Letter to the Editor

Remarks on Some Inequalities for Polynomials

M. A. Hachani

Presented by Virginia Kiryakova

In the present article, I point out serious errors in a paper published in *Mathematica Balkanica* three years ago. These errors seem to go unnoticed because some mathematicians are applying the results stated in this paper to prove other results, which should not continue.

The following result was proved by Govil (see [4, p. 51]).

Theorem A. *Let $p(z)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, and let $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on the circle $|z| = 1$, then*

$$(1) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|.$$

The result is best possible with equality holding for the polynomial $p(z) = z^n + k^n$.

Aziz and Ahmad [1] proved that if p satisfies the conditions of Theorem A, then

$$(2) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.$$

which is stronger than (1).

The following result of Govil, which is clearly related to Theorem A, appears in [5] as Theorem D on p. 184.

Theorem B. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$(3) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|.$$

In [2] the authors state and I quote: "In this paper, we consider polynomials of the form

$$p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \ 1 \leq \mu \leq n$$

and obtain generalization as well as improvement of (1). Also we generalize Theorem B". They state their so-called generalizations of (1), (2) and (3) as follows.

Theorem 1. *Let $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \ 1 \leq \mu < n$ be a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$ and $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, then*

$$(4) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \max_{|z|=1} |p(z)|.$$

Theorem 2. *Let $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \ 1 \leq \mu < n$ be a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$ and $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, then*

$$(5) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.$$

Theorem 2 is supposed to generalize Theorem B.

Theorem 3. *If $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \ 1 \leq \mu < n$ is a polynomial of degree n, having all its zeros on $|z| = k$, $k \leq 1$, then*

$$(6) \qquad \max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|.$$

Unfortunately, Theorems 1, 2 and 3 are false. To see that Theorem 1 is invalid, let us consider the example $p(z) := z^n + k^n$. It is of the form $c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} c_{n-\nu} z^{n-\nu}$ with

$$c_n = 1, \ c_{n-\nu} = 0 \ \text{for} \ \nu = \mu, \ldots, n-1 \ \text{and} \ c_0 = k^n,$$

where μ can be taken to be any integer in $\{1, 2, \ldots, n-1\}$. This polynomial has all its zeros on $|z| = k$. Since $p'(z) = nz^{n-1}$ and $q'(z) = nk^n z^{n-1}$, we see that $|p'(z)|$ and $|q'(z)|$ become maximum at every point on $|z| = 1$. Thus p satisfies the conditions of Theorem 1. Since

$$\max_{|z|=1} |p(z)| = 1 + k^n \quad \text{and} \quad \max_{|z|=1} |p'(z)| = n,$$

inequality (4) implies that

$$n \leq \frac{n}{1 + k^{n-\mu+1}}(1 + k^n) \qquad (\mu = 1, 2, \ldots, n-1,$$

which holds if and only if $k^{n-\mu+1} \leq k^n$, However, this is manifestly false for any $k \in (0, 1)$ and any $\mu \in \{2, \ldots, n-1\}$.

A naive reader might say that in this example $c_{n-\mu}$ is taken to be 0 and that, in Theorem 1, $c_{n-\mu}$ is supposed to be different from zero. No problem, we will modify our example.

Take any $a \in (0, 1)$ and consider the polynomial

$$p(z) := z^n + \delta z^{n-\mu} + a^n,$$

where δ is positive and small. Since the zeros of p are continuous functions [7, p. 9] of δ and those of $z^n + a^n$ all lie on $|z| = a$ the polynomial p has all its zeros in $|z| \geq k$, where $|a - k| \to 0$ as $\delta \to 0$. Now, note that

$$p'(z) = nz^{n-1} + \delta(n-\mu)z^{n-\mu-1} \qquad \text{and} \quad q'(z) = a^n nz^{n-1} + \delta\mu z^{\mu-1}.$$

So, both $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, namely the point 1. Thus, Theorem 1 applies and would imply that

$$n + \delta(n - \mu) \leq \frac{n}{1 + k^{n-\mu+1}}(1 + \delta + a^n),$$

where δ is any small positive number. Letting δ tend to 0, we would obtain

$$n \leq \frac{n}{1 + a^{n-\mu+1}}(1 + a^n),$$

where we have used the fact that $k \to a$ as $\delta \to 0$. This last inequality holds if and only if $a^{n-\mu+1} \leq a^n$, which is not true for any $\mu \in \{2, \ldots, n-1\}$ since $a \in (0\,,\,1)$.

Now we know that Theorem 1 is incorrect, but then Theorem 2, being stronger than Theorem 1 cannot be true either.

According to the statement of Theorem 3, the coefficients $c_{n-\mu}, \ldots, c_1$ can be anything as long as the polynomial p has all its zeros on $|z| = k$. So, there is nothing to prevent us from taking $p(z) = z^n + k^n$ and applying (6) with $\mu = n-1$. We would then obtain

$$n \leq \frac{n}{k^{3-n} + k^2}(1 + k^n)\,,$$

which is true if and only if

$$k^3 + k^{n+2} \leq nk^n\,(1 + k^n).$$

Fixing $k \in (0\,,\,1)$ and letting $n \to \infty$ the left-hand side of this inequality tends to k^3 whereas the right-hand side tends to 0, which is a contradiction. This shows that Theorem 3, as stated by the authors, cannot be true.

The proof of Theorem 3 uses the following faulty statement which appears as Lemma 3 in their paper:

Proposition 1.   *If $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$, then*

(7) $$k^{n-\mu+1} \max_{|z|=1} |p'(z)| \leq \max_{|z|=1} |q'(z)|\,,$$

where $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

In order to show that this is an invalid statement we may once again take any $a \in (0\,,\,1)$ and consider the polynomial $p(z) := z^n + \delta z^{n-\mu} + a^n$, where δ is positive and small. As observed above, it has all its zeros in $|z| \geq k$, where $|a - k| \to 0$ as $\delta \to 0$. Clearly,

$$\max_{|z|=1} |p'(z)| = |p'(1)| = n + \delta(n - \mu) \quad \text{and} \quad \max_{|z|=1} |q'(z)| = |q'(1)| = a^n n + \delta\mu\,.$$

Hence, if Proposition 1 was true then (7) would imply that

$$k^{n-\mu+1}\big(n + \delta(n - \mu)\big) \leq a^n n + \delta\mu\,.$$

Letting δ tend to 0, we would obtain

$$a^{n-\mu+1}n \le a^n n\,.$$

This holds if and only if $a^{n-\mu+1} \le a^n$, which is not true for any μ in $\{2,\ldots,n-1\}$ since $a \in (0\,,\,1)$.

The proof of Proposition 1 ($\equiv$ Lemma 3 in the paper of Dewan and Hans [2]) makes use of Lemma 2 in [2], which in turn uses the the following invalid statement presented by them as Lemma 1.

Proposition 2.   *If $p(z) = c_0 z^n + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| < k$, $k \ge 1$, then for $|z| = 1$*

(8) $$k^{n+\mu-3}\max_{|z|=1}|q'(z)| \le \max_{|z|=1}|p'(k^2 z)|\,,$$

where $q(z) := z^n \overline{p\left(\frac{1}{\bar z}\right)}$.

In order to see that Proposition 2 ($\equiv$ Lemma 1 in [2]) is false, let us take any $b > 1$ and consider the polyomial $p(z) := b^n + \delta z^\mu + z^n$, where δ is positive and small. The polynomial p all its zeros in $|z| < k$, where $|b - k| \to 0$ as $\delta \to 0$. We have

$$p'(k^2 z) = nk^{2n-2}z^{n-1} + \delta\mu k^{2\mu-2}z^{\mu-1} \ \text{ and } \ q'(z) = nb^n z^{n-1} + \delta(n-\mu)z^{n-\mu-1}\,.$$

Clearly then

$$\max_{|z|=1}|p'(k^2 z)| = nk^{2n-2} + \delta\mu k^{2\mu-2} \ \text{ and } \ \max_{|z|=1}|q'(z)| = nb^n + \delta(n-\mu)\,.$$

Hence, if Proposition 2 was true then from (8) we would obtain

$$k^{n+\mu-3}\big(nb^n + \delta(n-\mu)\big) \le nk^{2n-2} + \delta\mu k^{2\mu-2}\,.$$

Letting δ tend to 0, we would obtain $b^2 \le b^{3-\mu}$, which is not true for any $\mu \in \{2,\ldots,n-1\}$ since $b > 1$.

All the three theorems in [2] are wrong because of a serious mistake in the proof of Proposition 2 ($\equiv$ Lemma 1 in [2]). In particular, inequality (2.4) on page 30 of [2] is wrong. It says:

(2.4) $$k^{n-1}\left|q'\left(\frac{z}{k}\right)\right| \le k^\mu \left|\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu}\right| \quad \text{for} \quad |z| \ge 1\,.$$

They justify this inequality *essentially* as follows. By (2.3), inequality (2.4) holds for $|z| = 1$. We agree with this. Then they seem to consider the function

$$\phi(z) := \frac{k^{n-1}Q'(z)}{k^{\mu}\sum_{\nu=\mu}^{n}\nu c_{\nu}(kz)^{\nu-\mu}}$$

and note that $\sum_{\nu=\mu}^{n}\nu c_{\nu}(kz)^{\nu-\mu} \neq 0$ in $|z| > 1$ and so $\phi(z)$ is holomorphic in $|z| > 1$. We agree with this also. Since $|\phi(z)| \leq 1$ for $|z| = 1$ they think that "by maximum modulus principle, $|\phi(z)| \leq 1$ for $|z| > 1$". Not so fast! The maximim modulus principle, in the case they are in, requires the function to tend to a finite limit as $|z|$ tends to infinity but the function $\phi(z)$ to which the maximim modulus principle is being applied tends to infinity as $|z| \to \infty$ if $\mu \in \{2, \ldots, n-1\}$. Thus (2.4) is not true for $\mu \in \{2, \ldots, n-1\}$.

We wonder how Dewan and Hans could have overlooked the fact that in inequality (1) of Govil as well as in inequality (2) of Aziz and Ahmad, equality holds for $p(z) := z^n + k^n$, which is a polynomial of the form $p(z) := c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$. To think that they could improve upon (1) and (2), by considering polynomials which are of the form $p(z) := c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, was a hopeless idea to start with. They could do something better than Theorems A and B only if they considered a class of polynomials which did not contain $z^n + k^n$. In fact, there is no *raison d' ètre* for Theorems 1 and 2. Not only their proofs are wrong, these statements are simply false. The problem with Theorem 3 is of a different nature; namely, its proof uses Lemma 3, which is faulty.

Erroneous though they are, Theorems 1, 2, 3 and Lemmas 1, 2, 3 appearing in [2], have been used, as if they were "true", to draw conclusions, which cannot be taken seriously. For example in [6], the authors use Lemma 3 from [2] (quoted as Lemma 2.1 in [6]) to generalize Theorem 3 of Dewan and Hans from [2]. In [3], Dewan and Ahuja generalize Theorem 3 from [2] to polar derivatives; like Pukhta, Mir and Raja, they use Lemma 3 from [2] in their proof. One can easily construct counter-examples to the result which Dewan and Ahuja think they have proved in [3]. What a terrible waste of time to "generalize" something that is invalid to startwith!

As indicated above, there are people who have already lost time applying and generalizing Theorems 1, 2, 3 and Lemmas 1, 2, 3 of [2]. So, it seems desirable that the truth about these *so-called* theorems and lemmas be made known.

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Département de Mathématiques et de Statistique   *Received: July 17, 2013*
Université de Montréal
Montréal, Québec   H3C 3J7
CANADA
E-mail: hachani@dms.umontreal.ca