamical processes that are widely used in different areas like electrical engineering, electronics, and chemical and biological kinetics to mention only few of them. Depending on an application, nonlinearities in these models can be of different kind and range from polynomials and rational functions in chemical and biochemical models to complex exponential functions in electronics industry.

Even if nonlinear phenomena and mechanisms of their formation have been extensive investigated within the last decades, their adequate mathematical theory is still not completed. In the recent publications (see e.g. [1, 4, 10, 12, 13, 21] and references therein), differential equations of fractional order have been suggested for modelling of some systems and processes and in particular for the so called anomalous phenomena in the complex systems. The main reason for utility of the fractional dynamical systems in applications is that they can adequately represent some long-memory and non-local effects that are typical for many anomalous processes. During the last few years, fractional dynamical systems and their applications have been discussed by many authors. We refer the reader e.g. to the recent books [11], [15], [18] and to the references therein.

In applications, mathematical models in form of dynamical systems usually consist of many coupled material balance equations. Nevertheless, the two-component non-linear systems with positive and negative feedbacks are very important from the viewpoint of understanding of basic properties of non-linear dynamics and the role of feedbacks (see e.g. [9]). In this paper, we deal with the two-component incommensurate fractional dynamical systems:

$$\tau \cdot D_{\alpha} u = f\left(u, \mathcal{A}\right), \tag{1}$$

where $u(t) = \left(u_1\left(t\right), u_2\left(t\right)\right)^T$, $u_1(t), u_2(t)$ being the variables with a positive and a negative feedback, respectively, $D_{\alpha} u = \left(d^{\alpha_1} u_1 / dt^{\alpha_1}, d^{\alpha_2} u_2 / dt^{\alpha_2}\right)^T$ is the incommensurate fractional differential operator, $\alpha_1,\ \alpha_2 \in (0,\ 2)$, $f\left(u, \mathcal{A}\right) = \left(f_1(u_1, u_2, \mathcal{A}), f_2(u_1, u_2, \mathcal{A})\right)^T$ is a non-linear vector-function that depends on a external parameter $\mathcal{A}$, and the matrix $\tau = diag(\tau_1, \tau_2)$ represents the characteristic times of the system. The fractional derivatives of non-integer order α $(m-1 < \alpha < m,\ m \in N)$ are understood in the Caputo sense:

$$(D^{\alpha} x)(t) \equiv \frac{d^{\alpha}}{dt^{\alpha}} x\left(t\right) := \frac{1}{\Gamma\left(m-\alpha\right)} \int_0^t \left(t-\tau\right)^{m-\alpha-1} x^{(m)}\left(\tau\right)\, d\tau.$$

If $\alpha = m,\ m \in N$, the Caputo fractional derivative coincides with the common derivative of the order m. For the fractional dynamical system (1), an initial-value problem will be considered. We note, that the number of initial conditions depends on the orders of fractional derivatives: whereas for $0 < \alpha_i \leq 1,\ i = 1, 2$ one initial condition for $u_i,\ i = 1, 2$ is posed, for $1 < \alpha_i \leq 2,\ i = 1, 2$ two initial conditions for $u_i,\ i = 1, 2$ are required. In what follows, we restrict ourselves to the case, when the ratio α_1/α_2 of orders of the fractional derivatives is a rational number.

Let us note that models of type (1) provide a continuous transition between systems of coupled oscillators ($\alpha_1 = \alpha_2 = 2$) and algebraic systems ($\alpha_1 = \alpha_2 = 0$). Analysis of non-linear fractional dynamical systems with derivatives of different orders is thus an important topic that can help both in understanding of the role of derivative orders in system dynamics and open new insights regarding classical non-linear problems.

2. Linear stability analysis for stationary points

When the ratio α_1/α_2 of orders of the fractional derivatives in (1) is a rational number, the two-component incommensurate system (1) can be transformed under some reasonable constrains to a system of fractional differential equations of the same order γ ($\alpha_1 = p\gamma,\ \ \alpha_2 = r\gamma,\ p, r \in N$):

$$\tau^e \cdot D^e_\alpha u^e = F^e\left(u^e, \mathcal{A}\right), \tag{2}$$

where $D^e_\alpha u^e = \left(d^\gamma u^e_{11}/dt^\gamma, ..., d^\gamma u^e_{1p}/dt^\gamma, d^\gamma u^e_{21}/dt^\gamma, ..., d^\gamma u^e_{2r}/dt^\gamma\right)^T$, $u^e_{11} \equiv u_1$, $u^e_{21} \equiv u_2$, $u^e = \left(u_{11}, ..., u_{1p,}, u_{21}, ..., u_{2r}\right)^T$, $\tau^e = diag\left(\tau^\gamma_{11}, ..., \tau^\gamma_{1p,}, \tau^\gamma_{21}, ..., \tau^\gamma_{2r}\right)$, $F^e\left(u^e, \mathcal{A}\right) = \left(u^e_{12}, ..., u^e_{1p}, f_1(u^e_{11}, u^e_{21}, \mathcal{A}), u^e_{22}, ..., u^e_{2r}, f_2(u^e_{11}, u^e_{21}, \mathcal{A})\right)^T$. This representation is obtained my means of the semigroup property of the Caputo fractional derivative that is valid under some suitable conditions on the solution u of the problem under consideration.

The representation (2) of the system (1) makes it possible to use results already obtained for commensurate fractional dynamical systems and to demonstrate the influence of the relation between derivative orders on stability properties of solutions to the incommensurate systems.

First let us recall some basic facts regarding the linear stability analysis for the integer-order two-component systems ($\alpha_1 = \alpha_2 = 1$ in (1)). The stationary solutions $\bar{u} = (\bar{u}_1, \bar{u}_2)$ of the system (1) of differential equations are obtained as solutions of the system of algebraic equations $f_1(u_1, u_2, \mathcal{A}) = 0$, $f_2(u_1, u_2, \mathcal{A}) = 0$. To analyze the stability of these steady-state solutions, let us consider the linearization of the non-linear system. Expanding the right-hand side of the system (1) in powers of small perturbations $\Delta u = (\Delta u_1, \Delta u_2)$ of a steady-state solution $\bar{u}$ ($\Delta u_1 = u_1 - \bar{u}_1$, $\Delta u_2 = u_2 - \bar{u}_2$), we get the following system of linear differential equations:

$$D_\alpha \Delta u = J \cdot \Delta u \tag{3}$$

with the Jacobi matrix

$$J = \begin{pmatrix} a_{11}/\tau_1 & a_{12}/\tau_1 \\ a_{21}/\tau_2 & a_{22}/\tau_2 \end{pmatrix}, \tag{4}$$

where $a_{11} = \partial f_1/\partial u_1$, $a_{12} = \partial f_1/\partial u_2$, $a_{21} = \partial f_2/\partial u_1$, $a_{22} = \partial f_2/\partial u_2$ and all derivatives are evaluated at the fixed point $\bar{u} = (\bar{u}_1, \bar{u}_2)$. In some situations, stability (instability) of the steady-state solutions of the non-linear system (1) follows from stability (instability) of trivial solutions of the system (3). To judge the stability of the linear system, we have to analyze the eigenvalues of the Jacobi matrix (4). They are determined by the characteristic equation

$$\lambda^2 - trJ \cdot \lambda + \det J = 0 \tag{5}$$

with the roots $\lambda_{1,2} = \frac{1}{2}(trJ \pm \sqrt{tr^2 J - 4\det J})$ (here $trJ \equiv a_{11}/\tau_1 + a_{22}/\tau_2$, $\det J \equiv a_{11} \cdot a_{22}/\tau_1\tau_2 - a_{12} \cdot a_{21}/\tau_1\tau_2$). If at least one eigenvalue has a positive real part, i.e. if one of the conditions $trJ < 0$, $\det J < 0$ or $trJ > 0$ holds true, then both the trivial solution of the linear system (3) and the corresponding steady-state solution of the non-linear system (1) are unstable.

Let us now consider (1) with the fractional derivatives of the same order $\alpha = \alpha_1 = \alpha_2 \neq 1$. In this case, a similar procedure can be applied to analyze stability of the steady-state solutions. Because the Caputo derivative applied to a constant function is identically equal to zero, the linearization of (1) for small perturbations Δu of a steady-state solution $\bar{u}$ leads to the same linear system (3). Stability of solutions to this system of linear fractional differential equations is described by theorem of Matignon [14]:

Theorem 1. *The linear autonomous system*

$$D_\alpha x = Ax, \quad x(0) = x_0, \quad 0 < \alpha < 1 \tag{6}$$

with $x = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $A = \|a_{i,j}\|_{n\times n}$, $a_{i,j} = const$ *for* $\forall i, j$ *is asymptotically stable if and only if the condition* $\alpha < \frac{2}{\pi}|Arg(\lambda_i)|$ *is satisfied for all eigenvalues* λ_i *of the matrix* A*.*

This system is stable if and only if the condition $\alpha \leq \frac{2}{\pi}|Arg(\lambda_i)|$ *is satisfied for all eigenvalues* λ_i *of the matrix* A *and all critical eigenvalues that satisfy the condition* $\alpha = \frac{2}{\pi}|Arg(\lambda_i)|$ *have geometric multiplicity of one.*

To illustrate the theorem of Matignon, we note that the solution to the initial-value problem (6) can be represented in the form

$$x(t) = E_\alpha(At)x_0$$

with the Mittag-Leffler function of the matrix argument:

$$E_\alpha(M) = \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(\alpha k + 1)}.$$

In the case of a diagonalizable matrix A, i.e. a matrix that can be represented in the form $A = Q^{-1}BQ$ with a diagonal matrix B that contains eigenvalues $\lambda_i,\ i = 1, \ldots, n$ of A on its diagonal, the Mittag-Leffler function $E_\alpha(At)$ can be easy evaluated:

$$E_\alpha(At) = \sum_{k=0}^{\infty} \frac{(Q^{-1}BQ\,t)^k}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{(Q^{-1}BQ)^k t^k}{\Gamma(\alpha k+1)} = Q^{-1}B_{ML}(t)Q,$$

where $B_{ML}(t)$ is a diagonal matrix that contains the elements $E_\alpha(\lambda_i t)$ on its diagonal, $\lambda_i,\ i = 1,\dots,n$ being the eigenvalues of the matrix A. The Matignon theorem follows then from the well-known asymptotics of the Mittag-Leffler function $E_\alpha(\lambda_i t)$ in the complex plane (see e.g. [17]).

In fact, the Matignon theorem says, that for the linear stability analysis of the fractional dynamical systems we have to take into account not just the real parts of the eigenvalues as for the conventional dynamical systems but the relationship between the imaginary and the real parts of eigenvalues, too. For the system (3) in the case $\alpha_1 = \alpha_2 = \alpha$ $(0 < \alpha < 2)$ for every point inside the parabola $\det J = tr^2 J/4$ (Fig. 1(a)) there exists a marginal value of $\alpha_0 = \frac{2}{\pi}|Arg(\lambda_i)| = \frac{2}{\pi}\left|Arctg\left(\frac{Im\lambda_i}{Re\lambda_i}\right)\right|$ defined by the formula

$$\alpha_0 = \begin{cases} \frac{2}{\pi}\arctan\sqrt{4\det J/tr^2 J - 1}, & tr J > 0, \\ 2 - \frac{2}{\pi}\arctan\sqrt{4\det J/tr^2 J - 1}, & tr J < 0, \end{cases} \tag{7}$$

that determines the stability domain of the system (see [7]). In other words, the order α of the fractional derivatives in the system (1) is an additional bifurcation parameter that can change behavior of the system.

Another approach to linear stability analysis of the commensurate system (1) is in employing the Laplace transform technique. The formula of the Laplace transform of the Caputo fractional derivative is well-known (see e.g. [17, 21]):

$$(\widetilde{D^\alpha x})(s) := \int_0^\infty e^{-st}(D^\alpha x)(t)\,dt = s^\alpha \tilde{x}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}x^{(k)}(0_+), \quad n-1 < \alpha \le n,$$

where $\tilde{x}(s)$ denotes the Laplace transform of the function x at the point s. Applying this formula to the linearized system (3) in the case $0 < \alpha < 1$ leads to a system of linear equations in the frequency domain:

$$(\tau_1 s^\alpha - a_{11})\Delta\tilde{u}_1(s) - a_{12}\Delta\tilde{u}_2(s) = \tau_1 s^{\alpha-1}\Delta u_1(0), \tag{8}$$

$$-a_{21}\Delta\tilde{u}_1(s) + (\tau_2 s^\alpha - a_{22})\Delta\tilde{u}_2(s) = \tau_2 s^{\alpha-1}\Delta u_2(0). \tag{9}$$

This system can be easily solved in explicit form:

$$\Delta\tilde{u}_1(s) = \frac{\tau_1(\tau_2 s^\alpha - a_{22})s^{\alpha-1}\Delta u_1(0) + \tau_2 a_{12} s^{\alpha-1}\Delta u_2(0)}{(\tau_1 s^\alpha - a_{11})(\tau_2 s^\alpha - a_{22}) - a_{12}a_{21}}, \tag{10}$$

$$\Delta\tilde{u}_2(s) = \frac{\tau_1 a_{21} s^{\alpha-1}\Delta u_1(0) + \tau_2(\tau_1 s^\alpha - a_{11})s^{\alpha-1}\Delta u_2(0)}{(\tau_1 s^\alpha - a_{11})(\tau_2 s^\alpha - a_{22}) - a_{12}a_{21}}. \tag{11}$$

Let us now consider the case of α being a rational number, i.e. $\alpha = \frac{n}{m}$, $n < m$, $n, m \in \mathbf{N}$. The solution (10)-(11) can be then represented in the vector form as follows:

$$\Delta \widetilde{u}(s) = \frac{G \cdot s^{(2n-m)/m} + H \cdot s^{(n-m)/m}}{\prod_{i=1}^{2n} (s^{1/m} - z_i)}, \tag{12}$$

where G, H are certain constant vectors that are determined by the parameters of the system and z_i are zeros of the polynomial

$$P_{2n} = (\tau_1 z^n - a_{11})(\tau_2 z^n - a_{22}) - a_{12}a_{21}. \tag{13}$$

Conditions for stability (instability) of the trivial solution to the linearized system (3) are in fact determined by the asymptotic behavior of the inverse Laplace transform of (12) (see [8]):

$$\Delta u(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G\, s^{(2n-m)/m} + H\, s^{(n-m)/m}}{\prod_{i=1}^{2n} (s^{1/m} - z_i)} e^{st}\, ds. \tag{14}$$

The integral in (14) can be evaluated by considering an integral with the same integrand along a modified Bromwich contour that is shown in Fig. 1(b). It can be shown that this last integral vanishes along the circular arcs and along the cut, so that the integral (14) is equal to the sum of residiums in the poles of the integrand, i.e. in zeros of the polynomial (13) (for more details see e.g. [8]). This means that stability (instability) of the trivial solution to the linearized system (3) is determined by the roots of the polynomial (13) and can be again formulated in form of the quadratic equation (5).

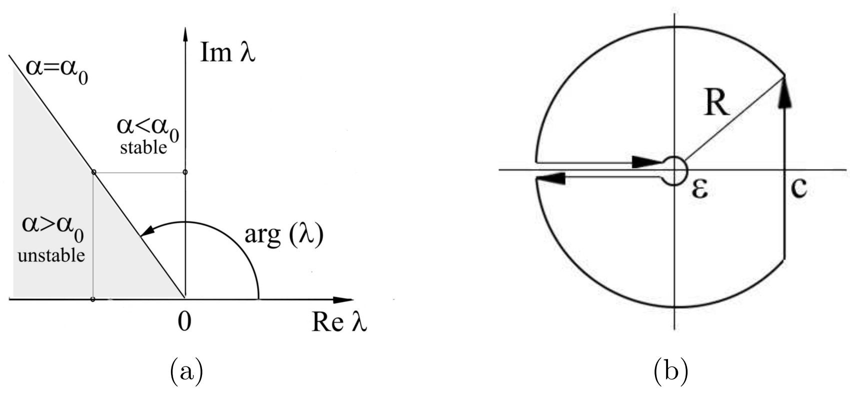

Fig. 1: Marginal value α_0 – (a); modified Bromwich contour – (b)

In the case $1 < \alpha < 2$, applying the Laplace transform to the linearized system leads to additional terms in the equations because of the second initial conditions for the unknown functions. Nevertheless, analysis of the inverse Laplace transform shows that these additional terms do not lead to any additional conditions for stability (instability) of the system and as before the instability is determined by the roots of the polynomial (13). Thus, the Laplace transform approach leads to the same stability conditions for the linearized system (3) as the ones given in the Matignon theorem, but in this case a geometric interpretation of the stability conditions can be given.

Indeed, let us write down the imaginary and real parts of the complex roots of the polynomial (13)

$$Y = \frac{a_{22}\tau_1 + a_{11}\tau_2}{2\tau_1\tau_2} \mp i \cdot \sqrt{-\left(\frac{a_{22}\tau_1 - a_{11}\tau_2}{2\tau_1\tau_2}\right)^2 - \frac{a_{12}a_{21}}{\tau_1\tau_2}} \tag{15}$$

and represent them in trigonometric form $Y = |Y|(\cos\varphi + i\sin\varphi)$, where

$$|Y| = \sqrt{\frac{a_{11}a_{22} - a_{12}a_{21}}{\tau_1\tau_2}}, \tag{16}$$

$$\cos\varphi = \frac{a_{22}\tau_1 + a_{11}\tau_2}{2\sqrt{\tau_1\tau_2(a_{11}a_{22} - a_{12}a_{21})}}, \quad \sin\varphi = \sqrt{\frac{-(a_{22}\tau_1 - a_{11}\tau_2)^2 - 4\tau_1\tau_2 a_{12}a_{21}}{4\tau_1\tau_2(a_{11}a_{22} - a_{12}a_{21})}}.$$

Making the inverse substitution $Y = s^{n/m}$ and applying the de Moivre formula, we obtain the following relations

$$s^{1/n} = |Y|^{1/n}\left(\cos\frac{\varphi + 2k\pi}{n} \pm i\sin\frac{\varphi + 2k\pi}{n}\right), \tag{17}$$

$$s = |Y|^{m/n}\left(\cos\frac{m(\varphi + 2k\pi)}{n} \pm i\sin\frac{m(\varphi + 2k\pi)}{n}\right) \tag{18}$$

with $k = 0, \ldots, n-1$.

For the instability of the system, at least one of the real parts of the m-th degree of the polynomial roots (13) has to be greater than zero, i.e, the condition

$$\cos\frac{m(\varphi + 2k\pi)}{n} > 0 \tag{19}$$

has to be fulfilled. The geometrical interpretation of condition (19) is that it verifies if some roots of the n-th degree of radical of the complex radius-vector raised to the m-th power are located in the right-hand side complex half-plane.

Now let us return to the two-component incommensurate fractional dynamical system (1) that can be represented in form (2) when the ratio α_1/α_2 of orders of the fractional derivatives in (1) is a rational number. The characteristic equation of the linearized system for (2) can be explicitly written down (see [5]) in the form

$$(-\lambda)^{p+r} + (-1)^{r-1}\frac{a_{22}}{\tau_2}(-\lambda)^p + (-1)^{p+1}\frac{a_{11}}{\tau_1}(-\lambda)^r + (-1)^{p+r}\det J = 0, \quad (20)$$

where a_{ij} are defined as before. Analysis of the eigenvalue spectrum that is determined by the equation (20) allows us to construct instability domains for all stationary points of the system (see [5] for details). The form of the characteristic equation shows that its roots distribution and thus the stability behavior of the linearized system strongly depend both on orders of the fractional derivatives and on their ratio. In general, solutions of the characteristic equation that is an algebraic equation of the $(p+r)$-th degree cannot be obtained in explicit form. Instead, we have to compute them numerically with a certain accuracy. Having found these eigenvalues, stability condition from the Matignon theorem is given by the inequality

$$\frac{\pi}{2(p+r)} - \min_i\{|Arg(\lambda_i)|\} > 0. \quad (21)$$

Finally, it should be noted that the results of linear stability analysis are valid only locally. They describe the system's behavior only in a small vicinity of a stationary state. In the non-linear case, new branches of solutions can appear because of bifurcations in the system. If the system parameters are close to the critical values, the minimum set of variables (order parameters) can be determined for these new branches and then used to express the type of system dynamics in the normal form on the basis of the Lyapunov-Schmidt procedure. In more complex cases, say, if the system parameters are far from the critical values or an interaction between the branched solutions leads to repeated bifurcations, construction of the normal form becomes a very complex task. In this case, numerical methods and computer simulations are practically the only tools for study of these non-linear systems. In the next section, we use computer simulations to analyze the behavior of solutions to a special class of fractional dynamical systems.

3. Stability analysis and nonlinear dynamics of fractional van der Pol-FitzHugh-Nagumo model

For practical applications of models formulated as dynamical systems with fractional derivatives, their auto-oscillation behavior, i.e. undamped oscillations raised by an external non-periodical source, are very important. Fractional oscillators of this type have attracted considerable interest during last years. Well-known examples are the fractional Lotka-Volterra model [2, 13], the fractional van der Pol oscillator [19], the fractional-order Chuas, Chen, Lorenz, and Lu systems [16, 20], the fractional Brusselator oscillator [6], etc. From the viewpoint of the qualitative theory of dynamical systems, auto-oscillations are special attractors in the phase space. Periodic auto-oscillations correspond to a simple attractor, a so called limit cycle. In is well-known that in the 2nd order dynamical systems with integer derivatives no strange attractors exist and non-trivial attractors can be always represented as some closed paths in the phase space. For the two-component dynamical systems with fractional derivatives of different orders, the situation with attractors is qualitatively different.

To illustrate qualitative behavior of solutions to the fractional dynamical systems, we consider in this section the nonlinear fractional model (1) with the nonlinear source term

$$f_1(u, \mathcal{A}) = u_1 - u_1^3/3 - u_2 \tag{22}$$

for the activator variable and with the linear source term

$$f_2(u, \mathcal{A}) = -u_2 + \beta u_1 + \mathcal{A} \tag{23}$$

for the inhibitor one (see [3, 9]). This model was proposed for the first time by R. FitzHugh (see e.g. [9]) as a generalization of the classical van der Pol equations. During the last decades, several nonlinear phenomena in physics, chemistry, and biology were explained on the basis of dynamical systems with nonlinearities of this type [3, 9].

In this section, we show that dynamics of the fractional dynamical system (1) with sources (22), (23) and different orders of fractional derivatives is qualitative different compared to one of the system with integer-order derivatives. The fixed points of this model correspond to the null-cline intersections (Fig. 2(a)) and can be determined as solutions to the cubic algebraic equation

$$(\beta - 1)\overline{u}_1 + \overline{u}_1^3/3 + \mathcal{A} = 0 \tag{24}$$

as functions of the external parameter A. Equation (20) with the coefficients $a_{11} = (1 - \bar{u}_1^2)$, $a_{12} = -1$, $a_{21} = \beta$, $a_{22} = -1$, equation (24), and the inequality (21) for the linearized system define the instability conditions for the null-cline intersection points (see [7]). In other words, solution of these algebraic systems allows us to obtain the eigenvalues of the linearized system for all fixed points and for arbitrary orders of the fractional derivatives.

In the simplest case $\alpha_1 = 2\alpha_2 = \alpha$, $0 < \alpha < 1$, the characteristic equation (20) takes the form $\lambda^3 + \lambda^2/\tau_2 - \lambda(1 - \bar{u}_1^2)/\tau_1 - \det J = 0$ and can be solved in explicit form by the Cardano formulas. In the general case, the instability conditions for the fractional van der Pol-FitzHugh-Nagumo model are determined by means of computer simulations.

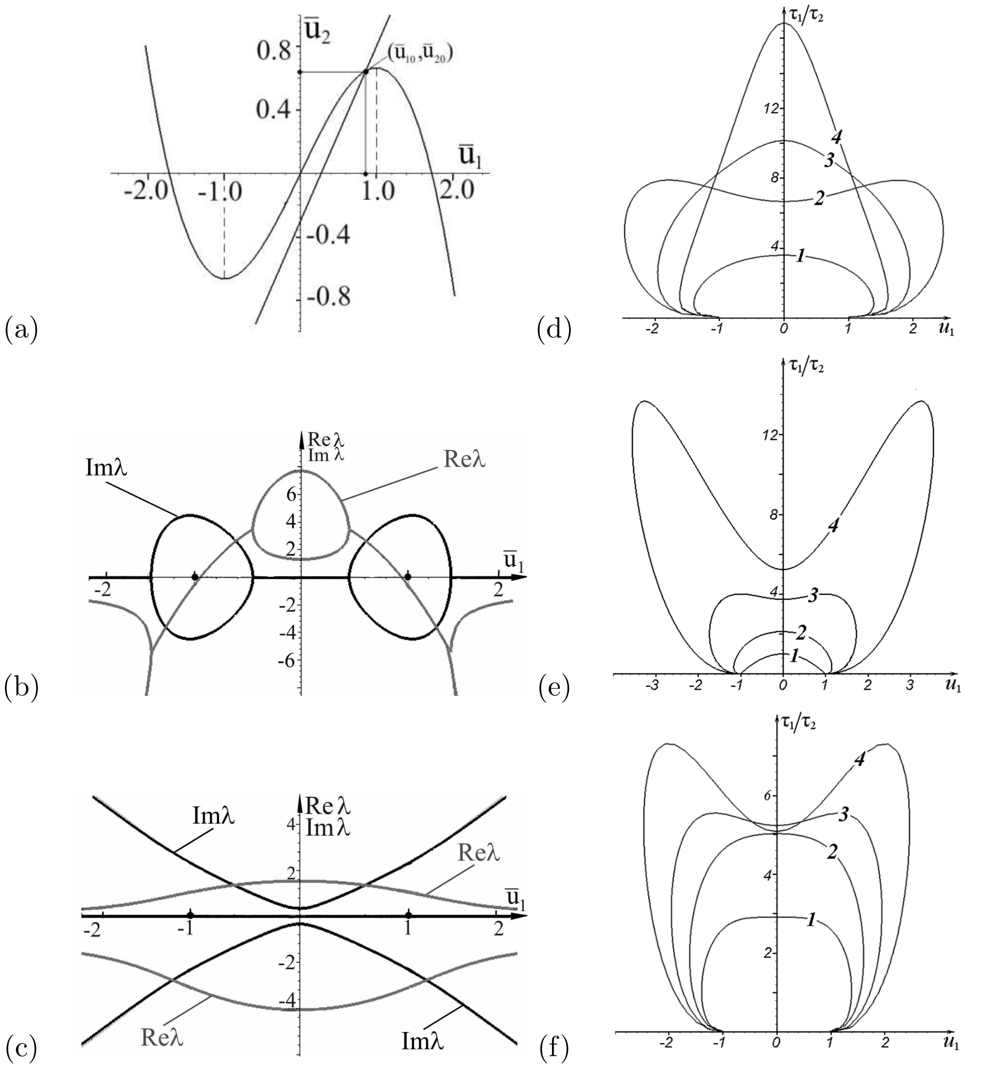

Fig. 2: Null-clines of the system (1) with sources (22), (23) – (a); Imaginary (gray) and real (black) parts of eigenvalues as functions of $\overline{u}_1$ for $\alpha_1 = \alpha_2$ – (b) and for $\alpha_1 = 2\alpha_2$ – (c). The parameters for the left column pictures are $\tau_1 = 0.1$, $\tau_2 = 1$, $\mathcal{A} = 0.5$, $\beta = 2$. Instability domains for different ratios between α_1, α_2: $\alpha_1 > \alpha_2$ - (d), $\alpha_1 = \alpha_2$ - (e), $\alpha_1 < \alpha_2$ - (f). The results of computer simulations for $\beta = 2.0$ and α_1=1.5, α_2=1.25 – **1(d);** α_1=1.75, α_2=1.5 – **2(d);** α_1=1.75, α_2=1.25 – **3(d);** α_1=1.75, α_2=1.0 – **4(d);** α_1=α_2=1.0 – **1(e);** α_1=α_2=1.25 – **2(e);** α_1=α_2= 1.5 – **3(e);** α_1=α_2= 1.75 – **4(e);** α_1=1.25, α_2=1.5 – **1(f);** α_1=1.0, α_2=1.75 –**2(f);** α_1=1.5, α_2=1.75 – **3(f);** α_1=1.25, α_2=1.75 – **4(f)**

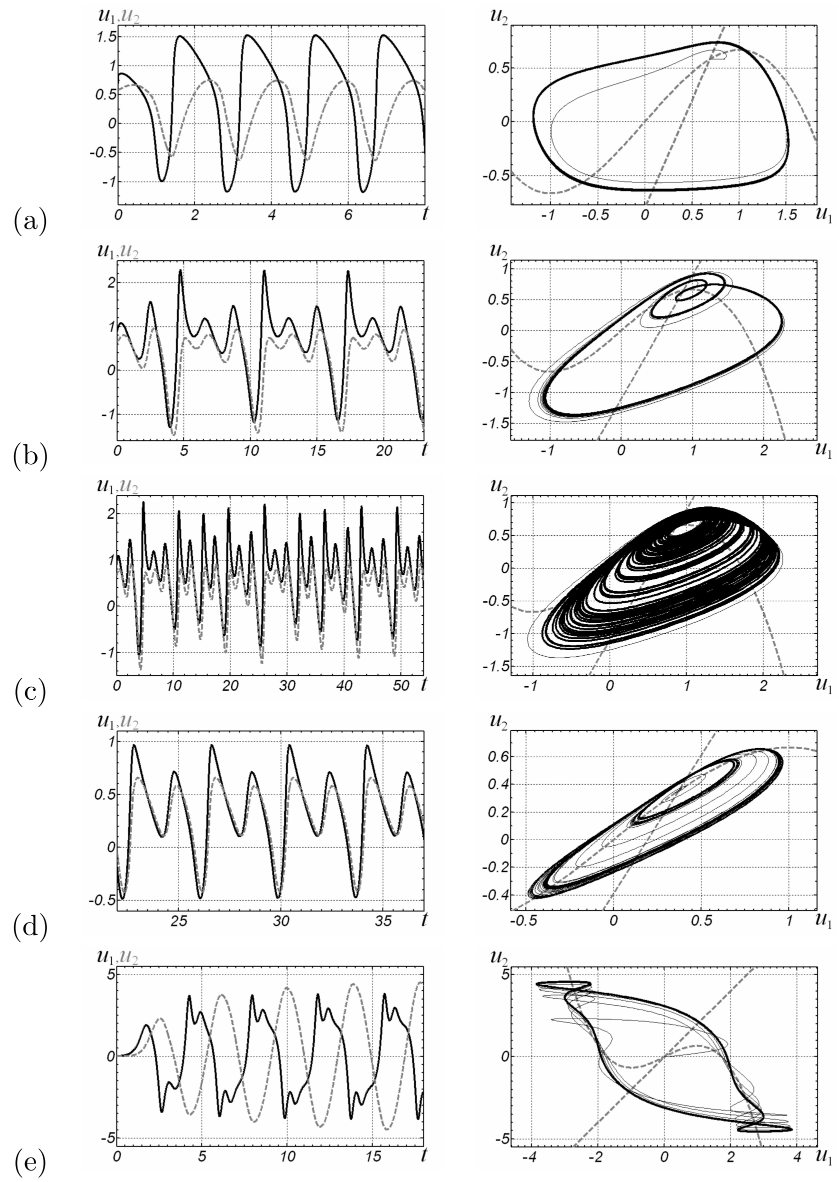

Fig. 3: Examples of the oscillations (left column) and the corresponding phase portraits (right column) for different values of fractional derivatives. Dynamics of variables u_1 (black lines) and u_2 (grey lines) for: α_1=0.5, α_2=1, $\beta = 2$, $\tau_1 = 0.25$, $\tau_2 = 1$, $\mathcal{A} = 0.8$ - **(a);** α_1=1.5, α_2=0.75, $\beta = 2$, $\tau_1 = 0.1$, $\tau_2 = 1$, $\mathcal{A} = -1.1$ - **(b);** α_1=1.5, α_2=0.75, $\beta = 2$, $\tau_1 = 0.1$, $\tau_2 = 1$, $\mathcal{A} = -1.145$ - **(c);** α_1=0.8, α_2=0.4, $\beta = 2$, $\tau_1 = 0.1$, $\tau_2 = 1$, $\mathcal{A} = -0.39$ - **(d);** α_1=1.8, α_2=1.8, $\beta = 2$, $\tau_1 = 0.1$, $\tau_2 = 1$, $\mathcal{A} = 0$ - **(e)**

Typical results of computer simulations of the eigenvalue spectrum and instability domains for different ratios between the orders of the fractional derivatives are presented in Fig. 2. Plots from Fig. 2(b) and 2(c) demonstrate that the spectrum of eigenvalues significantly depends on the ratio between the orders of the fractional derivatives. For a fractional dynamical system, the conditions for instability are strongly connected to the relationship between the real and the imaginary parts of the eigenvalues. Some typical examples of instability domains for the fractional dynamical system (1) with different orders of fractional derivatives are presented in Fig. 2(d)-(f). The curve on the middle plot that corresponds to the case $\alpha_1 = \alpha_2 = 1$ represents instability domain for the standard van der Pol-FitzHugh-Nagumo system. For all fractional derivatives orders $\alpha = (\alpha_1, \alpha_2)$ that are located in the region between the corresponding curve and the horizontal axis, the system (1) with sources (22), (23) is unstable. It is stable outside this region. As we see, both for $\alpha_1 > \alpha_2$ and for $\alpha_1 < \alpha_2$, the system can be unstable within a wide range of the governing parameters (Fig. 2(d),(f)).

It should be noted that the spectrum of the linearized problem is the same for a fixed ratio between the orders of the fractional derivatives. It therefore coincides with the spectrum of a system with the same non-linearities and integer-order derivatives of a higher order. Therefore, the time-evolution of a two-component fractional dynamical system can indicate behavior of dynamical systems with integer-order derivatives of higher order. In contrast to the integer-order dynamical systems, conditions of instability in fractional dynamical systems are realized in a qualitatively different manner and depend not only on the orders of the fractional derivatives but on the ratio between them, too. As a consequence, repeated bifurcations in these systems can occur under conditions that depend on the orders of fractional derivatives both in the case $\alpha_1 > \alpha_2$ and for $\alpha_1 < \alpha_2$. In addition, the "velocities" of the variables in the phase space are also different from the ones in the integer-order systems. As a result, we can expect that dynamics of the fractional order systems should be qualitatively different from and richer compared to one of the integer-order systems. Some results of numerical simulations for the fractional van der Pol-FitzHugh-Nahumo model are presented in Fig. 3.

To plot the pictures, the fractional van der Pol-FitzHugh-Nahumo system with the corresponding initial conditions was integrated numerically using numerical schemes on the basis of the Grünwald-Letnikov definition of the fractional derivative. As we can see, even in the case of a simple fractional dynamical model, a rich scope of different oscillations for different orders of the fractional derivatives is present. For many applications, dependence of the phase portrait of a dynamical system on the system parameters is an important topic. Of particular interest are the values of the parameters where the phase portrait changes qualitatively. For the incommensurate fractional dynamical systems there are two additional parameters that can influence long-time behavior of the system: the orders of the fractional derivatives and the ratio between them. Changing these parameters, we can modify system dynamics and observe both simple limit cycles, limit cycles with intersections, and limit cycles that look like strange attractors (see Fig. 3(c)). Let us finally note, that strange attractors cannot appear in the 2nd order dynamical systems with integer derivatives and thus are a new feature of two-component fractional order dynamical systems.

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