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FIRST-ORDER CONDITIONS FOR OPTIMIZATION PROBLEMS WITH QUASICONVEX INEQUALITY CONSTRAINTS

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Abstract. The constrained optimization problem $\min f(x)$, $g_j(x) \leq 0$ $(j = 1, \ldots, p)$ is considered, where $f : X \to \mathbb{R}$ and $g_j : X \to \mathbb{R}$ are nonsmooth functions with domain $X \subset \mathbb{R}^n$. First-order necessary and first-order sufficient optimality conditions are obtained when g_j are quasiconvex functions. Two are the main features of the paper: to treat nonsmooth problems it makes use of the Dini derivative; to obtain more sensitive conditions, it admits directionally dependent multipliers. The two cases, where the Lagrange function satisfies a non-strict and a strict inequality, are considered. In the case of a non-strict inequality pseudoconvex functions are involved and in their terms some properties of the convex programming problems are generalized. The efficiency of the obtained conditions is illustrated on an example.

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1. Introduction. The constrained optimization problem

$$\min f(x), \quad g_j(x) \le 0 \quad (j = 1, \dots, p) \tag{1}$$

is investigated, where $f : X \to \mathbb{R}$ and $g_j : X \to \mathbb{R}$ $(j = 1, \dots, p)$ are nonsmooth functions with domain $X \subset \mathbb{R}^n$. The scope of the paper is to obtain first-order necessary and sufficient optimality conditions of Kuhn-Tucker type for problems with nonsmooth quasiconvex constraints, and in particular ones with quasiconvex objective functions. Quasiconvex (quasiconcave) programming initiates in the well-known paper of Arrow, Enthoven [1] and has been studied thereafter by various authors, e. g. in [10], [2], [3], [4], [7], [8], [12]. The main features of the paper are the following: to treat nonsmooth problems it makes use of Dini directional derivatives; to obtain more sensitive conditions it admits directionally dependent multipliers. This approach has been used in [6] for problems with locally Lipschitz data, making use of the set-valued Dini derivative. Here we show, that for problems with quasiconvex constraints we can use instead the single-valued Dini derivative.

2. Basic definitions. For a set $X \subset \mathbb{R}^n$ and $x \in X$ we denote by $X(x)$ the set of the admissible directions, that is the set of all $u \in \mathbb{R}^n$ for which $t = 0$ is an accumulating point for the set $\{t \in \mathbb{R}_+ \mid x + tu \in X\}$. Consider the function $f : X \to \mathbb{R}$. The *lower Dini derivative* $f_-^{(1)}(x, u)$ of f at $x \in \operatorname{dom} f$ in direction $u \in X(x)$ is defined as an element of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ by

$$f_-^{(1)}(x, u) = \liminf_{t \to 0^+} \frac{1}{t}(f(x + tu) - f(x)).$$

The role of the Dini derivatives for quasiconvex programming is stressed in [4].

Recall that a function $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, is said *quasiconvex* (*strictly quasiconvex*) if X is convex and for all x^0, $x^1 \in X$, $x^0 \ne x^1$, such that $f(x^0) \ge f(x^1)$, and all $t \in (0,\, 1)$, it holds $f((1-t)x^0 + tx^1) \le f(x^0)$ $(f((1-t)x^0 + tx^1) < f(x^0))$. If these properties hold for a fixed $x^0 \in X$, we say that f is *quasiconvex* (*strictly quasiconvex*) at x^0. Moreover, in the last definition we will not suppose that X is convex, but the above properties will be assumed to hold only for those $t \in (0,\, 1)$, for which $(1-t)x^0 + tx^1 \in X$ (this relaxed definition allows for instance further to state Theorem 1 without the hypothesis that X is convex).

Following Diewert [5], we use the Dini derivative to introduce pseudoconvexity for nonsmooth functions. We call the set $X \subset \mathbb{R}^n$ convex-like at x^0 if for each $x^1 \in X$ it holds $x^1 - x^0 \in X(x^0)$. We say that set X is convex-like if it is convex-like for each $x^0 \in X$ (turn attention that the convex sets are convex-like). We say that the function $f : X \to \mathbb{R}$, where X is convex-like at x^0, is *pseudoconvex* (*strictly pseudoconvex*) at $x^0 \in X$, if $f(x^0) > f(x^1)$ $(f(x^0) \ge f(x^1))$ implies

$f_-^{(1)}(x^0, x^1 - x^0) < 0$. The function $f : X \to \mathbb{R}$, where X is convex-like, is said *pseudoconvex* (*strictly pseudoconvex*) if it is pseudoconvex (strictly pseudoconvex) at each point $x \in X$ (the definition of Diewert requires that the domain X is convex, here we relax this requirement to X is convex-like).

3. Conditions with non-strict inequalities We can write problem (1) in the form

$$\min f(x), \quad g(x) \leq 0, \tag{2}$$

accepting that $g(x) = (g_1(x), \dots, g_p(x))$ (the lower indexes will be used for the coordinates of a vector) and that $g(x) \leq 0$ means that the coordinates satisfy this inequality. We put $g_-^{(1)}(x,u) = ({g_1}_-^{(1)}(x,u), \dots, {g_p}_-^{(1)}(x,u))$. The scalar product in $\mathbb{R}^p$ is denoted $\langle \cdot, \cdot \rangle$, that is $\langle \eta, z \rangle = \sum_{j=1}^{p} \eta_j z_j$ for η, $z \in \mathbb{R}^p$. Besides the usual algebraic operations with infinities, we accept that $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$.

We write as usual $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. If $r \in \mathbb{R}_+$ we put

$$\mathbb{R}_+[r] = \begin{cases} \mathbb{R}, & r > 0, \\ \mathbb{R}_+, & r = 0, \end{cases} \qquad \overline{\mathbb{R}}_+[r] = \begin{cases} \overline{\mathbb{R}}, & r > 0, \\ \overline{\mathbb{R}}_+, & r = 0. \end{cases}$$

Given $z^0 \in \mathbb{R}_+^p$, we introduce the notations

$$\mathbb{R}_+^p[z^0] = \mathbb{R}_+[z_1^0] \times \dots \times \mathbb{R}_+[z_p^0],$$

$$\overline{\mathbb{R}}_+^p[z^0] = \overline{\mathbb{R}}_+[z_1^0] \times \dots \times \overline{\mathbb{R}}_+[z_p^0].$$

Recall that there exists a standard topology on $\overline{\mathbb{R}}$, in which a neighbourhood of $+\infty$ ($-\infty$) is any set $U \subset \overline{\mathbb{R}}$ containing an interval of the type $(a, +\infty]$ ($[-\infty, a)$) for some $a \in \mathbb{R}$. When $A_i \subset \overline{\mathbb{R}}$ ($i = 1, \dots, k$), then $\operatorname{int} \prod_{i=1}^{k} A_i = \prod_{i=1}^{k} \operatorname{int} A_i$ is the interior of $\prod_{i=1}^{k} A_i$ with respect to the product topology $\overline{\mathbb{R}}^k$. With these agreements the following lemma has place.

Lemma 1. *Let $z^0 \in \mathbb{R}_+^p$ and let $\bar{y} \in \overline{\mathbb{R}}$, $\bar{z} \in \overline{\mathbb{R}}^p$. Then the following two conditions are equivalent:*

$$(\bar{y}, \bar{z}) \notin -\operatorname{int}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^p[z^0]), \tag{3}$$

and

$$\begin{gathered} \exists (\xi^0, \eta^0) \in \mathbb{R}_+ \times \mathbb{R}_+^p : (\xi^0, \eta^0) \neq (0, 0), \\ \xi^0 = 0 \quad \text{if} \quad \bar{y} = -\infty, \quad \eta_j^0 = 0 \quad \text{if} \quad \bar{z}_j = -\infty, \\ \eta_j^0 z_j^0 = 0 \quad (j = 1, \dots, p) \quad \text{and} \quad \xi^0 \bar{y} + \langle \eta^0, \bar{z} \rangle \geq 0. \end{gathered} \tag{4}$$

P r o o f. If $r \in \overline{\mathbb{R}}$ we put $A(r) = \mathbb{R}_+$ when $r \geq 0$, and $A(r) = \mathbb{R}$ when $r < 0$. Now it is clear that condition (3) is satisfied if and only if the set

$A(\bar{y}) \times \prod_{j=1}^{p} A(\bar{z}_j)$ is separated from $-\mathrm{int}\,(\mathbb{R}_+ \times \mathbb{R}_+^p[z^0])$, the two sets are in $\mathbb{R}^{p+1}$. Applying the Separation theorem for these two sets, we see that (3) implies (4). Conversely, when condition (4) is satisfied, then (3) follows immediately, since $\xi^0\, y + \langle \eta^0,\, z\rangle < 0$ for $(y,z) \in -\mathrm{int}\,(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^p[z^0])$. □

Remark 1. In (4) due to $z^0 \in \mathbb{R}_+^p$ and $\eta_0 \in \mathbb{R}_+^p$ the slackness condition $\eta_j^0\, z_j^0 = 0$ $(j = 1,\dots,p)$ can be represented in the equivalent form $\langle \eta,\, z\rangle = 0$. The sum $\xi^0\,\bar{y} + \langle \eta^0,\, \bar{z}\rangle = \xi^0\,\bar{y} + \sum_{j=1}^{p} \eta_j^0 \bar{z}_j$ always has sense, since it has not addends equal to $-\infty$. The proof of Lemma 1 leads to a practical rule how to choose the multipliers ξ^0 and $\eta^0 = (\eta_1^0,\dots,\eta_p^0)$. Namely, we can put

$$\xi^0 = \begin{cases} 1, & \bar{y} \ge 0,\\ 0, & \bar{y} < 0, \end{cases} \qquad \text{and} \qquad \eta_j^0 = \begin{cases} 1, & z_j^0 = 0,\ \bar{z}_j \ge 0,\\ 0, & z_j^0 = 0,\ \bar{z}_j < 0,\\ 0, & z_j^0 > 0. \end{cases}$$

The next theorem uses the following notion of a minimizer. We say that the feasible point x^0 is a *radial minimizer* (*strict radial minimizer*) of (1), if for all admissible directions $u \in X(x^0)$, there exists $\delta(u) > 0$, such that $f(x^0) \le f(x^0 + tu)$ $(f(x^0) < f(x^0 + tu))$ whenever $0 < t < \delta(u)$ and the point $x^0 + tu$ is feasible. Obviously, each local (strict local) minimizer of (1) is its radial (strict radial) minimizer.

Recall that given a feasible point $x^0 \in X$, the set of the active indexes for (1) at x^0 is defined by $J(x^0) = \{j \mid g_j(x^0) = 0\}$.

Theorem 1 (Necessary conditions, non-strict inequalities)**.** *Consider problem* (1) *and let* x^0 *be a radial minimizer. Let the functions* g_j $(j = 1,\dots,p)$ *be continuous at* x^0 *when* $j \notin J(x^0)$ *and quasiconvex at* x^0 *when* $j \in J(x^0)$*. Then for each* $u \in X(x^0)$ *the following condition is satisfied:*

$$(f_-^{(1)}(x^0,u),\, g_-^{(1)}(x^0,u)) \notin -\mathrm{int}\,\left(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^p[-g(x^0)]\right). \tag{5}$$

Proof. Suppose on the contrary, that for some $u^0 \in X(x^0)$ we have $f_-^{(1)}(x^0,u^0) \in -\mathrm{int}\,\overline{\mathbb{R}}_+$ and $g_-^{(1)}(x^0,u^0) \in -\mathrm{int}\,\overline{\mathbb{R}}_+^p[-g(x^0)]$. Let $f_-^{(1)}(x^0,u^0) = \lim_k (1/t_k)(y^k - y^0)$ and $g_{j\,-}^{(1)}(x^0,u^0) = \lim_k (1/s_{jk})(z^{jk} - z^{j0})$ for some sequences $t_k \to 0^+$ and $s_{jk} \to 0^+$ $(j = 1,\dots,p)$, where $y^k = f(x^0 + t_k u^0)$, $y^0 = f(x^0)$, $z^{jk} = g_j(x^0 + s_{jk} u^0)$, $z^{j0} = g_j(x^0)$. Passing to a subsequence of $\{t_k\}$, we may assume that $t_k < \min(s_{1k},\dots,s_{pk})$. Now we prove that the points $x^0 + t_k u^0$ are feasible for all sufficiently large k. The condition $x^0 + t_k u^0 \in X$ is imposed implicitly taking the value $f(x^0 + t_k u^0)$. Since f and g_j $(j = 1,\dots,p)$ are supposed to have the same domain X, the values $g_j(x^0 + t_k u^0)$ are defined. It remains to prove that $g_j(x^0 + t_k u^0) \le 0$ $(j = 1, \dots, p)$ for all sufficiently large k. When $j \in J(x^0)$ we have $g_j(x^0 + s_{jk} u^0) \le 0 = g(x^0)$. Since g_j is quasiconvex at x^0, this gives $g_j(x^0 + t_k u^0) \le 0$. When $j \notin J(x^0)$, we have $g_j(x^0) < 0$. Now the continuity of g_j at x^0 implies $g_j(x^0 + t_k u^0) < 0$ for all sufficiently large k. Thus, for all sufficiently large k the point $x^0 + t_k u^0$ is feasible, and at the same time $f(x^0 + t_k u^0) - f(x^0) = y^k - y^0 < 0$, which contradicts the hypothesis that x^0 is a radial minimizer. □

Remark 2. Condition (5) will be referred as primal form condition. On the base of Lemma 1 it is equivalent to the following dual form condition:

$$
\begin{gathered}
\exists\, (\xi^0, \eta^0) \in \mathbb{R}_+ \times \mathbb{R}_+^p : \quad \langle \xi^0, \eta^0 \rangle \ne (0,\,0), \\
\xi^0 = 0 \quad \text{if} \quad f_-^{(1)}(x^0, u) = -\infty, \\
\eta_j^0 = 0 \quad \text{if} \quad g_{j\,-}^{(1)}(x^0, u) = -\infty \quad (j = 1, \dots, p), \\
\eta_j^0\, g_j(x^0) = 0 \quad (j = 1, \dots, p), \\
\text{and} \quad \xi^0\, f_-^{(1)}(x^0, u) + \textstyle\sum_{j=1}^p \eta_j^0\, g_{j\,-}^{(1)}(x^0, u) \ge 0.
\end{gathered} \tag{6}
$$

The multipliers ξ^0 and η_j^0 $(j = 1, \dots, p)$ can be chosen according to Remark 1.

Remark 3. The last row in (6) is a non-strict inequality. Since only such conditions are considered in this section, it is entitled "Conditions with non-strict inequalities". In the next section we will occupy with similar conditions, but with strict inequalities.

The following example shows, that without the hypothesis that g_j is continuous at x^0 when $j \notin J(x^0)$ Theorem 1 is not true.

Example 1. *Consider problem* (2) *with* $f,\, g : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = -x$ *and*

$$
g(x) = \begin{cases} -1, & x \le 0, \\ 1, & x > 0. \end{cases}
$$

The function g is quasiconvex. It holds $g(x^0) < 0$ and g is not continuous at x^0. The point $x^0 = 0$ is a radial (and global) minimizer, but condition (5) *is not satisfied. Indeed, for $u = 1$ it holds*

$$
(f_-^{(1)}(x^0, u), g_-^{(1)}(x^0, u)) = (-1, +\infty) \in -\operatorname{int}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}) = -\operatorname{int}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+[-g(x^0)]).
$$

If in Theorem 1 we replace the primal form condition (5) with the equivalent dual form condition (6), we observe that, in contrast to the classical theory, the multipliers depend on the directions. The next example shows that, when treating nonsmooth problems, the hypotheses of Theorem 1 do not imply condition (6) with directionally independent multipliers.

Example 2. *Consider problem* (2) *with* $f,\, g : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} x, & x \ge 0, \\ 2x, & x < 0, \end{cases} \qquad g(x) = \begin{cases} -2x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

The functions f *and* g *are continuous and strictly quasiconvex (also strictly pseudoconvex). The set of the feasible points is* $\mathbb{R}_+$. *Put* $x^0 = 0$. *Obviously* x^0 *is a global minimizer. Then condition* (6) *is satisfied in virtue of Theorem* 1, *but cannot be satisfied with directionally independent multipliers.*

Indeed, assume in the contrary, that condition (6) is satisfied with some directionally independent multipliers (ξ^0, η^0). For $u \ge 0$ it holds $f_-^{(1)}(x^0, u) = u$, $g_-^{(1)}(x^0, u) = -2u$, whence in particular we should have

$$\xi^0 \, f_-^{(1)}(x^0,\, 1) + \eta^0 \, g_-^{(1)}(x^0,\, 1) = \xi^0 - 2\eta^0 \ge 0.$$

Similarly, for $u \le 0$ it holds $f_-^{(1)}(x^0, u) = 2u$, $g_-^{(1)}(x^0, u) = -u$, whence in particular we should have

$$\xi^0 \, f_-^{(1)}(x^0,\, -1) + \eta^0 \, g_-^{(1)}(x^0,\, -1) = -2\xi^0 + \eta^0 \ge 0.$$

Adding the two inequalities we obtain $-(\xi^0 + \eta^0) \ge 0$, which obviously contradicts to $\xi^0 \ge 0$, $\eta^0 \ge 0$, $(\xi^0, \eta^0) \ne (0,\, 0)$.

Theorem 2 (Sufficient conditions, non-strict inequalities)**.** *Consider problem* (1) *with* X *convex-like at* x^0. *Let the functions* g_j, $j \in J(x^0)$, *be strictly pseudoconvex at* x^0, *and* f *be pseudoconvex (strictly pseudoconvex) at* x^0. *Suppose that for each* $u \in X(x^0)$ *condition* (5) *is satisfied. Then* x^0 *is a global minimizer (strict global minimizer).*

Proof. Assume on the contrary, that x^0 is not a global (strict global) minimizer. Then there exists a feasible point $x^1 \ne x^0$ such that $f(x^1) - f(x^0) < 0$ ($f(x^1) - f(x^0) \le 0$). Since f is pseudoconvex (strictly pseudoconvex) at x^0, it holds $f_-^{(1)}(x^0, u) < 0$ with $u = x^1 - x^0$. Therefore condition (5) gives that $g_-^{(1)}(x^0, u) \notin -\operatorname{int} \overline{\mathbb{R}}_+^p[-g(x^0)]$. On the other hand for $j \in J(x^0)$ we have $g_j(x^1) \le 0 = g_j(x^0)$. Since g_j is strictly pseudoconvex at x^0, we have ${g_j}_-^{(1)}(x^0, u) < 0$. This gives $g_-^{(1)}(x^0, u) \in -\operatorname{int} \overline{\mathbb{R}}_+^p[-g(x^0)]$, a contradiction. □

The following example shows that in Theorem 2 the strict pseudoconvexity requirements for the constraint functions g_j cannot be relaxed to only pseudoconvexity.

Example 3. *Consider problem* (2) *with* $f,\, g : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = -x$ *and* $g(x) = 0$. *Put* $x^0 = 0$. *The function* f *is strictly pseudoconvex, and* g *is pseudoconvex but not strictly pseudoconvex at* x^0. *The point* x^0 *is not a global* *minimizer, while condition* (5) *is satisfied, since*

$$(f_-^{(1)}(x^0,u),\, g_-^{(1)}(x^0,u)) = (-u,\, 0) \notin -\mathrm{int}\,(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+) = -\mathrm{int}\,(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+[-g(x^0)]).$$

The following example shows that also the pseudoconvexity requirements for the objective function are essential for Theorem 2 and cannot be reduced to (strict) quasiconvexity.

Example 4. *Consider problem* (2) *with* $f,\, g : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^3$ *and* $g(x) = x$. *Put* $x^0 = 0$. *The functions* f *and* g *are strictly quasiconvex,* g *is strictly pseudoconvex at* x^0*, but* f *is not so. Since* $f_-^{(1)}(x^0,u) = 0$ *and* $g_-^{(1)}(x^0,u) = u$*, condition* (5) *is satisfied (now* $f_-^{(1)}(x^0,u) \notin -\mathrm{int}\,\overline{\mathbb{R}}_+$*), but* x^0 *is not a global minimizer.*

The following theorem is a consequence of Theorems 1 and 2. Strengthening there the pseudoconvexity and the strict pseudoconvexity requirements respectively to convexity and strict convexity, we obtain a known classical result.

Theorem 3. *Let in problem* (1) *the set* X *be convex (or more generally convex-like), the functions* f *be pseudoconvex (strictly pseudoconvex), and* g_j $(j = 1,\dots,p)$ *be continuous and strictly pseudoconvex. Then a point* $x^0 \in X$ *is a global minimizer of problem* (1) *if and only if* x^0 *satisfies condition* (5)*.*

The given so far examples serve to clarify to what extend the hypotheses of the theorems are essential. Now we give an example to illustrate, that the obtained results are effective in solving complex nonsmooth problems (the nonsmoothness here is due to the appearance of the min function).

Example 5. *Solve problem* (2) *with* $f : \mathbb{R}^2 \to \mathbb{R}$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(x_1,x_2) = \min\left(x_1^2 + 8x_1x_2 + 16x_2^2 - 8x_1 - 32x_2 + 20,\; x_1 + 4x_2\right),$$

$$g(x_1,x_2) = -x_1 - x_2 + \sqrt{(x_1-x_2)^2+4}.$$

The function f can be written into the form

$$f(x_1,x_2) = \begin{cases} x_1^2 + 8x_1x_2 + 16x_2^2 - 8x_1 - 32x_2 + 20, & 4 \le x_1 + 4x_2 \le 5, \\ x_1 + 4x_2, & \text{otherwise.} \end{cases}$$

There are no solutions among the points outside the lines $\ell_1 : x_1 + 4x_2 = 4$ and $\ell_2 : x_1 + 4x_2 = 5$. We leave this case, since it can be checked easily with the given here theory, but also with a classical approach (near such points both f and g are smooth).

At the points $x \in \ell_1$ we have

$$f_-^{(1)}(x,u) = \begin{cases} 0, & u_1 + 4u_2 \ge 0, \\ u_1 + 4u_2, & u_1 + 4u_2 < 0, \end{cases}$$

$$g_-^{(1)}(x,u) = -u_1 - u_2 + \frac{(x_1 - x_2)(u_1 - u_2)}{\sqrt{(x_1 - x_2)^2 + 4}}\,.$$

Now the sign of $f_-^{(1)}(x,u)$ is easily estimated and from Remark 1 we can limit the choice of ξ^0 to:

$$\begin{aligned} f_-^{(1)}(x,u) \ge 0 \Rightarrow \xi^0 = 1 \quad &\text{for} \quad u_1 + 4u_2 \ge 0, \\ f_-^{(1)}(x,u) < 0 \Rightarrow \xi^0 = 0 \quad &\text{for} \quad u_1 + 4u_2 < 0. \end{aligned}$$

According to Remark 1 the choice of η^0 can be conditioned by the sign of $g_-^{(1)}(x,u)$ and the solution of the system

$$\begin{cases} -x_1 - x_2 + \sqrt{(x_1 - x_2)^2 + 4} = 0, \\ x_1 + 4x_2 = 4, \end{cases}$$

(that is $g(x) = 0$, $x \in \ell_1$). The latter has the unique solution $x^0 = (2,\, 1/2)$. At x^0 we have $g_-^{(1)}(x^0,u) = -\frac{2}{5}u_1 - \frac{8}{5}u_2$, which gives $g_-^{(1)}(x^0,u) \ge 0$ for $u_1 + 4u_2 \le 0$. Therefore the choice of η^0 can be restricted to:

$$\eta^0 = \begin{cases} 1, & x = x^0, \quad u_1 + 4u_2 \le 0, \\ 0, & x = x^0, \quad u_1 + 4u_2 < 0, \\ 0, & x \in \ell_1 \setminus \{x^0\}. \end{cases}$$

Now we see that at $x^0 = (2,\, 1/2)$ for all directions $u \in \mathbb{R}^2$ condition (6) can be satisfied (in which case we call the point x^0 stationary) with the choice:

$$(\xi^0,\, \eta^0) = \begin{cases} (1,\, 0), & u_1 + 4u_2 \ge 0, \\ (0,\, 1), & u_1 + 4u_2 < 0. \end{cases}$$

All the remaining points $x \in \ell_1 \setminus \{x^0\}$ are not stationary, since for any such point x we can choose at least one direction u, for which the obtained ξ^0 and η^0 give the zero pair $(\xi^0,\, \eta^0) = (0,\, 0)$.

Similarly, in the case $x \in \ell_2$ we see that there are no stationary points.

Thus $x^0 = (2,\, 1/2)$ is the only point which satisfies the necessary condition from Theorem 1. Since, as it can be easily checked, the function f is pseudoconvex at x^0, and g is strictly pseudoconvex at x^0, we can draw the conclusion that $x^0 = (2,\, 1/2)$ is a global minimizer for the considered problem, and its only radial minimizer.

Let us note, that f is pseudoconvex and g is strictly pseudoconvex (not only at x^0), therefore looking for solutions of the considered problem, we can refer to Theorem 3.

4. Conditions with strict inequalities. In this section we show that the first-order conditions with strict inequalities are related to the radial isolated minimizer.

Let k be a positive real. We say that the feasible point $x^0 \in X$ is a *radial isolated minimizer* (of order 1) for problem (1) if for all $u \in X(x^0)$, there exist positive reals $\delta = \delta(u)$ and $A = A(u)$, such that the inequality

$$f(x^0 + tu) \geq f(x^0) + A\,t\,\|u\|$$

is satisfied for all feasible points $x^0 + tu$ such that $0 \leq t < \delta(u)$. If the reals δ and A can be chosen to be independent on u, then x^0 is called an *isolated minimizer* for (1).

The following lemma is analogous of Lemma 1 and is proved in a similar way.

Lemma 2. *Let* $z^0 \in \mathbb{R}^p_+$ *and let* $\bar{y} \in \overline{\mathbb{R}}$, $\bar{z} \in \overline{\mathbb{R}}$. *Then the following two conditions are equivalent:*

$$(\bar{y},\, \bar{z}) \notin -(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^p_+[z^0]) \tag{7}$$

and

$$\begin{gathered} \exists(\xi^0,\, \eta^0) \in \mathbb{R}_+ \times \mathbb{R}^p_+ :\, (\xi^0,\, \eta^0) \neq (0,\, 0), \\ \xi^0 = 0 \quad \text{if} \quad \bar{y} = -\infty, \quad \eta^0_j = 0 \quad \text{if} \quad \bar{z}_j = -\infty, \\ \langle \eta^0,\, z^0 \rangle = 0, \quad \text{and} \quad \xi^0\, \bar{y} + \langle \eta^0,\, \bar{z} \rangle > 0. \end{gathered} \tag{8}$$

Theorem 4 (Sufficient conditions, strict inequalities)**.** *Let* $x^0 \in X$ *be a feasible point for problem* (1). *Suppose that for all* $u \in X(x^0) \setminus \{0\}$ *the following condition is satisfied:*

$$(f^{(1)}_-(x^0, u),\, g^{(1)}_-(x^0, u)) \notin -(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^p_+[-g(x^0)]). \tag{9}$$

Then x^0 *is a radial isolated minimizer of* (1). *Under the additional assumption that* X *is convex,* f *is quasiconvex (strictly quasiconvex) and* g_j $(j = 1, \ldots, p)$ *are quasiconvex, then* x^0 *is a global (strict global) minimizer of* (1).

P r o o f. Assume on the contrary, that x^0 is not a radial isolated minimizer of (1). Choose a sequence $\varepsilon_k \to 0^+$. ¿From the made assumption, there exists $u \in X(x^0) \setminus \{0\}$ and a sequence $t_k \to 0^+$, such that the points $x^0 + t_k u$ are feasible and $(1/t_k)\big(f(x^0 + t_k u) - f(x^0)\big) < \varepsilon_k\, \|u\|$. The latter gives $f^{(1)}_-(x^0, u) \leq 0$, that is $f^{(1)}_-(x^0, u) \in -\overline{\mathbb{R}}_+$. When $g_j(x^0) = 0$ we have similarly

$(1/t_k)\,\big(g_j(x^0+t_k u)-g_j(x^0)\big) \le 0$. Hence ${g_j}_-^{(1)}(x^0,u) \le 0$, that is ${g_j}_-^{(1)}(x^0,u) \in -\overline{\mathbb{R}}_+ = -\overline{\mathbb{R}}_+[-g_j(x^0)]$. When $g_j(x^0) < 0$, then $\overline{\mathbb{R}}_+[-g_j(x^0)] = \overline{\mathbb{R}}$ and again ${g_j}_-^{(1)}(x^0,u) \in -\overline{\mathbb{R}}_+[-g_j(x^0)]$. These reasonings show that $(f_-^{(1)}(x^0,u),\, g_-^{(1)}(x^0,u)) \in -(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^p[-g(x^0)])$, which contradicts the hypothesis (9).

Let the mentioned additional assumption are fulfilled. Suppose on the contrary, that x^0 is not a global (strict global) minimizer. Then there exists a feasible point $x^1 \in X \setminus \{x^0\}$, such that $f(x^0) > f(x^1)$ $(f(x^0) \ge f(x^1))$. Since g_j $(j=1,\dots,p)$ are quasiconvex, the points x^0+tu with $u = x^1 - x^0$ are feasible. Since x^0, as proved above, is a radial minimizer of (1), the point $t^0 = 0$ is a local minimizer for the quasiconvex (strictly quasiconvex) function $\phi(t) = f(x^0+tu)$, $0 \le t \le 1$, and therefore its global (strict global) minimizer. In particular $f(x^0) = \phi(0) \le \phi(1) = f(x^1)$ $(f(x^0) = \phi(0) < \phi(1) = f(x^1))$, a contradiction. $\square$

Remark 4. On the base of Lemma 2, the primal form condition (9) is equivalent to the following dual form condition:

$$
\begin{aligned}
&\exists\, (\xi^0,\eta^0) \in \mathbb{R}_+ \times \mathbb{R}_+^p : \quad \langle \xi^0,\eta^0\rangle \ne (0,\,0),\\
&\xi^0 = 0 \quad \text{if} \quad f_-^{(1)}(x^0,u) = -\infty,\\
&\eta_j^0 = 0 \quad \text{if} \quad {g_j}_-^{(1)}(x^0,u) = -\infty \quad (j=1,\dots,p),\\
&\eta_j^0\, g_j(x^0) = 0 \quad (j=1,\dots,p),\\
&\text{and} \quad \xi^0\, f_-^{(1)}(x^0,u) + \textstyle\sum_{j=1}^p \eta_j^0\, {g_j}_-^{(1)}(x^0,u) > 0.
\end{aligned}
\tag{10}
$$

As an application consider the problem in Example 2 for $x^0 = 0$ putting $\xi^0 = 3$, $\eta^0 = 1$ when $u > 0$, and $\xi^0 = 1$, $\eta^0 = 3$ when $u < 0$. Now it is easy to verify that

$$
\xi^0 f_-^{(1)}(x^0,u) + \eta^0 g_-^{(1)}(x^0,u) = |u| > 0 \quad \text{for all} \quad u \in \mathbb{R}^n \setminus \{0\}. \tag{11}
$$

On the base of Theorem 4 we conclude that x^0 is a strict global minimizer, hence the unique minimizer, of the considered problem.

For the next theorem, being a reversal of Theorem 4, we need the following constraint qualification of Kuhn-Tucker type:

$\mathbb{Q}'_-(x^0)$: If x^0 is feasible and ${g_j}_-^{(1)}(x^0,u) \in -\overline{\mathbb{R}}_+[-g_j(x^0)]$ for $j=1,\dots,p$, then exists $\bar t > 0$ such that $x^0+\bar t u$ is a feasible point for (1)).

Theorem 5 (Necessary conditions, strict inequalities)**.** *Let the set X be convex, the functions g_j $(j=1,\dots,p)$ be quasiconvex, and the feasible point x^0 be a radial isolated minimizer of problem* (1)*. Suppose that the constraint qualification $\mathbb{Q}'_-(x^0)$ has place. Then for all $u \in X(x^0) \setminus \{0\}$ condition* (9) *is satisfied.*

Proof. Condition (9) is certainly true when ${g_j}_-^{(1)}(x^0, u) \notin \overline{\mathbb{R}}_+[-g_j(x^0)]$ for some j. The alternative is that ${g_j}_-^{(1)}(x^0, u) \in -\overline{\mathbb{R}}_+[-g_j(x^0)]$ for $j = 1, \ldots, p$. Then the assumed constraint qualification implies the existence of a positive real $\bar{t}$, such that the point $x^0 + \bar{t}u$ is feasible. From the quasiconvexity of g_j it follows that all points $x^0 + tu$, $0 \le t \le \bar{t}$, are feasible. Since x^0 is a radial isolated minimizer, there exists a real $A > 0$, such that $(1/t)\,\big(f(x^0 + tu) - f(x^0)\big) \ge A\,\|u\|$ is satisfied for all sufficiently small positive t. This gives $f_-^{(1)}(x^0, u) \ge A\,\|u\|$. Hence $f_-^{(1)}(x^0, u) \notin -\overline{\mathbb{R}}_+$, which verifies (9) in this case. □

Like in the classical Kuhn-Tucker condition [9] (compare also with Mangasarian [11]) the sense of the constraint qualification $\mathbb{Q}'_-(x^0)$ is roughly speaking that a point cannot leave the set of the feasible points at x^0 in tangent directions. The following example shows that $\mathbb{Q}'_-(x^0)$ is essential for Theorem 5.

Example 6. *Consider problem* (2) *with* f, $g : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = g(x) = x^2$ *and let* $x^0 = 0$. *The function* g *is quasiconvex. The point* x^0, *as the only feasible point, is a radial isolated minimizer. It holds* $f_-^{(1)}(x^0, u) = g_-^{(1)}(x^0, u) = 0$ *for all* $u \in \mathbb{R}^n$, *whence obviously condition* (9) *is not satisfied.*

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