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THE LEGENDRE FORMULA IN CLIFFORD ANALYSIS

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Abstract. Let $\mathbb{R}_{0,2m+1}$ be the Clifford algebra of the antieuclidean $2m+1$ dimensional space. The elliptic Cliffordian functions may be generated by the ζ_{2m+2} function, analogous to the well-known Weierstrass ζ-function. The latter satisfies a Legendre equality. We prove a corresponding formula at the level of the monogenic function $\Delta^m \zeta_{2m+2}$.

Introduction. In the theory of elliptic functions, the classical Legendre formula for the Weierstrass ζ-function $\zeta(\omega_1)\omega_2 - \zeta(\omega_2)\omega_1 = i\,\pi/2$ has many uses among others in Number Theory. Going from $\mathbb{C}$ to a Clifford algebra we have also a ζ_{2m+2} function which is holomorphic Cliffordian and has the same structure. We are in a multidimensional space, then it is natural to fetch algebraic equality between π, the periods and the values of the function.

Closely related works are written by R. Fueter [2, 3], J. Ryan [11], C. Saçlioglu [12]. The latter stress the fact that in Physics many recent theories

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need dimensions beyond four and compactifications via periodic functions. Here we prove an equality at the level of monogenic functions, stressing how these and holomorphic Cliffordian functions are reinforcing each other. Of course, it is just the beginning of the true story!

1. On the Legendre formula in Complex Analysis. The aim of this section is to recall the Legendre formula for the ζ function of Weierstrass in $\mathbb{C}$ and to give a sketch of a modified classical proof which will be easier to be generalized in the Clifford case.

Start with the notation $2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$ for a lattice, generated by the two complex numbers ω_1, ω_2, $\mathbb{R}$-linearly independent, and rearranged in a set noted $\{w_\rho\}$, $p \in \mathbb{N}$, $w_0 = 0$. Thus, one can define the ζ function of Weierstrass:

$$\zeta(z) = \frac{1}{z} + \sum_{p=1}^{\infty} \left\{ \frac{1}{z - w_p} + \frac{1}{w_p} + \frac{z}{w_p^2} \right\},$$

which is not itself an elliptic function, but inherits the following quasi-periodicity property:

$$\zeta(z + \omega_j) = \zeta(z - \omega_j) + 2\zeta(\omega_j), \ j = 1, 2.$$

In such a way, we have the Legendre formula:

$$i \ \frac{\pi}{2} = \zeta(\omega_1)\omega_2 - \zeta(\omega_2)\omega_1.$$

The classical proof, given in almost all books on the subject, consists of an application of the residues theorem to ζ on the parallelogram R centered at the origin and spanned by the two half periods ω_1, ω_2, i.e. such that the boundary ∂R of R has four sides: $F^+_{+\omega_1} = [\omega_1 - \omega_2, \omega_1 + \omega_2]$, $F^-_{+\omega_2} = [\omega_1 + \omega_2, -\omega_1 + \omega_2]$, $F^-_{-\omega_1} = [-\omega_1 + \omega_2, -\omega_1 - \omega_2]$, $F^+_{-\omega_2} = [-\omega_1 - \omega_2, \omega_1 - \omega_2]$. Because of the unique pole of ζ in R, situated at 0, one has the value of the integral:

$$\int_{\partial R} \zeta(z) dz = 2\pi i.$$

On the other hand, one can make use of the decomposition of the boundary $\partial R = F^+_{+\omega_1} \cup F^-_{+\omega_2} \cup F^-_{-\omega_1} \cup F^+_{-\omega_2}$ and be able to apply the quasi-periodicity property on the two couples of opposite sides: $(F^+_{+\omega_1}, F^-_{-\omega_1})$ and $(F^-_{+\omega_2}, F^+_{-\omega_2})$. Let us mention our notations for the sides obey to the following rules:

(i) The lower index means that the side passes through the indicated point.

(ii) The upper index gives the orientation of the side: $+$, because the variable of integration over this sides is going from $-\omega_j$ to $+\omega_j$, $j = 1, 2$ and $-$ for the opposite cases.

The last argument for the second rule (ii) is not logically very strong. We will make this procedure stronger, even in $\mathbb{R}^n$, in section 3. Here, just mention the following: consider that the surrounding space $\mathbb{R}^2$ is referred to the frame $O\omega_1\omega_2$. Associate to the parallelogram R the two sets:

$$R \cap \{\omega_1 = 0\} \quad \text{and} \quad R \cap \{\omega_2 = 0\}.$$

The natural orientation of the surrounding space $\mathbb{R}^2$, given by $O\omega_1\omega_2$, induces on the first set a natural orientation given by ω_2 and this set will be denoted by F_1^+, whereas the natural orientation of the second set coming from the frame $O\omega_1\omega_2$, is $-\omega_1$. This set will be denoted F_2^-. Let us call F_1^+ and F_2^- the *canonical sides* of ∂R.

In fact ∂R is composed by two pairs of translated canonical sides. When one acts on F_1^+ with the translation $+\omega_1$, the orientation is keeped and we get $F_{+\omega_1}^+$, whereas one acts with the translation $-\omega_1$, the orientation changes: $F_{-\omega_1}^-$.

By the substitutions $z = w + \omega_1$ and $z = w - \omega_1$, respectively, one get:

$$\int_{F_{+\omega_1}^+} \zeta(z)dz = \int_{F_1^+} \zeta(w + \omega_1)dw \qquad \text{and}$$

$$\int_{F_{-\omega_1}^-} \zeta(z)dz = -\int_{F_1^+} \zeta(w - \omega_1)dw,$$

so that

$$\int_{F_{+\omega_1}^+} \zeta(z)dz + \int_{F_{-\omega_1}^-} \zeta(z)dz = 2\zeta(\omega_1)\int_{F_1^+} dw,$$

where we have made use of the quasi-periodicity property. Then, one deduces:

$$2\pi i = 2\zeta(\omega_1)\int_{F_1^+} dz - 2\zeta(\omega_2)\int_{F_2^+} dz,$$

putting again z as the variable of integration. It remains to compute the last two integrals. In the complex case this computation is obvious and it gives $2\omega_2$ and $2\omega_1$, respectively, hence the Legendre formula is obtained.

There is also another modification of the classical proof to do. Introduce the differential form $\gamma(z) = dy - idx$, so that:

$$\int_{F_j^+} dz = i \int_{F_j^+} \gamma(z), \quad j = 1, 2.$$

Let us make now the computation of $\int_{F_j^+} \gamma(z)$ following the method given in [1]. For the $\mathbb{C}$-valued differential form γ, it is true that:

$$\gamma(z) = nds,$$

where n means the outward pointing unit normal and ds is the classical linear element. As far as $\int_{F_1^+} \gamma(z)$ is concerned, we have:

$$\int_{F_1^+} \gamma(z) = (-i) \frac{\omega_2}{\|\omega_2\|} \int_{F_1^+} ds = -2i\omega_2.$$

2. On the Cauchy theory in $\mathbb{R}_{0,2m+1}$. Consider a function which is *holomorphic Cliffordian* ([6, 7]) excepting in a pointwise singularity, namely 0, ([8, 9]), i.e. $f : \mathbb{R}_*^{2m+2} \longrightarrow \mathbb{R}_{0,2m+1}$ and $D\Delta^m f(x) = 0$ for $x \in \mathbb{R}_*^{2m+2}$. Here, D is the Dirac operator $D = \sum_{i=0}^{2m+1} e_i \frac{\partial}{\partial x_i}$ and Δ^m the usual Laplacian iterated m times. Take an open set Ω of $\mathbb{R}^{2m+2}$, containing 0, and let B be a ball, centered at the origin, such that $\overline{B} \subset \Omega$. So we have

$$D\Delta^m f(x) = 0 \quad \text{for} \quad x \in \Omega \setminus B.$$

Hence:

$$\int_{\Omega \setminus B} D\Delta^m f(x)\omega(x) = 0,$$

where $\omega(x) = dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{2m+1}$.

Applying the Stokes formula, one has:

$$\int_{\Omega\setminus B} D\Delta^m f(x)\omega(x) = \int_{\partial(\Omega\setminus B)} \gamma(x)\big(\Delta^m f(x)\big),$$

where $\gamma(x) = \sum_{j=0}^{2m+1} (-1)^{j-1} e_j \, dx_0 \wedge \ldots \wedge dx_{j-1} \wedge dx_j \wedge \ldots \wedge dx_{2m+1}$.

Thus:

$$\int_{\partial\Omega} \gamma(x)\Delta^m f(x) = \int_{\partial B} \gamma(x)\Delta^m f(x).$$

Take an example: let Ω be the hyperparallelogram R centered at the origin, spanned by the paravectors $2\omega_1, 2\omega_2, \ldots, 2\omega_{2m+2}$ and take the function $\zeta = \zeta_{2m+2}$ the meromorphic elliptic Cliffordian function associated to this periods, [8], namely:

$$\zeta(x) = x^{-1} + \sum_{p=1}^{\infty} \left\{ (x - w_p)^{-1} + \sum_{\mu=0}^{2m+1} (w_p^{-1} x)^{\mu} w_p^{-1} \right\},$$

after having rearranged the lattice $2\mathbb{Z}^{2m+2}\omega$ in a countable set $\{w_p\}_0^{\infty}$, with $w_0 = (0, 0, \ldots, 0)$. The Laurent expansion of ζ on a neighborhoud of the origin (valid even in the whole R) is known, [8, 9]:

$$\zeta(x) = x^{-1} + \varphi(x),$$

where φ is a holomorphic Cliffordian function in R. Thus, we have:

$$\int_{\partial B} \gamma(x)\Delta^m \zeta(x) = \int_{\partial B} \gamma(x)\big(\Delta^m x^{-1} + \Delta^m \varphi(x)\big)$$

$$= \int_{\partial B} \gamma(x)\Delta^m (x^{-1}) + \int_{\partial B} \gamma(x)\Delta^m \varphi(x).$$

The last integral is zero, because:

$$\int_{\partial B} \gamma(x)\Delta^m \varphi(x) = \int_{B} D\Delta^m \varphi(x)\omega(x).$$

On the other hand, we know, [7], that:

$$\Delta^m(x^{-1}) = (-1)^m\ 2^{2m}(m!)^2\varpi_m E(x),$$

where $\varpi_m = \dfrac{2\pi^{m+1}}{m!}$ and $E(x) = \dfrac{x^*}{|x|^{2m+2}}$ is the Cauchy kernel of monogenic functions, [1]. In such a way, we get:

$$\int_{\partial B} \gamma(x)\ \Delta^m(x^{-1}) = (-1)^m\ 2^{2m+1}m!\ \pi^{m+1} \int_{\partial B} E(x)\gamma(x).$$

But the Cauchy formula for monogenic functions, [1]:

$$\int_{\partial B} E(y-x)\gamma(y)f(y) = \begin{cases} f(x), & x \in \overset{\circ}{B} \\ 0, & x \notin \overline{B} \end{cases}$$

with $f \equiv 1$, gives:

$$\int_{\partial B} E(y)\gamma(y) = 1.$$

Finally:

$$\int_{\partial R} \gamma(x)\ \Delta^m\zeta(x) = (-1)^m\ 2^{2m+1}(m!)\pi^{m+1}.$$

This formula can be viewed as the natural generalization to higher dimensions of the residues theorem in $\mathbb{C}$ applied to ζ in R, see section 1, $(m = 0)$:

$$\int_{\partial R} \zeta(z)(dy - idx) = 2\pi,$$

which is equivalent to $\displaystyle\int_{\partial R} \zeta(z)dz = 2\pi i.$

3. Some elements of the geometry of $\mathbb{R}^n$. Consider the Euclidean space $\mathbb{R}^n$ with its canonical basis $\{e_1, e_2, \ldots, e_n\}$ and let G be an orthogonal hyperparallelogram, centered at the origin, which sides are $2\lambda_1 e_1, \ldots, 2\lambda_n e_n$, $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, n$.

Introduce the *canonical faces*:

$$G_j = G \cap \{x_j = 0\}, \quad j = 1, \dots, n.$$

The aim is to oriented suitably the G_j. Each G_j is a hyperparallelogram in $\mathbb{R}^{n-1}$ possessing a natural frame

$$O\ e_{j+1}e_{j+2}\dots e_n\ e_1\ e_2\dots e_{j-1}.$$

Let us say G_j be oriented positively, noted G_j^+, when the permutation:

$$(e_j, e_{j+1}, \dots, e_n, e_1, \dots, e_{j-1}) \longrightarrow (e_1, e_2, \dots, e_n)$$

is of positive signature and G_j oriented negatively, noted G_j^-, when the signature of the permutation is -1.

In $\mathbb{R}^2$, we have two sets G_1, G_2. G_1 possesses its own natural frame Oe_2 and will be positively oriented, whereas G_2 will be negatively oriented.

It is remarkable that in $\mathbb{R}^3$, the orientations of the three sets G_1, G_2, G_3 will be positive, because all the permutations $\{(e_1, e_2, e_3) \longrightarrow (e_1, e_2, e_3)\}$, $\{(e_2, e_3, e_1) \longrightarrow (e_1, e_2, e_3)\}$, $\{(e_3, e_1, e_2) \longrightarrow (e_1, e_2, e_3)\}$ are always of positive signature.

As far as $\mathbb{R}^4$ is concerned, the situation is: $G_1^+, G_2^-, G_3^+, G_4^-$. This phenomenon is general: in $\mathbb{R}^{2m+1}$ all the faces (let us tell them again "*faces*") are positively oriented, while in $\mathbb{R}^{2m+2}$ one has an alternated change of the signs, ending with G_{2m+2}^-.

Note also then when we translate every canonical face G_j, the orientation is keeped if the translation is realized in the direction of the vector e_j and changes its sign if we move in the direction of $-e_j$. So, if we start from G_j^+, then we have $G_{+\lambda_j e_j}^+$ and $G_{-\lambda_j e_j}^-$, and if G_k^-, then $G_{+\lambda_k e_k}^-$ and $G_{-\lambda_k e_k}^+$ are obtained.

Important remark. The same procedure can be applied to the case of a hyperparallelogram R, centered at the origin and spanned by the vectors $2\omega_1, \dots, 2\omega_n$ under the condition they are $\mathbb{R}$-linearly independent.

Come back to the Cliffordian case. We are studying the function $\zeta = \zeta_{2m+2}$ in the hyperparallelogram R, centered at the origin, generated by the paravectors $2\omega_1, 2\omega_2, \ldots, 2\omega_{2m+2}$ belonging to $\mathbb{R} \oplus \mathbb{R}^{2m+1}$ whose basis is $e_0 = 1$, $e_1, e_2, \ldots, e_{2m+1}, e_j^2 = -1$, $j = 1, \ldots, 2m+1$.

Such a R possesses 2^{2m+2} vertices and $4m+4$ faces. The $2m+2$ canonical faces are $F_1^+, F_2^-, F_3^+, F_4^-, \ldots, F_{2m+2}^-$ and the oriented boundary of R should be decomposed as follows:

$$\partial R = \sum_{k=0}^{m} \Big\{ \big(F^+_{+\omega_{2k+1}} + F^-_{-\omega_{2k+1}}\big) + \big(F^-_{+\omega_{2k+2}} + F^+_{-\omega_{2k+2}}\big) \Big\}.$$

4. The Legendre formula in $\mathbb{R}_{0,2m+1}$. Repeating the classical proof of the Legendre formula, our first ingredient is:

$$(-1)^m \; 2^{2m+1}(m!)\pi^{m+1} = \int_{\partial R} \gamma(x) \; \Delta^m \zeta(x).$$

Let us compute the right-hand side term using the decomposition of the boundary ∂R of R given in the end of section 3. Start with:

$$\int_{F^+_{+\omega_{2k+1}}} \gamma(x) \; \Delta^m \zeta(x) + \int_{F^-_{-\omega_{2k+1}}} \gamma(x) \; \Delta^m \zeta(x).$$

Set $x = y + \omega_{2k+1}$ and $x = y - \omega_{2k+1}$, respectively. So we get:

$$\int_{F^+_{2k+1}} \gamma(x) \; \Delta^m \zeta(x + \omega_{2k+1}) + \int_{F^-_{2k+1}} \gamma(x) \; \Delta^m \zeta(x - \omega_{2k+1}),$$

where the integrations are carried over the canonical faces and we have denoted the variable again by x. Further, this is equal to:

$$\int_{F^+_{2k+1}} \gamma(x)[\Delta^m \zeta(x + \omega_{2k+1}) - \Delta^m \zeta(x - \omega_{2k+1})]$$

$$= \int_{F^+_{2k+1}} \gamma(x) \Delta^m \big(\zeta(x + \omega_{2k+1}) - \zeta(x - \omega_{2k+1})\big)$$

because the linearity of Δ^m. Now it is time to remember that the ζ Weierstrass meromorphic Cliffordian function, we are considering, possesses a quasi-periodicity property, formulated as follows, [8]:

$$\zeta(x+\omega_{2k+1})-\zeta(x-\omega_{2k+1})=2\zeta(\omega_{2k+1})+2\sum_{p=1}^{m}\frac{(x\mid\nabla_y)^{2p}}{(2p)!}\ \zeta(y)\left.\right|_{y=\omega_{2k+1}}$$

i.e. the quasi-periodicity polynomial is not of degree 0 as in the complex case, but is a holomorphic Cliffordian polynomial on x of degree $2m$.

As we see, we have to apply Δ_x^m on the polynomial of quasi-periodicity. For this purpose, let us mention the following:

Lemma 1.

(*a*) $$\Delta_x\ (x\mid\nabla_y)^2\ \ \varphi(y)=2\Delta_y\ \varphi(y)$$

(*b*) $$\Delta_x^m(x\mid\nabla_y)^{2m}\ \varphi(y)=(2m)!\Delta_y^m\ \varphi(y)$$

for any function $\varphi\in\mathcal{C}^{2m}(\mathbb{R}^{2m+2}), m\in\mathbb{N}$.

The proof of (*a*) is carried by a direct computation, those of (*b*): by a recurrence argument on $m\in\mathbb{N}$.

Applying the lemma, we have:

$$\Delta_x^m\left(2\zeta(\omega_{2k+1})+2\sum_{p=1}^{m}\frac{(x\mid\nabla_y)^{2p}}{(2p)!}\ \zeta(y)\left.\right|_{y=\omega_{2k+1}}\right)=2(\Delta^m\zeta)(\omega_{2k+1}),$$

so that:

$$\int_{F^+_{+\omega_{2k+1}}}\gamma(x)\Delta^m\zeta(x)+\int_{F^-_{-\omega_{2k+1}}}\gamma(x)\Delta^m\zeta(x)=2(\Delta^m\zeta)(\omega_{2k+1})\int_{F^+_{2k+1}}\gamma(x).$$

With the same procedure, one get:

$$\int_{F^-_{+\omega_{2k+2}}} \gamma(x)\Delta^m\zeta(x) + \int_{F^+_{-\omega_{2k+2}}} \gamma(x)\Delta^m\zeta(x) = -2(\Delta^m\zeta)(\omega_{2k+2}) \int_{F^+_{2k+2}} \gamma(x).$$

So, we can write down a first variant of the Legendre formula:

$$(-1)^m\ 2^{2m}(m!)\pi^{m+1} = \sum_{j=1}^{2m+2} (-1)^{j+1}(\Delta^m\zeta)(\omega_j) \int_{F_j^+} \gamma(x).$$

In order to obtain a more achieved form of the Legendre formula, we have to compute the integrals. This can be done by different manners. First of all, remark that each face F_j^+ can be decomposed in 2^{2m+1} elementary cells and, obviously:

$$\int_{F_j^+} \gamma(x) = 2^{2m+1} \int_{C_j} \gamma(x),$$

where C_j is the hyperparallelogram in $\mathbb{R}^{2m+1}$ spanned by $\omega_1, \omega_2, \ldots, \widehat{\omega}_j, \ldots,$ ω_{2m+2} – the sign $\wedge$ means an omission.

Let us compute $\int_{C_j} \gamma(x)$. The unit normal vector pointing outward C_j is the paravector:

$$n_j = \frac{(-1)^{m+1}\ i\ (\omega_1 \wedge \ldots \wedge \widehat{\omega}_j \wedge \ldots \wedge \omega_{2m+2})}{\|\omega_1 \wedge \ldots \wedge \widehat{\omega}_j \wedge \ldots \wedge \omega_{2m+2}\|},$$

where $i = e_1 e_2 \ldots e_{2m+1}$ is the pseudoscalar of $\mathbb{R}_{0,2m+1}$. The expression on the numerator can be viewed as a generalization of the usual vector product, while the real positive number on the denominator is nothing else than the volume of C_j. So, we can apply the method described at the end of section 1, and thus:

$$\begin{aligned} \int_{C_j} \gamma(x) &= n_j \int_{C_j} ds = n_j \times \text{volume of } C_j \\ &= (-1)^{m+1} i\ (\omega_1 \wedge \ldots \wedge \widehat{\omega}_j \wedge \ldots \wedge \omega_{2m+2}). \end{aligned}$$

Finally, we obtained the following version of the Legendre formula in $\mathbb{R}_{0,2m+1}$:

$$(m!)\ \frac{\pi}{2}^{m+1} = \sum_{j=1}^{2m+2} (-1)^j (\Delta^m \zeta)(\omega_j)\ i(\ \omega_1 \wedge \ldots \wedge \widehat{\omega}_j \wedge \ldots \wedge \omega_{2m+2}).$$

Let us mention we could compute the integrals $\int\limits_{C_j} \gamma(x)$ following another method. For this, it suffices to parametrize C_j:

Lemma 2. *Consider in* $\mathbb{R}^{n+1} = \{x = (x_0, x_1, \ldots, x_n)\}$, *with the usual basis* $\{e_0, e_1, \ldots, e_n\}$, *the hyperparallelogram* C *spanned by* $\omega_1, \ldots, \omega_n$, *where* $\omega_j \in \mathbb{R}^{n+1}$, *i.e.* $\omega_j = \sum\limits_{k=0}^{n} \langle \omega_j \rangle_k\ e_k$, $j = 1, \ldots, n$. *Then:*

$$\int\limits_C \gamma(x) = \det \begin{pmatrix} e_0 & e_1 & \ldots & e_n \\ \langle \omega_1 \rangle_0 & \langle \omega_1 \rangle_1 & \ldots & \langle \omega_1 \rangle_n \\ \ldots & \ldots & \ldots & \ldots \\ \langle \omega_n \rangle_0 & \langle \omega_n \rangle_1 & \ldots & \langle \omega_n \rangle_n \end{pmatrix}.$$

Proof. C can be parametrized:

$$[0,1]^n \xrightarrow{\ \phi\ } \mathbb{R}^{n+1}$$

$$(t_1, t_2, \ldots, t_n) \longmapsto x = \phi(t_1, t_2, \ldots, t_n),$$

with $\phi(t_1, t_2, \ldots, t_n) = \sum\limits_{j=1}^{n} t_j \omega_j$, which means that

$$x_k = x_k(t_1, t_2, \ldots, t_n) = \sum_{j=1}^{n}\ t_j\ \langle \omega_j \rangle_k,$$

for $k = 0, \ldots, n$.

Remember that

$$\gamma(x) = \sum_{i=0}^{n} (-1)^i e_i\ dx_0 \wedge \ldots \wedge\ d\widehat{x}_i \wedge \ldots \wedge\ dx_n.$$

Thus:

$$e_i\ dx_0 \wedge \ldots \wedge\ d\widehat{x}_i \wedge \ldots \wedge\ dx_n = e_i\ \frac{D(x_0,\ldots,\widehat{x}_i,\ldots,x_n)}{D(t_1,t_2,\ldots,t_n)}\ dt_1 \wedge \ldots \wedge dt_n$$

$$= e_i\ \det\left(\frac{\partial x_k}{\partial t_\ell}\right)_{\substack{k=0,\ldots,n,\ k\neq i\\ \ell=1,\ldots,n}} dt_1 \wedge \ldots \wedge dt_n$$

$$= e_i\ \det\left(\langle\omega_j\rangle_k\right)_{\substack{k=0,\ldots,n,\ k\neq i\\ j=1,\ldots,n}} dt_1 \wedge \ldots \wedge dt_n.$$

Now:

$$\begin{aligned}\int\limits_C \gamma(x) &= \sum_{i=0}^{n}(-1)^i\ e_i \det\left(\langle\omega_j\rangle_k\right) \int\limits_0^1 \ldots \int\limits_0^1 dt_1 \wedge \ldots \wedge dt_n \\ &= \det\begin{pmatrix} e_0 & e_1 & \ldots & e_n \\ \langle\omega_1\rangle_0 & \langle\omega_1\rangle_1 & \ldots & \langle\omega_1\rangle_n \\ \ldots & \ldots & \ldots & \ldots \\ \langle\omega_n\rangle_0 & \langle\omega_n\rangle_1 & \ldots & \langle\omega_n\rangle_n \end{pmatrix}.\end{aligned}$$

Remark that when $n = 2$, i.e. in $\mathbb{R}^3$, this determinant is nothing else that the vector product of ω_1 and ω_2.

Apply the lemma 2 to all the C_j, $j = 1,\ldots,2m+2$ in our case, we get

$$\int\limits_{F_j^+} \gamma(x) = 2^{2m+1}\ \det\ E_j,$$

where we have noted

$$E_j = \begin{pmatrix} e_0 & e_1 & \ldots & e_{2m+1} \\ \langle\omega_1\rangle_0 & \langle\omega_1\rangle_1 & \ldots & \langle\omega_1\rangle_{2m+1} \\ \ldots & \ldots & \ldots & \ldots \\ \langle\omega_{j-1}\rangle_0 & \ldots & \ldots & \langle\omega_{j-1}\rangle_{2m+1} \\ \langle\omega_{j+1}\rangle_0 & \ldots & \ldots & \langle\omega_{j+1}\rangle_{2m+1} \\ \ldots & \ldots & \ldots & \ldots \\ \langle\omega_{2m+2}\rangle_0 & \ldots & \ldots & \langle\omega_{2m+2}\rangle_{2m+1} \end{pmatrix}$$

and thus:

$$(-1)^m \ (m!) \ \frac{\pi}{2}^{m+1} = \sum_{j=1}^{2m+2} (-1)^{j+1} (\Delta^m \zeta)(\omega_j) \ \det E_j,$$

which gives, for $m = 0$:

$$\frac{\pi}{2} = \zeta(\omega_1)(Im \ \omega_2 - i \ Re \ \omega_2) - \zeta(\omega_2)(Im \ \omega_1 - i \ Re \ \omega_1),$$

equivalent to $i \ \frac{\pi}{2} = \zeta(\omega_1)\omega_2 - \zeta(\omega_2)\omega_1$.

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