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Solutions of Fractional Diffusion-Wave Equations in Terms of H-functions

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The method of integral transforms based on joint application of a fractional generalization of the Fourier transform and the classical Laplace transform is utilized for solving Cauchy-type problems for the time-space fractional diffusion-wave equations expressed in terms of the Caputo time-fractional derivative and the Weyl space-fractional operator. The solutions obtained are in integral form whose kernels are Green functions expressed in terms of the Fox H-functions. The results derived are of general nature and include already known results as particular cases.

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1. Introduction

The modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [10], [23]. An illustration of this are the generalization of the fractional partial difference equation suggested as a replacement of Fick's law [22], the fractional-order diffusion equation studied by Metzler, Glöckle and Nonnenmacher [16], and the fractional diffusion equation in the form

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = a^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < \beta < \frac{1}{2}, \tag{1}$$

introduced by Nigmatullin [20], [21]. The equation (1) is also known as the fractional diffusion-wave equation [11], [12]. When the order of the fractional derivative is $2\beta = 1$, the equation becomes the classical diffusion equation, and if $2\beta = 2$ it becomes the classical wave equation. The case $0 < 2\beta < 1$ was employed for studying the so-called ultraslow diffusion, whereas the case $1 < 2\beta < 2$ corresponds to the intermediate processes [6].

A space-time fractional diffusion equation, obtained from the standard diffusion equation by replacing the second order space-derivative by a fractional Riesz derivative, and the first order time-derivative by a Caputo fractional derivative, has been treated by Saichev and Zaslavsky [26], Uchajkin and Zolotarev [34], Gorenflo, Iskenderov and Luchko [7], Scalas, Gorenflo and Mainardi [33], Metzler and Klafter [17]. The results obtained in [7], are complemented in [13] where the fundamental solution of the corresponding Cauchy problem is found by means of the Fourier-Laplace transform. Based on Mellin-Barnes integral representation, the fundamental solutions of the problem under question are also expressed in terms of proper Fox H-functions [14].

The Fourier-Laplace transform method was adopted also in a number of papers by Saxena et al. [30], [31], [32] and Haubold et al. [8]. The same approach was also implemented in [25], where solutions of generalized fractional partial differential equations involving the Caputo time-fractional derivative and the Weyl space-fractional derivative are obtained.

To avoid the utility of a convention to suppress the imaginary unit in the Fourier transform of the Weyl fractional operator as in [30]-[32] and [25], we employ in this paper a fractional generalization of the Fourier transform and Laplace transform for solving Cauchy-type problems for the time-space fractional diffusion-wave equation expressed in terms of the Caputo time-fractional derivative of order γ and the Weyl space-fractional operator. We also distinguish the cases of ultraslow diffusion ($0 < \gamma < 1$) and the intermediate processes ($1 < \gamma < 2$) to obtain the Green functions presented in the formal solutions, in terms of Fox H-functions. Some of the already known results are also included as particular cases.

2. Preliminaries

For a function u of the class S of a rapidly decreasing test functions on the real axis R the Fourier transform is defined as

$$\hat{u}(\omega) = F[u(x);\omega] = \int_{-\infty}^{\infty} e^{i\omega x} u(x)\, dx, \quad \omega \in \mathbf{R}, \tag{2}$$

whereas the inverse Fourier transform has the form

$$u(x) = F^{-1}[\hat{u}(\omega);x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} \hat{u}(\omega)\, d\omega, \quad x \in \mathbf{R}. \tag{3}$$

Denote by $V(\mathbf{R})$ the set of functions $\nu(x) \in S$ satisfying

$$\left.\frac{d^n \nu}{dx^n}\right|_{x=0} = 0, \quad n = 0, 1, 2, \ldots$$

Then the Fourier pre-image of the space $V(\mathbf{R})$

$$\Phi(\mathbf{R}) = \{\varphi \in S : \hat{\varphi} \in V(\mathbf{R})\}$$

is called the Lizorkin space. As it is stated in [9], the space $\Phi(\mathbf{R})$ is invariant with respect to the fractional integration and differentiation operators.

In this paper we adopt the following fractional generalization of the Fourier transform called Fractional Fourier Transform (FRFT), as introduced in [9].

Definition 2.1. For a function $u \in \Phi(\mathbf{R})$ the FRFT of the order α $(0 < \alpha < 1)$ is defined as

$$\hat{u}_\alpha(\omega) = F_\alpha[u(x); \omega] = \int_{-\infty}^{\infty} e_\alpha(\omega, x) u(x)\, dx, \quad \omega \in \mathbf{R}, \tag{4}$$

where

$$e_\alpha(\omega, x) := \begin{cases} e^{-i|\omega|^{\frac{1}{\alpha}} x}, & \omega \le 0 \\ e^{i|\omega|^{\frac{1}{\alpha}} x}, & \omega > 0 \end{cases}. \tag{5}$$

Evidently, if $\alpha = 1$ the kernel (5) of the FRFT (4), reduces to the kernel of (2), that leads to the relation

$$\hat{u}_\alpha(\omega) = F_\alpha[u(x); \omega] = F_\alpha[u(x); k] = \hat{u}_\alpha(k), \tag{6}$$

where

$$k = \begin{cases} -|\omega|^{\frac{1}{\alpha}}, & \omega \le 0 \\ |\omega|^{\frac{1}{\alpha}}, & \omega > 0 \end{cases}. \tag{7}$$

Thus, if

$$F_\alpha[u(x); \omega] = F[u(x); k] = \hat{u}(k),$$

then

$$u(x) = F_\alpha^{-1}[\hat{u}_\alpha(\omega); x] = F_\alpha^{-1}[\hat{u}(k); x]. \tag{8}$$

The Caputo fractional derivative is defined as (see [2])

$$D_*^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau, & n-1 < \alpha < n \\ \dfrac{d^n f(t)}{dt^n}, & \alpha = n \end{cases}, \tag{9}$$

where $n > 0$ is integer.

The method we follow makes the rule of the Laplace transform

$$L[f(t);s] = \int_0^\infty e^{-st} f(t)\,dt \tag{10}$$

of the Caputo derivative of key importance (Podlubny [23]),

$$L[D_*^\alpha f(t);s] = s^\alpha L[f(t);s] - \sum_{k=0}^{n-1} f^{(k)}(t) s^{n-1-k}, \quad n-1 < \alpha \le n. \tag{11}$$

The Weyl fractional operator of order α is defined by ([27])

$$D_*^\alpha f(x) = \frac{d^n}{dx^n}\left[I_+^{n-\alpha} f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\,dt, \tag{12}$$

where $x \in \mathbf{R}$, $\alpha > 0$, $n = [\alpha] + 1$ and

$$I_+^n f(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1} f(t)\,dt$$

is the left-sided Riemann-Liouville integral operator.

The one-parameter generalization of the exponential function was introduced by Mittag-Leffler [18] as

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}\,.$$

Its further generalization was done by Agarwal [1] who defined the two-parameter function of the Mittag-Leffler type in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \;\; \alpha > 0, \;\; \beta > 0. \tag{13}$$

Let us notice also that the effect of the application of the Laplace transform (10) on the function (13) is given by the formulas [23, 1.2.2., (1.80)],

$$L\left[t^{\alpha m+\beta-1}\frac{d^m}{dt^m}E_{\alpha,\beta}(\pm\alpha t^\alpha);s\right] = \frac{m!\,s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}, \quad \operatorname{Re} s > |a|^{\frac{1}{\alpha}}. \tag{14}$$

By the Fox H-function we mean a generalized hypergeometric function, represented by the Mellin-Barnes type integral

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right.\right]$$

$$:= H^{m,n}_{p,q}\left[z\,\middle|\, \begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_L \theta(\xi)\, z^{-\xi} d\xi,$$

where

$$\theta(\xi) = \frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\right]\left[\prod_{i=1}^{n}\Gamma(1-a_i-A_i\xi)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j\xi)\right]\left[\prod_{i=n+1}^{p}\Gamma(a_i+A_i\xi)\right]},$$

and the contour L is defined as in [30]. In terms of the usual notations, $N_0 = (0,1,2,\ldots)$, $\mathbf{R} = (-\infty,\infty)$, $\mathbf{R}_+ = (0,\infty)$ and $\mathbf{C}$ being the complex numbers field, the orders $m,n,p,q \in N_0$ with $1 \le n \le p$, $1 \le m \le q$, $A_j, B_j \in \mathbf{R}_+$, $a_j, b_j \in \mathbf{R}_+$ or $\mathbf{C}$ ($i = 1,2,\ldots,p$; $j = 1,2,\ldots q$; such that

$$A_i(b_j+k) \neq B_j(a_i-l-1),\ \ k,l \in N_0,\ \ i = 1,2,\ldots,n, j = 1,2,\ldots m.$$

The empty product is always interpreted as unity.

It has been established in [28] that, if $\alpha \in \mathbf{C}$ for Re $\alpha > 0$

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z\,\middle|^{(0,1)}_{(0,1),(1-\beta,\alpha)}\right]. \tag{15}$$

If we set in (15) $\beta = 1$ we see that

$$E_{\alpha,1}(z) = E_\alpha(z) = H^{1,1}_{1,2}\left[-z\,\middle|^{(0,1)}_{(0,1),(0,\alpha)}\right]. \tag{16}$$

According to [24], [29], the cosine transform of the H-function is given by

$$\int_0^\infty t^{p-1}\cos(kt)\, H^{m,n}_{p,q}\left[at^\mu\,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] dt \tag{17}$$

$$= \frac{\pi}{k^\rho} H^{n+1,m}_{q+1,p+2}\left[\frac{k^\mu}{a}\,\middle|\, \begin{matrix}(1-b_q,B_q),(\frac{1+\rho}{2},\frac{\mu}{2})\\(\rho,\mu),(1-a_p,A_p),(\frac{1+\rho}{2},\frac{\mu}{2})\end{matrix}\right],$$

where

$$\mathbf{Re}\left[\rho + \mu\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right] > 1; \quad k^\mu > 0;$$

$$\mathbf{Re}\left[\rho + \mu\max_{1\le j\le n}\left(\frac{a_i-1}{A_i}\right)\right] < \frac{3}{2}; \quad |\arg\, a| < \frac{1}{2}\pi a;$$

$$\theta > 0 \ \text{ and } \ \theta = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j.$$

We also use the following property of the H-function ([15], [24])

$$H^{m,n}_{p,q}\left[x^\delta\,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = \frac{1}{\delta}H^{m,n}_{p,q}\left[x\,\middle|\, \begin{matrix}(a_p,A_p/\delta)\\(b_q,B_q/\delta)\end{matrix}\right], \quad \text{where } \ \delta > 0. \tag{18}$$

3. FRFT of Weyl operator

The application of the conventional Fourier transform (2) for solving fractional differential equations encounters in most of the cases inconveniences caused by multi-valued complex factors that the transform produces when applied on a fractional derivative [13], [17]. To prevail over complications of this type, it is reasonable to employ fractional Fourier transforms as (4) that act on a fractional derivative exactly the same way as the Fourier transform (2) does. To describe the effect of the application of the FRFT (4) on the Weyl operator (12) we use that if $x \in \mathbf{R}$, $\omega \in \mathbf{R}$, $\omega \neq 0$ and $0 < \sigma < 1$ (see [9]),

$$I_-^\sigma[e^{i\omega t}; x] = e^{i\omega x}|\omega|^{-\sigma}\left[\cos\frac{\sigma\pi}{2} + i\ \mathrm{sign}\ \omega\ \ \sin\frac{\sigma\pi}{2}\right], \tag{19}$$

where

$$I_-^\sigma f(x) = \frac{1}{\Gamma(\sigma)}\int_x^\infty (t-x)^{\sigma-1} f(t)\,dt \tag{20}$$

is the right-sided Riemann-Liouville fractional integral operator.

We also take the advantage of the rule for integration by parts ([27]), according to which for the functions u and v from the Lizorkin space $\Phi(\mathbf{R})$:

$$\int_{-\infty}^\infty v(x) D_+^\alpha[u;x]\,dx = \int_{-\infty}^\infty u(x) D_-^\alpha[v;x]\,dx, \tag{21}$$

where D_-^α is the Weyl right-sided differential operator defined for $x \in \mathbf{R}$, $\alpha > 0$ and $n = [\alpha]+1$, defined as

$$D_-^\alpha f(x) = (-1)^n \frac{d^n}{dx^n}[I_-^{n-\alpha}u(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^\infty \frac{f(t)dt}{(t-x)^{\alpha-n+1}}\,.$$

Lemma 3.1. *Let* $\omega \in \mathbf{R}$, $\omega \neq 0$ *and* $0 < \sigma < 1$. *Then*

$$D_-^{\sigma+1}[e^{i\omega t}; x] = -\omega^2|\omega|^{\sigma-1}\left[\sin\frac{\sigma\pi}{2} + i\ \mathrm{sign}\ \omega\ \ \cos\frac{\sigma\pi}{2}\right] e^{i\omega x}.$$

P r o o f. From (19) it follows that

$$D_-^{\sigma+1}[e^{i\omega t}; x] = \frac{d^2}{dx^2}\left\{[I_-^{1-\sigma}[e^{i\omega t}; x]\right\}$$

$$= \frac{d^2}{dx^2}\left\{e^{i\omega x}|\omega|^{\sigma-1}\left[\sin\frac{\sigma\pi}{2} + i\mathrm{sign}\omega\ \ \cos\frac{\sigma\pi}{2}\right]\right\}$$

$$= -\omega^2|\omega|^{\sigma-1}\left[\sin\frac{\sigma\pi}{2} + i \text{ sign } \omega \ \cos\frac{\sigma\pi}{2}\right] e^{i\omega x}.$$

■

Theorem 3.1. *If* $0 < \alpha \geq 1,\ 0 \leq \sigma < 1$ *and* $u \in \Phi(\mathbf{R})$*, then*

$$F_\alpha[D_+^{\sigma+1}u(x);\omega] = c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}} F_\alpha[u(x);\omega],$$

where

$$c(\sigma) = -\sin\frac{\sigma\pi}{2} - i \text{ sign } \omega \ \cos\frac{\sigma\pi}{2}.$$

Proof. If $\alpha = 1$ and $\sigma = 0$ according to (6),

$$F[D_+^1 u(x);\omega] = F[u'(x);\omega] = -i\omega F[u(x);\omega]$$

and thus the statement of the theorem reduces to the classical result for the conventional Fourier transform (2).

Consider now the case $0 < \alpha < 1$, $0 < \sigma < 1$ and $\omega = 0$. Since $\Phi(\mathbf{R})$ is closed with respect to fractional differentiation it becomes clear from (4) that

$$F_\alpha[D_+^{\sigma+1}u(x);0] = \int_{-\infty}^{\infty} D_+^{\sigma+1}u(x)\,dx = D_+^{\sigma}u(x)|_{-\infty}^{\infty} = 0.$$

Let $0 < \alpha < 1$, $0 < \sigma < 1$ and $\omega > 0$. Then (4), (5), (21) and Lemma 3.1 yield

$$F_\alpha[D_+^{\sigma+1}u(x);\omega] = \int_{-\infty}^{\infty} e^{i|\omega|^{\frac{1}{\alpha}}x}\{D_+^{\sigma+1}[u;x]\}\,dx = \int_{-\infty}^{\infty} u(x)\{D_-^{\sigma+1}[e^{i|\omega|^{\frac{1}{\alpha}}t};x]\}\,dx$$

$$= \int_{-\infty}^{\infty} u(x)\left\{-|\omega|^{\frac{2}{\alpha}}|\omega|^{\frac{\sigma-1}{\alpha}}\left[\sin\frac{\sigma\pi}{2} + i\cos\frac{\sigma\pi}{2}\right]e^{i|\omega|^{\frac{1}{\alpha}}x}\right\}dx$$

$$= -|\omega|^{\frac{\sigma+1}{\alpha}}\left(\sin\frac{\sigma\pi}{2} + i\cos\frac{\sigma\pi}{2}\right)\int_{-\infty}^{\infty} e^{i|\omega|^{\frac{1}{\alpha}}x}u(x)\,dx$$

$$= c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}F_\alpha[u(x);\omega].$$

Likewise we consider the remaining case $0 < \alpha < 1$, $0 < \sigma < 1$, $\omega < 0$. Using again (21) and Lemma 3.1, we get

$$F_\alpha[D_+^{\sigma+1}u(x);\omega] = \int_{-\infty}^{\infty} e^{-i|\omega|^{\frac{1}{\alpha}}x}\{D_+^{\sigma+1}[u;x]\}\,dx = \int_{-\infty}^{\infty} u(x)\{D_-^{\sigma+1}[e^{-i|\omega|^{\frac{1}{\alpha}}t};x]\}dx$$

$$= \int_{-\infty}^{\infty} u(x)\left\{-|\omega|^{\frac{2}{\alpha}}|\omega|^{\frac{\sigma-1}{\alpha}}\left[\sin\frac{\sigma\pi}{2} - i\cos\frac{\sigma\pi}{2}\right]e^{-i|\omega|^{\frac{1}{\alpha}}x}\right\}dx$$

$$= -|\omega|^{\frac{\sigma+1}{\alpha}} \left(\sin\frac{\sigma\pi}{2} - i\cos\frac{\sigma\pi}{2} \right) \int_{-\infty}^{\infty} u(x) e^{-i|\omega|^{\frac{1}{\alpha}} x}\, dx$$

$$= c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}} F_\alpha[u(x);\omega],$$

that accomplishes the proof. ■

4. Fractional diffusion equation

In this section we apply the FRFT (4) for solving the Cauchy-type problem for the fractional diffusion equation

$$D_*^\gamma u(x,t) - \mu^2 D_+^{\sigma+1} u(x,t) = q(x,t),\ x \in \mathbf{R},\ t > 0, \tag{22}$$

subject to the initial condition

$$u(x,t)|_{t=0} = f(x), \tag{23}$$

when $0 \le \gamma \le 1$, $f(x) \in \Phi(\mathbf{R})$ and μ is a diffusivity constant.

Theorem 4.1. *If $0 < \gamma \le 1$ and $0 < \sigma \le 1$ the Cauchy-type problem (22)-(23) is solvable and the solution $u(x,t)$ is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t) f(\xi) d\xi + \int_0^t (t-\tau)^{\gamma-1} \left\{ \int_{-\infty}^{\infty} G_2(x-\xi,t-\tau) q(\xi,\tau) d\xi \right\} d\tau,$$

where

$$G_1(x,t) = \frac{1}{(\sigma+1)x} H_{3,3}^{2,1} \left[\frac{|x|}{(-\mu^2 c(\sigma) t^\gamma)^{\frac{1}{\sigma+1}}} \middle| \begin{matrix} (1,\frac{1}{\sigma+1}), (1,\frac{\gamma}{\sigma+1}), (1,\frac{1}{2}) \\ (1,1), (1,\frac{1}{\sigma+1}), (1,\frac{1}{2}) \end{matrix} \right],$$

$$G_2(x,t) = \frac{1}{(\sigma+1)x} H_{3,3}^{2,1} \left[\frac{|x|}{(-\mu^2 c(\sigma) t^\gamma)^{\frac{1}{\sigma+1}}} \middle| \begin{matrix} (1,\frac{1}{\sigma+1}), (\gamma,\frac{\gamma}{\sigma+1}), (1,\frac{1}{2}) \\ (1,1), (1,\frac{1}{\sigma+1}), (1,\frac{1}{2}) \end{matrix} \right].$$

Proof. Denote $L[u(x,t);s] = \bar{u}(x,s)$ and $F_\alpha[u(x,t);\omega] = \hat{u}_\alpha(\omega,t)$. First consider the case $0 < \gamma \le 1$ and $0 < \sigma < 1$. According to (11) and Theorem 3.1, the application of the Laplace transform (10) followed by the FRFT (4) to the equation (22) and the initial condition (23) leads to the following representation of the Laplace-FRFT transform of the solution

$$\hat{\bar{u}}_\alpha(\omega,s) = \frac{s^{\gamma-1}}{s^\gamma - \mu^2 c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}} \hat{f}_\alpha(\omega) + \frac{\hat{\bar{q}}_\alpha(\omega,s)}{s^\gamma - \mu^2 c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}}\,. \tag{24}$$

Using now (6), (8) and (14), the equation (24) converts into

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k)\, E_{\gamma,1}[\mu^2 c(\sigma)|k|^{\sigma+1} t^\gamma]\, dk$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left\{\int_0^t (t-\tau)^{\gamma-1}E_{\gamma,\gamma}[\mu^2 c(\sigma)|k|^{\sigma+1}(t-\tau)^{\gamma}]\hat{q}(k,\tau)\,d\tau\right\}\,dk.$$

By means of the convolution theorem for the Fourier transform (2), the above representation leads to

$$u(x,t)=\int_{-\infty}^{\infty} G_1(x-\xi,t)f(\xi)d\xi+\int_0^t (t-\tau)^{\gamma-1}\left\{\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)q(\xi,\tau)d\xi\right\}d\tau,$$

where

$$G_1(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\gamma,1}[\mu^2 c(\sigma)|k|^{\sigma+1}t^{\gamma}]\,dk \tag{25}$$

and

$$G_2(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\gamma,\gamma}[\mu^2 c(\sigma)|k|^{\sigma+1}t^{\gamma}]\,dk. \tag{26}$$

The formulas (16) and (18) with allow from (25) to obtain

$$G_1(x,t)=\frac{2}{(\sigma+1)\pi}\int_0^{\infty}\cos kx\, H_{1,2}^{1,1}\left[(-\mu^2 c(\sigma)t^{\gamma})^{\frac{2}{\sigma+1}}\,\middle|\,\begin{matrix}(0,\frac{1}{\sigma+1})\\(0,\frac{2}{\sigma+1}),(0,\frac{2\gamma}{\sigma+1})\end{matrix}\right]dk.$$

Taking finally into account (17) and (18) again, we get

$$G_1(x,t)=\frac{1}{(\sigma+1)x}H_{3,3}^{2,1}\left[\frac{|x|}{(-\mu^2 c(\sigma)t^{\gamma})^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(1,\frac{\gamma}{\sigma+1}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac{1}{2})\end{matrix}\right].$$

Similarly by (15), (17) and (18), we obtain from (26),

$$G_2(x,t)=\frac{1}{(\sigma+1)x}H_{3,3}^{2,1}\left[\frac{|x|}{(-\mu^2 c(\sigma)t^{\gamma})^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(\gamma,\frac{\gamma}{\sigma+1}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac{1}{2})\end{matrix}\right].$$

We accomplish the proof of the statement with the remark that its validity in the case $0<\gamma\le 1$ and $\sigma=1$ was confirmed by the results obtained in [3] and [30]. ■

Corollary 4.1. ([19], [30]) *If $0<\gamma\le 1$, $\sigma=1$, $f(x)\in\Phi(\mathbf{R})$ and $q(x,t)\equiv 0$ the solution of the Cauchy-type problem (22)-(23) is given by the integral*

$$u(x,t)=\int_{-\infty}^{\infty} G(x-\xi,t)f(\xi)\,d\xi,$$

where

$$G(x,t)=\frac{1}{2x}H_{3,3}^{2,1}\left[\frac{|x|}{(\mu^2 t^{\gamma})^{1/2}}\,\middle|\,\begin{matrix}(1,\frac{1}{2}),(1,\frac{\gamma}{2}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{2}),(1,\frac{1}{2})\end{matrix}\right].$$

By means of (15), (17) and the formula [5, p.611, (5)],

$$F^{-1}\left[\sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4a};x\right]=e^{-ax^2},$$

it might be seen that the solution provided by Theorem 4.1 occurs as a generalization of the fundamental solution of the classical diffusion problem.

Corollary 4.2. *If $\gamma=1$, $\sigma=1$, $f(x)\in\Phi(\mathbf{R})$ and $q(x,t)\equiv 0$ the solution of the Cauchy-type problem (22)-(23) is given by the integral*

$$u(x,t)=\frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/4\mu t}f(\xi)\,d\xi.$$

5. Fractional wave equation

We consider a Cauchy-type problem for the equation (22), but under the assumptions $1<\gamma\le 2$ and $0<\sigma\le 1$ subject to the initial conditions

$$u(x,t)|_{t=0}=f(x),\quad u_t(x,t)|_{t=0}=g(x),\quad x\in\mathbf{R}. \tag{27}$$

Theorem 5.1. *If $1<\gamma\le 2$, $0<\sigma\le 1$ $f(x)\in\Phi(\mathbf{R})$ and $g(x)\in\Phi(\mathbf{R})$, then the Cauchy-type problem (22)–(27) is solvable and its solution is given by*

$$u(x,t)=\int_{-\infty}^{\infty}G_1(x-\xi,t)f(\xi)\,d\xi+\int_{-\infty}^{\infty}G_2(x-\xi,t)g(\xi)\,d\xi$$
$$+\int_0^t(t-\tau)^{\gamma-1}\left\{\int_{-\infty}^{\infty}G_3(x-\xi,t-\tau)q(\xi,\tau)\,d\xi\right\}d\tau,$$

where

$$G_1(x,t)=\frac{1}{(\sigma+1)x}H_{3,3}^{2,1}\left[\frac{|x|}{(-\mu^2c(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(1,\frac{\gamma}{\sigma+1}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac{1}{2})\end{matrix}\right],$$

$$G_2(x,t)=\frac{1}{(\sigma+1)x}H_{3,3}^{2,1}\left[\frac{|x|}{(-\mu^2c(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(2,\frac{\gamma}{\sigma+1}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac{1}{2})\end{matrix}\right],$$

$$G_3(x,t)=\frac{1}{(\sigma+1)x}H_{3,3}^{2,1}\left[\frac{|x|}{(-\mu^2c(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(\gamma,\frac{\gamma}{\sigma+1}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac{1}{2})\end{matrix}\right].$$

P r o o f. As in Theorem 4.1, let us consider first the case $1 < \gamma \le 2$ and $0 < \sigma < 1$. Then the application of the Laplace transform (10) followed by the FRFT (4) to the equation (22) and the initial conditions (27) leads, because of (11) and Theorem 3.1, to the representation of the joint Laplace-FRFT transform of the solution

$$\begin{aligned}\hat{\tilde{u}}_\alpha(\omega,s) &= \frac{s^{\gamma-1}}{s^\gamma-\mu^2c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}}\hat{f}_\alpha(\omega)+\frac{s^{\gamma-2}}{s^\gamma-\mu^2c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}}\hat{g}_\alpha(\omega)\\ &+\frac{\hat{\tilde{q}}_\alpha(\omega,s)}{s^\gamma-\mu^2c(\sigma)|\omega|^{\frac{\sigma+1}{\alpha}}}.\end{aligned} \tag{28}$$

By the formulas (6), (8), (14) and the convolution theorem for the Fourier transform (2), the equation (28) becomes

$$u(x,t)=\int_{-\infty}^{\infty}G_1(x-\xi,t)f(\xi)\,d\xi+\int_{-\infty}^{\infty}G_2(x-\xi,t)g(\xi)\,d\xi$$

$$+\int_0^t(t-\tau)^{\gamma-1}\left\{\int_{-\infty}^{\infty}G_3(x-\xi,t-\tau)q(\xi,\tau)\,d\xi\right\}d\tau,$$

where

$$G_1(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}E_{\gamma,1}[\mu^2c(\sigma)|k|^{\sigma+1}t^\gamma]\,dk,$$

$$G_2(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}tE_{\gamma,2}[\mu^2c(\sigma)|k|^{\sigma+1}t^\gamma]\,dk,$$

$$G_3(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}E_{\gamma,\gamma}[\mu^2c(\sigma)|k|^{\sigma+1}t^\gamma]\,dk.$$

In proving Theorem 4.1, we have already deduced that

$$G_1(x,t)=G_1(x,t)=\frac{1}{(\sigma+1)x}H^{2,1}_{3,3}\left[\frac{|x|}{(-\mu^2c(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(1,\frac{\gamma}{\sigma+1}),(1,\frac12)\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac12)\end{matrix}\right],$$

and

$$G_3(x,t)=G_1(x,t)=\frac{1}{(\sigma+1)x}H^{2,1}_{3,3}\left[\frac{|x|}{(-\mu^2c(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}}\,\middle|\,\begin{matrix}(1,\frac{1}{\sigma+1}),(\gamma,\frac{\gamma}{\sigma+1}),(1,\frac12)\\(1,1),(1,\frac{1}{\sigma+1}),(1,\frac12)\end{matrix}\right],$$

It remains simply to apply (15) and (18) with $\delta=\frac{2}{\sigma+1}$ in order to obtain

$$G_2(x,t)=\frac{2t}{(\sigma+1)\pi}\int\cos kx\,H^{1,1}_{1,2}\left[(-\mu^2c(\sigma)t^\gamma)^{\frac{2}{\sigma+1}}k^2\,\middle|\,\begin{matrix}(0,\frac{2}{\sigma+1})\\(0,\frac{2}{\sigma+1}),(-1,\frac{2\gamma}{\sigma+1})\end{matrix}\right]dk.$$

Then from (17) and (18) with $\delta = \frac{1}{2}$ it follows immediately that

$$G_2(x,t) = \frac{1}{(\sigma+1)x} H^{2,1}_{3,3} \left[\frac{|x|}{(-\mu^2 c(\sigma) t^\gamma)^{\frac{1}{\sigma+1}}} \,\middle|\, \begin{matrix} (1, \frac{1}{\sigma+1}), (2, \frac{\gamma}{\sigma+1}), (1, \frac{1}{2}) \\ (1,1), (1, \frac{1}{\sigma+1}), (1, \frac{1}{2}) \end{matrix} \right].$$

The validity of the theorem for the case as $1 < \gamma \leq 2$ and $\sigma = 1$ is confirmed by the results obtained in [5, 6.7, (b)]. ■

Corollary 5.1. ([4]) *If $\gamma = 2$, $\sigma = 1$, $f(x) \in \Phi(\mathbf{R})$ and $g(x) \in \Phi(\mathbf{R})$, the Cauchy-type problem (22)–(27) has a solution of the form*

$$u(x,t) = \frac{1}{2}[f(x-\mu t) + f(x+\mu t)] + \frac{1}{2\mu}\int_{x-\mu t}^{x+\mu t} g(\eta)\, d\eta$$

$$+ \frac{1}{2\mu}\int_0^t \left[\int_{x-\mu(t-\tau)}^{x+\mu(t-\tau)} q(\eta,\tau)\, d\eta \right] d\tau.$$

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