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Hyperbolic Fourth-R Quadratic Equation and Holomorphic Fourth-R Polynomials

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The algebra $\mathbf{R}(1, j, j^2, j^3)$, $j^4 = -1$ of the fourth-**R** numbers, or in other words the algebra of the double-complex numbers $\mathbf{C}(1, j)$ and the corresponding functions, were studied in the papers of S. Dimiev and al. (see [1], [2], [3], [4]). The hyperbolic fourth-**R** numbers form other similar to $\mathbf{C}(1, j)$ algebra with zero divisors. In this note the square roots of hyperbolic fourth-**R** numbers and hyperbolic complex numbers are found. The quadratic equation with hyperbolic fourth-**R** coefficients and variables is solved. The Cauchy-Riemann system for holomorphicity of fourth-**R** functions is recalled. Holomorphic homogeneous polynomials of fourth-**R** variables are listed.

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1. Fourth-R numbers and hyperbolic fourth-R numbers

The algebra of fourth-**R** numbers is defined as follows

$$\mathbf{R}(1, j, j^2, j^3) = \{x = x_0 + jx_1 + j^2x_2 + j^3x_3,\ j^4 = -1\},$$

where x_0, x_1, x_2, x_3 are real numbers and j is a formal symbol with the property $j^4 = -1,\ j^0 = 1,\ j \notin \mathbf{C}$. The addition and multiplication with scalar are componentwise, and multiplication of fourth-**R** numbers is made by the rule of opening of brackets, using the identities for the symbol j, namely

$$xy = x_0y_0 - x_1y_3 - x_2y_2 - x_3y_1 + (x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2)j$$

$$+ (x_0y_2 + x_1y_1 + x_2y_0 - x_3y_3)j^2 + (x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0)j^3.$$

This algebra is associative and commutative one, but it has zero divisors. Such are for example the numbers $x(1 + j^2 + \sqrt{2}j)$ and $x(1 + j^2 - \sqrt{2}j)$ for each fourth-$\mathbf{R}$ number x. The algebra $\mathbf{R}(1,j,j^2,j^3)$ is isomorphic to the algebra of double-complex numbers $\mathbf{C}(1,j)$, which consists of couples of complex numbers $(z,w)=z+jw$ with specific multiplication, ruled by the same formal symbol j. More precisely,

$$\mathbf{C}(1,j)=\{\alpha=z+jw,\ j^4=-1,\ z,w\in\mathbf{C}\},$$

i.e. z,w are complex numbers and j is a formal symbol with the property $j^4=-1$, $j^0=1$, $j\notin\mathbf{C}$.

The algebra of the hyperbolic fourth-$\mathbf{R}$ numbers is the following one

$$\mathbf{R}(1,j,j^2,j^3)=\{x=x_0+jx_1+j^2x_2+j^3x_3,\ j^4=+1\},$$

where x_0,x_1,x_2,x_3 are real numbers and j is a formal symbol with the property $j^4=+1$, $j^0=1$, $j\notin\mathbf{C}$. It is endowed with a structure of a commutative and associative algebra with respect to the (open brackets) multiplication using the identities $j^{k+4}=j^k$, $k=0,1,\dots$ and $j^0=1$, $j^4=+1$ for the symbol j:

$$xy=x_0y_0+x_1y_3+x_2y_2+x_3y_1+(x_0y_1+x_1y_0+x_2y_3+x_3y_2)j$$

$$+(x_0y_2+x_1y_1+x_2y_0+x_3y_3)j^2+(x_0y_3+x_1y_2+x_2y_1+x_3y_0)j^3,$$

where $x=x_0+jx_1+j^2x_2+j^3x_3$ and $y=y_0+jy_1+j^2y_2+j^3y_3$, $x_k,y_k\in\mathbf{R}$ for $k=0,1,2,3$. Here the addition and the multiplication with scalar from $\mathbf{R}$ are componentwise. This algebra has divisors of zero, too.

The idea of fourth-$\mathbf{R}$ numbers, in more general form, appears for the first time in the framework of Kazan geometric school [5] with formal level for j. A matrix representation of the formal symbols j, j^2, j^3, j^4 was developed in [3], [4] and in definite form, separating the hyperbolic case $j^4=+1$ from the elliptic one $j^4=-1$, in [1].

2. Hyperbolic fourth-R square roots and hyperbolic fourth-R quadratic equation

First we shall consider the hyperbolic fourth-$\mathbf{R}$ square root

$$\sqrt{m_0+jm_1+j^2m_2+j^3m_3}, \tag{1}$$

of the hyperbolic fourth-$\mathbf{R}$ number $m=m_0+jm_1+j^2m_2+j^3m_3\in\mathbf{R}(1,j,j^2,j^3)$, $j^4=+1$, $m_k\in\mathbf{R}$ for $k=0,1,2,3$. For this purpose we shall consider the quadratic equation

$$a^2=m_0+jm_1+j^2m_2+j^3m_3, \tag{2}$$

where $a=a_0+ja_1+j^2a_2+j^3a_3\in\mathbf{R}(1,j,j^2,j^3)$ $j^4=+1$, $a_k\in\mathbf{R}$ for $k=0,1,2,3$ is a hyperbolic fourth-$\mathbf{R}$ number. The equation (2) is equivalent to the following system of four second degree equations with real variables:

$$\begin{aligned} &\text{I) } a_0^2 + a_2^2 + 2a_1a_3 = m_0, \qquad &&\text{II) } 2a_0a_1 + 2a_2a_3 = m_1, \\ &\text{III) } 2a_0a_2 + a_3^2 + a_1^2 = m_2, \qquad &&\text{IV) } 2a_0a_3 + 2a_1a_2 = m_3, \end{aligned} \tag{3}$$

which arises from the equalities $a^2 = (a_0 + ja_1 + j^2a_2 + j^3a_3)^2 = a_0^2 + a_2^2 + 2a_1a_3 + j(2a_0a_1 + 2a_2a_3) + j^2(2a_0a_2 + a_3^2 + a_1^2) + j^3(2a_0a_3 + 2a_1a_2) = m_0 + jm_1 + j^2m_2 + j^3m_3$.

The sums and the differences of the equations I) and III), and II) and IV) in (3), respectively, gives the following system of four real quadratic equations

$$\begin{aligned} \text{i) } &(a_0 + a_2)^2 + (a_1 + a_3)^2 = m_0 + m_2, \\ \text{ii) } &2(a_0 + a_2)(a_1 + a_3) = m_1 + m_3. \\ \text{iii) } &(a_0 - a_2)^2 - (a_1 - a_3)^2 = m_0 - m_2, \\ \text{iv) } &2(a_0 - a_2)(a_1 - a_3) = m_1 - m_3. \end{aligned} \tag{4}$$

Then the sum of the equations i) and ii) and the difference between i) and ii) in (4) gives, respectively,

$$\begin{aligned} \text{a) } &(a_0 + a_2 + a_1 + a_3)^2 = m_0 + m_1 + m_2 + m_3 \\ \text{b) } &(a_0 + a_2 - a_1 - a_3)^2 = m_0 - m_1 + m_2 - m_3. \end{aligned} \tag{5}$$

The necessary condition for existing of the square root

$$m_0 + m_2 \geq |m_1 + m_3| \tag{6}$$

arises from (5).

Let us introduce the variables X and Y in the following way:

$$X := a_0 - a_2, \qquad Y := a_1 - a_3. \tag{7}$$

The equations iii) and iv) from (4) in these variables seems as follows:

$$X^2 - Y^2 = m_0 - m_2, \qquad 2XY = m_1 - m_3. \tag{8}$$

Solving this auxiliary system, we obtain the equations

$$Y = \frac{m_1 - m_3}{2X}, \quad X^2 - \frac{(m_1 - m_3)^2}{4X^2} = m_0 - m_2, \quad \text{for} \quad m_1 \neq m_3. \tag{9}$$

The equation of fourth degree $4X^4 - 4(m_0 - m_2)X^2 = (m_1 - m_3)^2$ can be written also as $(2X^2 - m_0 + m_2)^2 = (m_1 - m_3)^2 + (m_0 - m_2)^2$, from where we obtain

$$2X^2 = m_0 - m_2 \pm \sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2}. \tag{10}$$

As we look for the real solutions of the system (3) of 4 equations of second degree, the sign minus in the formula (10) does not give a solution. So we obtain the following two real solutions for X and two real solutions for Y, respectively,

$$X = \varepsilon \frac{1}{\sqrt{2}} \sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} + m_0 - m_2}, \qquad (11)$$
$$Y = \varepsilon \ \mathrm{sign}(m_1 - m_3) \frac{1}{\sqrt{2}} \sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} - (m_0 - m_2)},$$

where $\varepsilon = \pm 1$.

We consider again the system of equations (5). As the condition (6) holds, it is fulfilled $m_0 + m_1 + m_2 + m_3 \geq 0$ and $m_0 - m_1 + m_2 - m_3 \geq 0$ and

$$a_0 + a_2 + a_1 + a_3 = \varepsilon_1 \sqrt{m_0 + m_1 + m_2 + m_3} \qquad (12)$$
$$a_0 + a_2 - a_1 - a_3 = \varepsilon_2 \sqrt{m_0 - m_1 + m_2 - m_3},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$ and the roots in the right hand sides are the arithmetic roots of the corresponding real nonnegative numbers.

From the system (12), we obtain

$$a_0 + a_2 = \frac{\varepsilon_1}{2} \sqrt{m_0 + m_1 + m_2 + m_3} + \frac{\varepsilon_2}{2} \sqrt{m_0 - m_1 + m_2 - m_3}, \qquad (13)$$
$$a_1 + a_3 = \frac{\varepsilon_1}{2} \sqrt{m_0 + m_1 + m_2 + m_3} - \frac{\varepsilon_2}{2} \sqrt{m_0 - m_1 + m_2 - m_3},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$.

The following real numbers a_0, a_1, a_2 and a_3 are obtained by the expression for X and Y in (7) and the equalities (13)

$$a_0(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1}{4} \sqrt{m_0 + m_1 + m_2 + m_3} + \frac{\varepsilon_2}{4} \sqrt{m_0 - m_1 + m_2 - m_3}$$
$$+ \frac{\varepsilon}{2\sqrt{2}} \sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} + m_0 - m_2},$$
$$a_1(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1}{4} \sqrt{m_0 + m_1 + m_2 + m_3} - \frac{\varepsilon_2}{4} \sqrt{m_0 - m_1 + m_2 - m_3}$$
$$+ \frac{\varepsilon \ \mathrm{sign}(m_1 - m_3)}{2\sqrt{2}} \sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} - (m_0 - m_2)},$$
$$a_2(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1}{4} \sqrt{m_0 + m_1 + m_2 + m_3} + \frac{\varepsilon_2}{4} \sqrt{m_0 - m_1 + m_2 - m_3}$$
$$- \frac{\varepsilon}{2\sqrt{2}} \sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} + m_0 - m_2}$$

and

$$a_3(\varepsilon, \varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1}{4} \sqrt{m_0 + m_1 + m_2 + m_3} - \frac{\varepsilon_2}{4} \sqrt{m_0 - m_1 + m_2 - m_3}$$

$$-\frac{\varepsilon\ \mathrm{sign}(m_1-m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_0-m_2)^2+(m_1-m_3)^2}-(m_0-m_2)},$$

where $\varepsilon, \varepsilon_1$ and $\varepsilon_2 = \pm 1$ and $\mathrm{sign}(x)$ denotes the real-valued function of real variable: $\mathrm{sign}\ (x) = \begin{cases} 1 & \text{when} \quad x > 0, \\ 0 & \text{when} \quad x = 0, \\ -1 & \text{when} \quad x < 0. \end{cases}$

So we obtain the formula for the square root in the case $m_1 \neq m_3$ as follows:

$$a(\varepsilon_1, \varepsilon_2, \varepsilon) = \sqrt{m_0 + jm_1 + j^2 m_2 + j^3 m_3}$$
$$= \frac{\varepsilon_1(1+j+j^2+j^3)}{4}\sqrt{m_0+m_1+m_2+m_3}$$
$$+\frac{\varepsilon_2(1-j+j^2-j^3)}{4}\sqrt{m_0-m_1+m_2-m_3}$$
$$+\frac{\varepsilon(1-j^2)}{2\sqrt{2}}\sqrt{\sqrt{(m_0-m_2)^2+(m_1-m_3)^2}+m_0-m_2}$$
$$+\frac{\varepsilon j(1-j^2)\ \mathrm{sign}(m_1-m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_0-m_2)^2+(m_1-m_3)^2}-(m_0-m_2)}.$$

Remark 1. The case $m_1 = m_3 = 0$ is consider in part 3 below. In the case $m_1 = m_3 \neq 0$ we obtain the following square roots of the fourth-**R** number $m_0 + j^2 m_2 + j(1+j^2)m_1$:

$$a(\varepsilon_1, \varepsilon_2, \varepsilon) = \sqrt{m_0 + j^2 m_2 + j(1+j^2)m_1}$$
$$= \frac{\varepsilon j(1-j^2)}{2}\sqrt{m_2-m_0} + \frac{\varepsilon_1(1+j^2)}{2\sqrt{2}}\sqrt{m_0+m_2+\varepsilon_2\sqrt{(m_0+m_2)^2-4m_1^2}}$$
$$+\frac{\varepsilon_1 j(1-j^2)\ \ \mathrm{sign}\ \ m_1}{2\sqrt{2}}\sqrt{m_0+m_2-\varepsilon_2\sqrt{(m_0+m_2)^2-4m_1^2}}$$

in the case $m_2 > |m_0|$, $m_0 + m_2 > 2|m_1|$ and

$$a(\varepsilon_1, \varepsilon_2, \varepsilon) = \sqrt{m_0 + j^2 m_2 + j(1+j^2)m_1}$$
$$= \frac{\varepsilon(1+j^2)}{2}\sqrt{m_0-m_2} + \frac{\varepsilon_1 j(1+j^2)}{2\sqrt{2}}\sqrt{m_0+m_2+\varepsilon_2\sqrt{(m_0+m_2)^2-4m_1^2}}$$
$$+\frac{\varepsilon_1(1-j^2)\ \ \mathrm{sign}\ \ m_1}{2\sqrt{2}}\sqrt{m_0+m_2-\varepsilon_2\sqrt{(m_0+m_2)^2-4m_1^2}}$$

in the case $m_0 > |m_2|$, $m_0 + m_2 > 2|m_1|$, where the numbers $\varepsilon_1, \varepsilon_2$ and ε are equal to ± 1.

In the case $m_0 = m_1 = m_2 = m_3 > 0$, the square root $\sqrt{m_0(1+j+j^2+j^3)}$ is equal to $\varepsilon\dfrac{m_0}{2}(1+j+j^2+j^3)$.

In other cases square root of the corresponding fourth-**R** number does not exist.

Theorem 1. *The quadratic equation*

$$x^2 + px + q = 0$$

with hyperbolic fourth-$\mathbf{R}$ *coefficients* $p,\ q \in \mathbf{R}(1,j,j^2,j^3),\ j^4 = +1,\ p = p_0 + jp_1 + j^2p_2 + j^3p_3,\ \ q = q_0 + jq_1 + j^2q_2 + j^3q_3,\ \ p_k, q_k \in \mathbf{R}$ *for* $k = 0,1,2,3$, *which satisfies the conditions*

$$(p_0+p_1+p_2+p_3)^2 \geq 4\ (q_0+q_1+q_2+q_3),\ \ (p_0-p_1+p_2-p_3)^2 \geq\ 4\ (q_0-q_1+q_2-q_3)$$

has the following solutions

$$x_+(\varepsilon_1,\varepsilon_2,\varepsilon) = -\frac{p}{2} + a(\varepsilon_1,\varepsilon_2,\varepsilon) \quad \textit{and} \quad x_-(\varepsilon_1,\varepsilon_2,\varepsilon) = -\frac{p}{2} - a(\varepsilon_1,\varepsilon_2,\varepsilon),$$

where

$$a(\varepsilon_1,\varepsilon_2,\varepsilon) = \sqrt{\frac{p^2}{4} - q}$$

are the given above hyperbolic fourth-$\mathbf{R}$ *square roots of the discriminant* $\dfrac{p^2}{4} - q$.

3. Hyperbolic complex square root

Let us consider the hyperbolic complex numbers, forming two dimensional commutative, associative algebra with zero divisors

$$\tilde{\mathbf{C}} := \{x_0 + j^2x_2,\ j^4 = 1,\ x_0, x_2 \in \mathbf{R}\}.$$

The algebra $\tilde{\mathbf{C}}$ has as zero divisors the elements $x + j^2x$ and $x - j^2x$, where $x \in \mathbf{R}$. Let us consider the quadratic equation $(a_0 + j^2a_2)^2 = m_0 + j^2m_2$, i.e. this is the equation (2) with conditions $a_1 = a_3 = 0$ and $m_1 = m_3 = 0$. It is equivalent to the system of two equation of second order

$$a_0^2 + a_2^2 = m_0, \quad 2a_0a_2 = m_2. \tag{14}$$

Then, for the hyperbolic complex square roots are obtained the equations

$$(a_0 + a_2)^2 = m_0 + m_2, \qquad (a_0 - a_2)^2 = m_0 - m_2. \tag{15}$$

So $m_0 + m_2 \geq 0$ and $m_0 - m_2 \geq 0$, i.e. $m_0 \geq |m_2|$, is a necessary condition for the existing of the square root of the hyperbolic complex number $m_0 + j^2m_2$.

It is fulfilled

$$a_0 + a_2 = \varepsilon_1\sqrt{m_0 + m_2}, \qquad a_0 - a_2 = \varepsilon_2\sqrt{m_0 - m_2},$$

where ε_1 and $\varepsilon_2 = \pm 1$.

Then the asking square roots are the numbers

$$a(\varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1(1+j^2)}{2}\sqrt{m_0+m_2} + \frac{\varepsilon_2(1-j^2)}{2}\sqrt{m_0-m_2}.$$

So we obtain four different hyperbolic complex square roots of the hyperbolic complex number $m_0 + j^2 m_2$ such that $m_0 \geq |m_2|$.

4. Fourth-R holomorphy

The holomorphic fourth-**R**-functions

$$f : \mathbf{R}(1, j, j^2, j^3) \to \mathbf{R}(1, j, j^2, j^3),$$

$j^4 = -1$, are defined in terms of the classical conditions about the differential df. Namely, it is fulfilled

$$df = \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta + \frac{\partial f}{\partial \alpha^*} d\alpha + \frac{\partial f}{\partial \beta^*} d\beta^*$$

where

$$\frac{\partial f}{\partial \alpha} := \frac{1}{2}\left(\frac{\partial f}{\partial x_0} - j^2 \frac{\partial f}{\partial x_2}\right), \qquad \frac{\partial f}{\partial \beta} := \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - j^2 \frac{\partial f}{\partial x_3}\right), \tag{16}$$

$$\frac{\partial f}{\partial \alpha^*} := \frac{1}{2}\left(\frac{\partial f}{\partial x_0} + j^2 \frac{\partial f}{\partial x_2}\right), \qquad \frac{\partial f}{\partial \beta^*} := \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + j^2 \frac{\partial f}{\partial x_3}\right),$$

where $j^4 = -1$.

The condition for holomorphicity is the following equation

$$\frac{\partial f}{\partial \alpha^*} d\alpha^* + \frac{\partial f}{\partial \beta^*} d\beta^* = 0, \tag{17}$$

which implies the basic holomorphicity Cauchy-Riemann type system of PDE of first order (see [2]):

$$\frac{\partial f}{\partial x_0} + j^2 \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_1} + j^2 \frac{\partial f}{\partial x_3} = 0, \quad x_k \in \mathbf{R}, \quad j^4 = -1. \tag{18}$$

5. Holomorphic homogeneous fourth-R polynomials

5.1. Holomorphic homogeneous fourth-R polynomials of first degree

Theorem 2. *A homogeneous fourth-***R** *polynomial of first degree P is holomorphic, iff it is of the kind*

$$P = P(x_0 + j^2 x_2) + Q(x_1 + j^2 x_3), \tag{19}$$

where $P, Q \in \mathbf{R}(1, j, j^2, j^3)$, $j^4 = -1$.

P r o o f. Let us consider the homogeneous fourth-**R** valued polynomial of first degree

$$P = Ax_0 + Bx_1 + Cx_2 + Dx_3,$$

where $A, B, C, D \in \mathbf{R}(1, j, j^2, j^3)$, $j^4 = -1$ and x_0, x_1, x_2 and x_3 are real variables. The conditions for holomorphicity (18) gives the following equalities

$$0 = \frac{\partial(Ax_0 + Bx_1 + Cx_2 + Dx_3)}{\partial x_0} + j^2 \frac{\partial(Ax_0 + Bx_1 + Cx_2 + Dx_3)}{\partial x_2} = A + j^2 C,$$

$$0 = \frac{\partial(Ax_0 + Bx_1 + Cx_2 + Dx_3)}{\partial x_1} + j^2 \frac{\partial(Ax_0 + Bx_1 + Cx_2 + Dx_3)}{\partial x_3} = B + j^2 D.$$

So the holomorphic homogeneous polynomials of first degree are of the kind

$$P = P(x_0 + j^2 x_2) + Q(x_1 + j^2 x_3),$$

where $P = A = -j^2 C$ and $Q = B = -j^2 D$. Conversely, it is clear that such polynomials satisfy the system (18). This proves Theorem 2. ■

5.2. Holomorphic homogeneous fourth-R polynomials of degree n

Theorem 3. *A homogeneous fourth-**R** polynomial P of degree n is holomorphic iff it is of the following kind*

$$P = \sum_{k=0}^{n} C_k (x_0 + j^2 x_2)^k (x_1 + j^2 x_3)^{n-k}, \tag{20}$$

where $C_k \in \mathbf{R}(1, j, j^2, j^3)$, $j^4 = -1$, *for* $k = 0, 1, \ldots, n$.

P r o o f. First let us check that the polynomial of n-th degree

$$P = \sum_{k=0}^{n} C_k (x_0 + j^2 x_2)^k (x_1 + j^2 x_3)^{n-k}$$

satisfies the system (18). It is fulfilled

$$\frac{\partial P}{\partial x_0} + j^2 \frac{\partial P}{\partial x_2} = \sum_{k=0}^{n} k C_k (x_0 + j^2 x_2)^{k-1} (x_1 + j^2 x_3)^{n-k}$$
$$+ j^2 \sum_{k=0}^{n} k C_k (x_0 + j^2 x_2)^{k-1} j^2 (x_1 + j^2 x_3)^{n-k} = 0$$

and

$$\frac{\partial P}{\partial x_1} + j^2 \frac{\partial P}{\partial x_3} = \sum_{k=0}^{n} (n-k) C_k (x_0 + j^2 x_2)^k (x_1 + j^2 x_3)^{n-k-1}$$
$$+ j^2 \sum_{k=0}^{n} (n-k) C_k (x_0 + j^2 x_2)^k (x_1 + j^2 x_3)^{n-k-1} j^2 = 0.$$

So all polynomials of the considered kind are solutions of the system (18).

Now let $P(x_0+jx_1+j^2x_2+j^3x_3)$ be a holomorphic homogeneous fourth-**R** polynomial of degree n, i.e.

$$P=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=0}^{n-k-l}a_{klm}x_0^kx_1^lx_2^mx_3^{n-k-l-m},$$

and let it satisfy the system (18), from where

$$0=\frac{\partial P}{\partial x_0}+j^2\frac{\partial P}{\partial x_2}=\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{m=0}^{n-k-l}a_{klm}kx_0^{k-1}x_1^lx_2^mx_3^{n-k-l-m}$$
$$+j^2\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{m=0}^{n-k-l}a_{klm}mx_0^kx_1^lx_2^{m-1}x_3^{n-k-l-m}$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{n-l-1}\sum_{m=0}^{n-k-l-1}((k+1)a_{k+1lm}+j^2(m+1)a_{klm+1})x_0^kx_1^lx_2^mx_3^{n-k-l-m}.$$

Then

$$(k+1)a_{k+1lm}+j^2(m+1)a_{klm+1}=0$$

and

$$a_{k+1lm}=-j^2\frac{m+1}{k+1}a_{klm+1}\quad\text{for}\quad k=0,1,\dots,n-1\quad\text{and}\quad m=0,1,\dots,n-k-l-1.$$

Repeating this calculation, we obtain

$$a_{klm}=(-j^2)^k\frac{(m+1)(m+2)\dots(m+k)}{k(k-1)\dots1}a_{0lm+k}=\binom{m+k}{k}(-j^2)^ka_{0lm+k}.$$

Setting $p=m+k$, the polynomial P looks like as follows

$$P=\sum_{l=0}^{n}\sum_{p=0}^{n-l}a_{0lp}\sum_{k=0}^{p}\binom{p}{k}(-j^2x_0)^kx_2^{p-k}x_1^lx_3^{n-l-p}$$
$$=\sum_{l=0}^{n}\sum_{p=0}^{n-l}a_{0lp}(-j^2)^p(x_0+j^2x_2)^px_1^lx_3^{n-l-p}.$$

Using the second equation $\frac{\partial P}{\partial x_1}+j^2\frac{\partial P}{\partial x_3}=0$ from (18) for the so obtained polynomial P, we obtain a second term $(x_1+j^2x_3)^{n-p}$ for the variable $x_1+j^2x_3$ and this complete the proof of the theorem. ■

Remark. The algebra of a kind of quaternion polynomials was studied by the Bulgarian mathematician L. Tchakalov (1924) in [6].

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