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Exact Solutions of Nonlocal Pluriparabolic Problems

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A generalization of the classical Duhamel principle for the pluriparabolic equation

$$\frac{\partial u}{\partial t_1} + \cdots + \frac{\partial u}{\partial t_n} = \frac{\partial^2 u}{\partial x^2} + F(x, t_1, \ldots, t_n) \quad \text{in} \quad 0 \le x \le a,\ 0 \le t_k \le T_k$$

with time-nonlocal initial value conditions of the form

$$\chi_{k,\tau}\{u(x, t_1, \ldots, t_{k-1}, \tau, t_{k+1}, \ldots, t_n)\} = f_k(x, t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$$

with linear functionals χ_k on $C[0, T_k]$ $(k = 1, \ldots, n)$, a space-local boundary value condition of the form

$$u(0, t_1, \ldots, t_n) = \psi(t_1, \ldots, t_n)$$

and a space-nonlocal boundary value condition of the form

$$\Phi_\xi\{u(\xi, t_1, \ldots, t_n)\} = \phi(t_1, \ldots, t_n)$$

with a linear functional Φ on $C^1[0, a]$ is proposed. To this end two non-classical convolutions $\phi \overset{t_1 \ldots t_n}{*} \psi$ and $F \overset{x t_1 \ldots t_n}{*} G$ are used: the first one for functions of $t_1, \ldots, t_n$ only and the second – for functions of $x, t_1, \ldots, t_n$. The corresponding Duhamel representation takes the following form: If $\Omega(x, t_1, \ldots, t_n)$ is a solution for the boundary value problem for the special choice $F \equiv 0$, $f_k \equiv 0$, $\psi \equiv 0$ and $\phi \equiv 1$, then for $\psi \equiv 0, f_k \equiv 0, k = 1, \ldots, n$ (under some additional assumptions for smoothness of the boundary function ϕ and the function F)

$$u(x, t_1, \ldots, t_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n}(\Omega \overset{t_1 \ldots t_n}{*} \phi) + \frac{\partial^n}{\partial t_1 \ldots \partial t_n}(\Omega \overset{x t_1 \ldots t_n}{*} F).$$

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1. Introduction

In the present paper it is proposed a generalization of the classical Duhamel principle for the pluriparabolic equation

$$\frac{\partial u}{\partial t_1}+\cdots+\frac{\partial u}{\partial t_n}=\frac{\partial^2 u}{\partial x^2}+F(x,t_1,\ldots,t_n) \quad \text{in} \quad 0\le x\le a,\ 0\le t_k\le T_k \tag{1}$$

with local and nonlocal boundary value conditions (BVCs) of the form: nonlocal initial conditions

$$\chi_{k,\tau}\{u(x,t_1,\ldots,t_{k-1},\tau,t_{k+1},\ldots,t_n)\}=f_k(x,t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n) \tag{2}$$

$(k=1,\ldots,n)$, and local and nonlocal boundary conditions

$$u(0,t_1,\ldots,t_n)=\psi(t_1,\ldots,t_n), \qquad \Phi_\xi\{u(\xi,t_1,\ldots,t_n)\}=\phi(t_1,\ldots,t_n). \tag{3}$$

Here χ_k, $k=1,\ldots,n$ are linear functionals on $C[0,T_k]$ and Φ is a linear functional on $C^1[0,a]$.

Such problems for a pluriparabolic equation with energy functional of the form

$$\Phi_\xi\{u(\xi,t_1,\ldots,t_n)\}=\int_0^a u(\xi,t_1,\ldots,t_n)d\xi \tag{4}$$

are considered by J.R. Cannon [3], A. Bouziani [2], S. Mesloub [9].

2. Convolutions

Our basic tool for obtaining explicit solutions of the problem considered are some multidimensional non-classical convolutions. Their construction begins with the simplest one-dimensional case (Dimovski [5]).

Consider the elementary one-dimensional BVP in $C[0,T_k]$

$$y'-\mu y=f(t), \qquad \chi_k\{y\}=0. \tag{5}$$

For the sake of some technical simplifications we assume that the constant function $\{1\}$ does not belong to the kernel of the functional χ_k, $k=1,\ldots,n$, i.e. $\chi_k\{1\}\ne 0$. Then without any loss of generality we can assume $\chi_k\{1\}=1$. Its solution is

$$y=r_k(f,\mu)(t_k)=\int_0^{t_k} e^{\mu(t_k-\sigma)}f(\sigma)d\sigma-\chi_k\left\{\int_0^{\sigma} e^{\mu(\tau-\sigma)}f(\xi)d\xi d\sigma\right\}\frac{e^{\mu t_k}}{G_k(\mu)}, \tag{6}$$

where $G_k(\mu) = \chi_{k,\tau}\{e^{\mu\tau}\}$ is the *exponential indicatrix* of the functional χ_k. Our assumption $\chi_k\{1\} \neq 0$ is equivalent to $G_k(0) \neq 0$, i.e. $\mu = 0$ is not an eigenvalue of BVP (5). Then instead of (6) we may consider the special case

$$r_k(f,0) = l_k f = \int_0^{t_k} f(\sigma)d\sigma - \chi_k\left(\int_0^{\sigma} f(\xi)d\xi d\sigma\right) \tag{7}$$

which defines a right inverse operator l_k of $\dfrac{d}{dt_k}$ on the space $C[0,T_k]$ satisfying the following identity

$$l_k f'(t_k) = f(t_k) - \chi_{k,\tau} f(\tau). \tag{8}$$

Theorem 1. (Dimovski [5]) *The operation*

$$(f \overset{t_k}{*} g)(t_k) = \chi_{k,\tau}\left(\int_{\tau}^{t_k} f(t_k+\tau-\sigma)g(\sigma)d\sigma\right), \tag{9}$$

where the subscript τ means that χ_k acts on the variable τ only, is a commutative and associative in $C[0,T_k]$ such that

$$l_k f(t_k) = \{1\} \overset{t_k}{*} f(t_k) \tag{10}$$

and

$$r_k(f,\mu)(t_k) = \left\{\frac{e^{\mu t_k}}{G_k(\mu)}\right\} \overset{t_k}{*} f(t_k). \tag{11}$$

Next we need an one-dimensional convolution, connected with $\dfrac{d^2}{dx^2}$ in $C^1[0,a]$. Consider the elementary BVP

$$y'' + \lambda^2 y = f(x), \qquad y(0) = 0, \quad \Phi\{y\} = 0 \tag{12}$$

with a non-zero linear functional Φ on $C^1[0,a]$. In order it to have a solution, it is necessary to assume $\Phi_\xi\{\xi\} \neq 0$. Again, without essential loss of generality, one can assume $\Phi_\xi\{\xi\} = 1$. The solution is

$$y = R_{-\lambda^2} f(x) = \frac{1}{\lambda}\int_0^x \sin\lambda(x-\xi)f(\xi)d\xi - \Phi_\xi\left\{\frac{1}{\lambda}\int_0^x \sin\lambda(a-\xi)f(\xi)d\xi\right\}\frac{\sin\lambda x}{\lambda E(\lambda)}, \tag{13}$$

where $E(\lambda) = \Phi_\xi\left\{\dfrac{\sin\lambda\xi}{\lambda}\right\}$ is the *sine-indicatrix* of the functional Φ. For a simplification of the next consideration it is useful to assume that $\lambda = 0$ is not an eigenvalue of (12). Since $E(0) = \Phi_\xi\{\xi\}$ by the above assumptions we have $E(0) = 1$. Now

$$R_0 f(x) = Lf(x) = \int_0^x (x-\xi)f(\xi)d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\} \tag{14}$$

defines a right inverse operator L of $\dfrac{d^2}{dx^2}$ on the space $C[0,a]$ satisfying the identity

$$Lf''(x) = f(x) + [x\Phi(1) - 1]f(0) - x\Phi_\xi[f(\xi)]. \tag{15}$$

Theorem 2. (Dimovski [5]) *The operation*

$$(f \overset{x}{*} g)(x) =$$
$$-\frac{1}{2}\widetilde{\Phi}_\xi\left[\int_x^\xi f(\xi+x-\eta)g(\eta)d\eta - \int_{-x}^\xi f(|\xi-x-\eta|)g(|\eta|)\mathrm{sgn}\,(\xi-x-\eta)\eta d\eta\right], \tag{16}$$

where $\widetilde{\Phi}_\xi = \Phi_\xi \circ \int_0^\xi$, *is commutative and associative in* $C[0,a]$ *such that*

$$Lf(x) = \{x\} \overset{x}{*} f(x) \tag{17}$$

and

$$R_{-\lambda^2} f(x) = \left\{\frac{\sin\lambda x}{\lambda E(\lambda)}\right\} \overset{x}{*} f(x). \tag{18}$$

Next the following multidimensional generalizations of the Duhamel convolution are given.

Theorem 3. *The operation*

$$(\phi \overset{t_1\ldots t_n}{*} \psi)(t_1,\ldots,t_n) =$$
$$\chi_{n,\tau_n}\cdots\chi_{1,\tau_1}\left[\int_{\tau_n}^{t_n}\!\!\cdots\int_{\tau_1}^{t_1}\phi(t_1+\tau_1-\sigma_1,\ldots,t_n+\tau_n-\sigma_n)\psi(\sigma_1,\ldots,\sigma_n)d\sigma_1\ldots d\sigma_n\right] \tag{19}$$

for $\phi,\psi \in C([0,T_1]\times\cdots\times[0,T_n])$ *is bilinear, commutative and associative and*

$$l_1\ldots l_n\phi = \{1\} \overset{t_1\ldots t_n}{*} \phi. \tag{20}$$

Using definition (19) of the operation $\phi \overset{t_1\ldots t_n}{*} \psi$ on $C([0,T_1]\times\cdots\times[0,T_n])$, we define a $(n+1)$-dimensional convolution in $C([0,a]\times[0,T_1]\times\cdots\times[0,T_n])$.

Definition 1. For $F, G \in C([0,a]\times[0,T_1]\times\cdots\times[0,T_n])$, let

$$\begin{aligned}
&(F \overset{xt_1\ldots t_n}{*} G)(x,t_1,\ldots,t_n)\\
&\quad = -\frac{1}{2}\widetilde{\Phi_\xi}\left[\int_x^\xi F(\xi+x-\eta,t_1,\ldots,t_n) \overset{t_1\ldots t_n}{*} G(\eta,t_1,\ldots,t_n)d\eta\right.\\
&\left. -\int_{-x}^{\xi} F(|\xi-x-\eta|,t_1,\ldots,t_n) \overset{t_1\ldots t_n}{*} G(|\eta|,t_1,\ldots,t_n)\operatorname{sgn}(\xi-x-\eta)\eta d\eta\right].
\end{aligned} \tag{21}$$

Theorem 4. *The operation defined by (21) is bilinear, commutative and associative in* $C([0,a]\times[0,T_1]\times\cdots\times[0,T_n])$ *such that*

$$Ll_1\ldots l_n f = \{x\} \overset{xt_1\ldots t_n}{*} f. \tag{22}$$

S k e t c h o f t h e p r o o f. The proof of both Theorems 1 and 2 goes along the same line. First, we verify the assertions for products

$$\phi(t_1,\ldots,t_n) = \phi_1(t_1)\ldots\phi_n(t_n) \quad \psi(t_1,\ldots,t_n) = \psi_1(t_1)\ldots\psi_n(t_n),$$

or

$$F(x,t_1,\ldots,t_n) = f(x)\phi_1(t_1)\ldots\phi_n(t_n) \quad G(x,t_1,\ldots,t_n) = g(x)\psi_1(t_1)\ldots\psi_n(t_n)$$

and reduce them to the one dimensional assertions. Next we approximate the arbitrary functions ϕ, ψ and F, G by products, e.g. by polynomials. ∎

The following analogues of the identities (8) and (15) hold for functions $u \in C([0,a]\times[0,T_1]\times\cdots\times[0,T_n])$:

$$l_k u_{t_k}(x,t_1,\ldots,t_n) = u(x,t_1,\ldots,t_n) - \chi_{k,\tau_k}[u(x,t_1,\ldots,\tau_k,\ldots,t_n)] \tag{23}$$

$(k=1,\ldots,n)$ and

$$\begin{aligned}
&Lu_{xx}(x,t_1,\ldots,t_n)\\
&\quad = u(x,t_1,\ldots,t_n) + [x\Phi(1)-1]u(0,t_1,\ldots,t_n) - x\Phi_\xi[u(\xi,t_1,\ldots,t_n)].
\end{aligned} \tag{24}$$

3. Rings of multipliers of convolution algebras

In what follows, let $C = C([0,a]\times[0,T_1]\times\cdots\times[0,T_n])$ and let $\big(C,*\big)$ be the respective convolution algebra. We follow a standard procedure for constructing of an operational calculus for BVP (1) – (3) based on convolution (21) and its multipliers as outlined in Dimovski [5].

Let us remind the notion of multiplier of the algebra $(C,*)$ (Larsen, [7]). An operator $M : C \to C$ is said to be a *multiplier* of the convolution algebra $(C, *)$, iff the relation

$$M(f*g) = (Mf)*g$$

holds for arbitrary $f, g \in C$.

The multipliers of $(C,*)$ form a commutative ring $\mathfrak{M}$ without annihilators with respect to the usual multiplication of operators. Let $\mathfrak{N}$ be the multiplicative set of the non-divisors of 0 of the ring $\mathfrak{M}$. $\mathfrak{N}$ evidently is nonempty since at least the identity operator and the multiplier convolution operator $L = \{x\}*$ are non-divisors of 0. Another examples are the operators l_k.

Consider the formal fractions A/B where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$.

Definition 2. The ring $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$ of the multiplier fractions is the quotient of the ring $\mathfrak{M}\times\mathfrak{N}$ with respect to the equivalence relation

$$(A,B) \sim (C,D) \Leftrightarrow AD = BC,$$

i.e. $\mathcal{M} = \mathfrak{M}\times\mathfrak{N}/\sim$.

Theorem 5. *The ring $\mathcal{M}$ of the multiplier fractions contains subrings isomorphic to: a) $\mathbb{R}$, b) $\Big(C[0,a], \overset{x}{*}\Big)$, c) $\Big(C[0,T_k], \overset{t_k}{*}\Big)$, d) $(C,*)$.*

P r o o f. a) The correspondence $\alpha \longmapsto \dfrac{\alpha L}{L}$, $\alpha \in \mathbb{R}$ is an embedding $\mathbb{R} \hookrightarrow \mathcal{M}$;

b) The correspondence $f \longmapsto \dfrac{(Lf)\overset{x}{*}}{L}$ is an embedding $\Big(C[0,a], \overset{x}{*}\Big) \hookrightarrow \mathcal{M}$;

c) The correspondence $\varphi \longmapsto \dfrac{(l_k\varphi)\overset{t_k}{*}}{l_k}$ is an embedding $\Big(C[0,T_k], \overset{t_k}{*}\Big) \hookrightarrow \mathcal{M}$;

d) The correspondence $u \longmapsto \dfrac{\{u\}*}{I}$ where I is the identity operator of C is an embedding $(C,*) \hookrightarrow \mathcal{M}$.

The verification is immediate. Let us prove for example b). Let $f, g \in C[0,a]$. We are to prove that

$$f \overset{x}{*} g \longmapsto \frac{(Lf)\overset{x}{*}}{L} \cdot \frac{(Lg)\overset{x}{*}}{L}$$

Indeed,

$$f \overset{x}{*} g \longmapsto \frac{\{L(f \overset{x}{*} g)\}*}{L} = \frac{L\left[\{L(f \overset{x}{*} g)\}*\right]}{L^2} = \frac{\left\{(Lf) \overset{x}{*} (Lg)\right\} \overset{x}{*}}{L^2}$$

$$= \left[\frac{(Lf) \overset{x}{*}}{L}\right] \left[\frac{(Lg) \overset{x}{*}}{L}\right].$$

Here we make use of the convolution property

$$L(f \overset{x}{*} g) = (Lf) \overset{x}{*} g = f \overset{x}{*} (Lg).$$

■

For every $\phi \in C([0, T_1] \times \cdots \times [0, T_n])$ the partial convolution (19) defines a multiplier acting on $F \in C$ as follows

$$\phi \overset{t_1 \dots t_n}{*} F. \tag{25}$$

The corresponding equivalence class in $\mathcal{M}$ is called *constant with respect to* x and is denoted by

$$[\phi]_x \tag{26}$$

Similarly, let $f \in C([0, a] \times [0, T_1] \times \cdots \times [0, T_{k-1}] \times [0, T_{k+1}] \times \cdots \times [0, T_n])$. The partial convolution operator

$$f \overset{xt_1 \dots t_{k-1} t_{k+1} \dots t_n}{*} \tag{27}$$

defines a multiplier in an obvious manner. Its class is called *constant with respect to* t_k and is denoted by

$$[f]_{t_k}. \tag{28}$$

4. Algebraization of the BVP (1)–(3)

Crucial for the algebraization of the problem are the reciprocal elements to L and $l_1, \dots, l_n$ in $\mathcal{M}$. Let they be denoted by S, s_1, ..., s_n, respectively. Now

$$S\{x\} = SL = 1, \qquad s_k l_k = 1 \quad (k = 1, \dots, n) \tag{29}$$

where 1 denotes the unit of the algebra $\mathcal{M}$. For a function $u = u(x, t_1, \dots, t_n)$ this together with (23) and (24) gives

$$u_{t_k} = s_k u - [\chi_{k,\tau}\{u(x, t_1, \dots, \tau \dots, t_n)\}]_{t_k} \qquad (k = 1, \dots, n). \tag{30}$$

and the last term is a constant with respect to the variable t_k. Similarly,

$$u_{xx} = Su + (x\Phi(1) - 1)u(0, t_1, \ldots, t_n) + [\phi(t_1, \ldots, t_n)]_x \tag{31}$$

and the last term is a constant with respect to the variable x. Suppose $u = u(x, t_1, \ldots, t_n)$ is a solution to the boundary value problem (1)–(3). Then (30) and (31) reduce the BVP (1)–(3) to a simple linear algebraic equation for the function u:

$$\begin{aligned}(s_1 + \cdots + s_n - S)u = (x\Phi(1) - 1)\psi(t_1, \ldots, t_n) + [\phi(t_1, \ldots, t_n)]_x \\ + \sum_{k=1}^{n} [f_k(x, t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)]_{t_k} + \{F(x, t_1, \ldots, t_n)\}.\end{aligned} \tag{32}$$

Definition 3. A function $u \in C([0, a] \times [0, T_1] \times \cdots \times [0, T_n])$ is a *weak solution* to BVP (1)–(3) if it satisfies (32).

In order to reveal the basic ideas, we restrict BVP (1)–(3) to the case

$$\chi_{k,\tau}\{u(x, t_1, \ldots, t_{k-1}, \tau, t_{k+1}, \ldots, t_n)\} = 0, \qquad k = 1, \ldots, n \tag{33}$$

and

$$u(0, t_1, \ldots, t_n) = 0, \qquad \Phi_\xi\{u(\xi, t_1, \ldots, t_n)\} = \phi(t_1, \ldots, t_n). \tag{34}$$

Suppose $s_1 + \cdots + s_n - S = \Sigma$ is a non-divisor of zero. Then $\frac{1}{\Sigma}$ is well defined. If u is a weak solution of BVP (1),(33) and (34) then formally we obtain

$$u = \frac{1}{\Sigma}[\phi(t_1, \ldots, t_n)]_x + \frac{1}{\Sigma}\{F(x, t_1, \ldots, t_n)\}. \tag{35}$$

In order to interpret (35) as a function, we need some algebraic manipulations:

$$u = (s_1 \ldots s_n)\frac{1}{(s_1 \ldots s_n)\Sigma}[\phi(t_1, \ldots, t_n)]_x + (s_1 \ldots s_n)\frac{1}{(s_1 \ldots s_n)\Sigma}\{F(x, t_1, \ldots, t_n)\}. \tag{36}$$

Assuming that $\frac{1}{(s_1 \ldots s_n)\Sigma}$ can be interpreted as a continuous function $\Omega(x, t_1, \ldots, t_n)$ then it can be considered as a weak solution of the homogeneous problem with $\phi \equiv 1$. Indeed, the product $l_1 \ldots l_n$ can be interpreted as the numerical multiplier $[1]_x$, i.e.

$$l_1 \ldots l_n = \{1\} \overset{t_1 \ldots t_n}{*} .$$

Hence

$$\Omega(x, t_1, \ldots, t_n) = (s_1 \ldots s_n) \frac{1}{(s_1 \ldots s_n)\Sigma} l_1 \ldots l_n. \tag{37}$$

Now we can formulate the following *conditional theorem of existence* (a generalization of Duhamel principle).

Theorem 6. *If BVP (1),(33) and (34) has a weak solution* Ω *for* $F \equiv 0$, $\phi \equiv 1$, *then*

$$\Omega(x, t_1, \ldots, t_n) = \frac{1}{(s_1 \ldots s_n)\Sigma} \tag{38}$$

and the BVP (1),(33) and (34) with "arbitrary" F *and* ϕ *also has a weak solution of the form*

$$u(x, t_1, \ldots, t_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} (\Omega \overset{t_1 \ldots t_n}{*} \phi) + \frac{\partial^n}{\partial t_1 \ldots \partial t_n} (\Omega \overset{x t_1 \ldots t_n}{*} F), \tag{39}$$

provided F *and* ϕ *have continuous partial derivatives*

$$\frac{\partial^n F}{\partial t_1 \ldots \partial t_n} \quad \text{and} \quad \frac{\partial^n \phi}{\partial t_1 \ldots \partial t_n}.$$

The proof requires some differentiation properties of the convolutions involved, but here we will not enter into details.

5. Uniqueness of the solution for BVP (1)–(3).

Theorem 6 is a conditional theorem of existence of solution of BVP (1)–(3). As for the uniqueness problem we can state a more definite assertion. To this end, we study the uniqueness for BVP (1)–(3) by means of the spectral properties of the one-dimensional problems that compose it, taking advantage from the fact that these problems are better studied. The eigenvalues $\mu_m^{(k)}$ ($k = 1, \ldots, n$; $m = 1, \ldots, \infty$) for (5) are the zeros of the indicatrices $G_k(\mu) = \chi_{k,\tau}\{e^{\mu\tau}\}$. The projections on the respective eigenspaces are

$$p_{k,\mu_m^{(k)}}(\phi) = -\frac{1}{2\pi i} \int_{\Gamma_{\mu_m^{(k)}}} r_k(\phi, \mu) d\mu = -\left\{ \frac{1}{2i\pi} \int_{\Gamma_{\mu_m^{(k)}}} \frac{e^{\mu t_k} d\mu}{G_k(\mu)} \right\} \overset{t_k}{*} \phi, \tag{40}$$

where $\Gamma_{\mu_m^{(k)}}$ is a small contour around the eigenvalue $\mu_m^{(k)}$ (see [6]).

We will prove a theorem for uniqueness of the solution of BVP (1)–(3) under some additional restrictions on the time-functionals χ_k ($k = 1, \ldots, n$).

Definition 4. A linear functional χ_k on $C[0, T_k]$ is called strongly nonlocal if its support includes the endpoints of the interval $[0, T_k]$, i.e. $0, T_k \in \mathrm{supp}\chi_k$. The corresponding BVPs are called *strongly nonlocal.*

Further we consider only strongly nonlocal BVPs with respect to the time variables. In the case of simple eigenvalues $\mu_m^{(k)}$ the respective eigenspaces are one-dimensional and spanned on the functions $e^{\mu_m^{(k)} t_k}$ and, moreover,

$$p_{k,\mu_m^{(k)}}(\phi) = \phi \overset{t_k}{*} \left\{ -\frac{e^{\mu_m^{(k)} t_k}}{G'_k(\mu_m^{(k)})} \right\}. \tag{41}$$

Similarly the eigenvalues λ_l ($l = 1, \ldots, \infty$) for (12) are the zeros of $E(\lambda) = \Phi \left\{ \dfrac{\sin \lambda \xi}{\lambda} \right\}$.

In order to state the uniqueness result, we need a lemma:

Lemma 1. *The following equalities hold*

$$S\{\sin \lambda_l x\} = -\lambda_l^2 \sin \lambda_l x \quad \text{and} \quad s_k\{e^{\mu_m^{(k)} t_k}\} = \mu_m^{(k)} e^{\mu_m^{(k)} t_k}, \tag{42}$$

$k = 1, \ldots, n;\ l, m = 1, \ldots, \infty.$

P r o o f. It is enough to apply (8) and (15). ∎

It is easily seen that

$$\Sigma \left\{ \sin \lambda_l x e^{\mu_{m_1}^{(1)} t_1} \ldots e^{\mu_{m_n}^{(n)} t_n} \right\} = (\mu_{m_1}^{(1)} + \cdots + \mu_{m_n}^{(n)} + (\lambda_l)^2) \left\{ \sin \lambda_l x e^{\mu_{m_1}^{(1)} t_1} \ldots e^{\mu_{m_n}^{(n)} t_n} \right\} \tag{43}$$

and if $\mu_{m_1}^{(1)} + \cdots + \mu_{m_n}^{(n)} + (\lambda_l)^2 = 0$, then Σ is a divisor of zero.

If there is no dispersion relation of this form, i.e if

$$\mu_{m_1}^{(1)} + \cdots + \mu_{m_n}^{(n)} + (\lambda_l)^2 \neq 0 \tag{44}$$

for each combination of eigenvalues $\mu_{m_1}^{(1)}, \ldots \mu_{m_n}^{(n)}, \lambda_l$, then Σ is a non-divisor of 0.

Lemma 2. (Multidimensional Schwartz-Leontiev theorem) *If* $\phi \in C([0, T_1] \times \ldots [0, T_k])$ *and*

$$\prod_{k=1}^{n} p_{k,\mu_m^{(k)}}(\phi) = 0 \tag{45}$$

for all combinations of eigenvalues, then $\phi \equiv 0$.

P r o o f. The proof of Lemma 2 follows from the one-dimensional Schwartz-Leontiev theorem (see [1], p. 198, [6], pp. 92-93, [10] and [8], pp. 260-261). ∎

P r o o f o f P r o p o s i t i o n 1. Suppose that Σ is a divisor of 0, i.e. that for some u

$$[s_1 + \cdots + s_n - S] u(x, t_1, \ldots, t_n) = 0.$$

Let

$$u_{m_1 \ldots m_n}(x, t_1, \ldots, t_n) = u(x, t_1, \ldots, t_n) \overset{t_1 \ldots t_n}{*} \{e^{\mu_{m_1}^{(1)} t_1} \ldots e^{\mu_{m_n}^{(n)} t_n}\}.$$

Now, from (41) it follows

$$u_{m_1\dots m_n}(x,t_1,\dots,t_n)=U(x)e^{\mu^{(1)}_{m_1}t_1}\dots e^{\mu^{(n)}_{m_n}t_n}$$

and the function $U(x)$ must satisfy

$$(\mu^{(1)}_{m_1}+\dots+\mu^{(n)}_{m_n}-S)U=0$$

Now (44) and [5], Theorem 1.4.1 guarantee the existence of $(\mu^{(1)}_{m_1}+\dots+\mu^{(n)}_{m_n}-S)^{-1}$ and hence $U=0$. Equivalently the function U must satisfy

$$\frac{d^2U}{dx^2}-(\mu^{(1)}_{m_1}+\dots+\mu^{(n)}_{m_n})U=0$$

and $U(0)=0$, $\Phi(U)=0$ and since (44) implies that $\mu^{(1)}_{m_1}+\dots+\mu^{(n)}_{m_n}$ is not an eigenvalue of (12), then $U\equiv 0$. ∎

Remark. The assumption that the eigenvalues are simple is a technical one and is used only to simplify the above argument. The proof for multiple eigenvalues $\mu^{(k)}_{m_k}$ is somewhat more involved, but the conclusion is the same.

Now we can state

Theorem 7. *The conditions that (44) holds for all possible combinations of eigenvalues of the one-dimensional eigenvalue problems are sufficient for uniqueness of the solution of BVP (1)–(3), provided all the time nonlocal BVCs are strong.*

Example. Consider the BVP

$$\frac{\partial u}{\partial t_1}+\frac{\partial u}{\partial t_2}=\frac{\partial^2 u}{\partial x^2}+F(x,t_1,t_2)\quad\text{in}\quad 0\le x\le 1,\ 0\le t_1,t_2\le T \tag{46}$$

with nonlocal initial condition of the form

$$\frac{1}{\mu-1}\left[\mu u(x,t_1,0)-u(x,t_1,T)\right]=0\qquad \frac{1}{\mu-1}\left[\mu u(x,0,t_2)-u(x,T,t_2)\right]=0 \tag{47}$$

(Dezin nonlocal conditions, see [4]) and local and nonlocal energy boundary conditions

$$u(0,t_1,t_2)=0\quad\text{and}\int_0^1 u(x,t_1,t_2)dx=\phi(t_1,t_2). \tag{48}$$

The spectral properties of the last problem (Samarskiy-Ionkin spectral problem) are studied from operational calculus point of view in [5], Theorem 3.4.4. The eigenvalues are $2n\pi$ with multiplicity two and the corresponding eigenspaces are spanned on the functions $\sin 2n\pi x$ and $x\cos 2n\pi x$.

Since the eigenvalues of the corresponding one dimensional problems with Dezin condition are of the form $\mu^{(k)}_m=\frac{1}{T}(\ln|\mu|+2m^{(k)}\pi i)$, $k=1,2$, $m^{(k)}\in\mathbb{Z}$ we can have dispersion relation only if the imaginary part (of the sum corresponding to (44)) is zero i.e. if $m^{(1)} = m^{(2)} = 0$ or $m^{(1)} = -m^{(2)}$. In both cases the condition $\frac{2}{T}\ln\mu \neq (2\pi n)^2$ for all $n \in \mathbb{N}$ guarantees that the real part is nonzero and hence that there is no dispersion relation of the form $\mu^{(1)}_{m_1} + \mu^{(2)}_{m_2} + (2n\pi)^2 = 0$.

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