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Sensor Location Problem for a Multigraph

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We introduce sparse linear underdetermined systems with embedded network structure. Their structure is inherited from the non-homogeneous network flow programming problems with nodes of variable intensities. One of the new applications of the researched underdetermined systems is the sensor location problem (SLP) for a multigraph. That is the location of the minimum number of sensors in the nodes of the multigraph, in order to determine the arcs flow volume and variable intensities of nodes for the whole multigraph. Research of the rank of the sparse matrix is based on the constructive theory of decomposition of sparse linear systems.

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1 Introduction

Let $G = (I, U)$ be a finite oriented connected multigraph without loops with set of nodes I and set of arcs U , $|U| \gg| I |$. Let $K(|K| < \infty)$ be a set of different types of flow transported through the network G. We assume that $K = \{1, \cdots, |K|\}$. Let us denote a connected network corresponding to a certain type of flow $k \in K$: $G^k = (I^k, U^k)$, $I^k \subseteq I$, $U^k = \{(i,j)^k : (i,j) \in \widehat{U}^k\}$, $\widehat{U}^k \subseteq U$ – a set of arcs of the network G carrying the flow of type $k \in K$. Also, we define for each multiarc $(i,j) \in U$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$

of types of flow transported through the multiarc (i, j). Consider the following sparse linear underdetermined system

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} x_{ji}^k = \begin{cases} a_i^k, & i \in I^k \backslash I_k^*, \\ x_i^k \cdot \mathrm{sign}[i], & i \in I_k^*, k \in K; \end{cases} \tag{1}$$

$$\sum_{k\in K} \sum_{(i,j)^k\in U^k} \lambda_{ij}^{k,p} x_{ij}^k + \sum_{k\in K} \sum_{i\in I_k^*} \lambda_i^{k,p} x_i^k = \beta_p, \quad \text{for} \quad p = \overline{1, q}, \tag{2}$$

where $I_i^+(U^k) = \{j \in I^k : (i, j)^k \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j, i)^k \in U^k\}$, x_{ij}^k – the flow along the arc $(i, j)^k$. Nodes $i^k \in I_k^*$ (further i), $k \in K$ are named dynamic (or nodes with variable intensities x_i^k), $sign[i^k] = 1$, if $i^k \in I_{k+}^*$, $sign[i^k] = -1$, if $i^k \in I_{k-}^*, I_{k+}^*, I_{k-}^* \subseteq I_k^*, I_{k+}^* \bigcap I_{k-}^* = \oslash$, a_i^k, $\lambda_{ij}^{k,p}$, $\lambda_i^{k,p}$ β_p – rational numbers.

The matrix of system (1) – (2) has the following block structure::

$$A = \begin{bmatrix} M & B \\ Q & T \end{bmatrix}$$

Here M is a sparse matrix with a block-diagonal structure of size $\sum_{k\in K} |I^k| \times \sum_{k\in K} |U^k|$ such that each block represents a $|I^k| \times |U^k|$ incidence matrix of the network $S^k = (I^k, U^k)$, $k \in K$, namely, $M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_{|K|}$, where $M_k, k = 1, \ldots, |K|$ are blocks of matrix M; Q is a $q \times \sum_{k\in K} |U^k|$ matrix (dense, in the general case) with elements $\lambda_{ij}^{k,p}, (i, j) \in U$,$k \in K(i, j)$,$p = \overline{1, q}$. Matrix B is a sparse matrix with a block-diagonal structure of size $\sum_{k\in K} |I^k| \times \sum_{k\in K} |I_k^*|$ such that each block represents a $|I^k| \times |I_k^*|$ matrix of the network $S^k = (I^k, U^k)$, $k \in K$, namely, $B = B_1 \bigoplus B_2 \bigoplus \cdots \bigoplus B_{|K|}$, where $B_k, k = 1, \ldots, |K|$ are blocks of matrix B. Each matrix B_k has in column j a nonzero element equal to $-\mathrm{sign}[i]$ in raw i for each $j \in I_k^*$ with other elements of the column being zeroes, $i \in I^k$, $k \in K$. T is a matrix of size $q \times \sum_{k\in K} |I_k^*|$ and consists of elements $\lambda_i^{k,p}$ for $i \in I_k^*, k \in K$, $p = \overline{1, q}$. In [1], [3] we considered sparse linear systems with embedded network structure. Their structure was inherited from the homogeneous network flow programming problems with nodes of variable intensity.

In [4] sparse linear systems for the fractal-like matrices were investigated. In this paper we consider algorithms for decomposition of sparse systems for a multigraph, which are used in the sensor location problem.

We assume that

$$\sum_{k\in K}|I^k| + q < \sum_{k\in K}|U^k| + \sum_{k\in K}|I^*_k|.$$

Theorem 1 *The rank of the matrix* $\begin{bmatrix} M_k & B_k \end{bmatrix}$ *of system (1) for fixed* $k \in K$ *for a connectivity graph* $G^k = (I^k, U^k)$, $I^*_k \neq \emptyset$, *is equal to* $|I^k|$ *[1].*

The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc are for fixed $k \in K$ constructed according to the rules [1].

Theorem 2 *The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc for fixed* $k \in K$ *satisfy the system (3) [1, 2].*

$$\sum_{j\in I_i^+(U^k)} x^k_{ij} - \sum_{j\in I_i^-(U^k)} x^k_{ji} = \begin{cases} 0, & i \in I^k \backslash I^*_k, \\ x^k_i \cdot \mathrm{sign}[i], & i \in I^*_k, k \in K. \end{cases} \tag{3}$$

Theorem 3 *Any solution of system (3) for fixed* $k \in K$ *is a linear combination of characteristic vectors [1].*

Definition 1 *We call an aggregate of sets* $R = \{U^k_R, I^{*k}_R, k \in K\}, U^k_R \subseteq U^k$ *and* $I^{*k}_R \subseteq I^*_k$ *a support of multigraph* G *for system (1) if for* $\widetilde{R} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}, \widetilde{U}^k = U^k_R, \widetilde{I}^*_k = I^{*k}_R$ *the system*

$$\sum_{j\in I_i^+(\widetilde{U}^k)} x^k_{ij} - \sum_{j\in I_i^-(\widetilde{U}^k)} x^k_{ji} = \begin{cases} 0, & i \in I^k \backslash \widetilde{I}^*_k \\ x^k_i \cdot \mathrm{sign}[i], & i \in \widetilde{I}^*_k, k \in K \end{cases} \tag{4}$$

has only a trivial solution, but has a nontrivial solution for any of the following set aggregations:

- $\widetilde{R} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}$, $\widetilde{U}^{k_0} = U^{k_0}_R \bigcup (i_0, j_0)^{k_0}$, *for* $(i_0, j_0)^{k_0} \in U^{k_0} \backslash U^{k_0}_R$; $\widetilde{U}^k = U^k_R$ *for* $k \in K \setminus k_0$ *and* $\widetilde{I}^*_k = I^{*k}_R$, $k \in K$;

- $\widetilde{R} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}$, $\widetilde{U}^k = U^k_R$, *for* $k \in K$ *and* $\widetilde{I}^*_{k_0} = I^{*k_0}_R \bigcup \{i_0\}$, $i_0 \in I^*_{k_0} \backslash I^{*k_0}_R$; $\widetilde{I}^*_k = I^{*k}_R$ $k \in K \setminus k_0$.

For a subset of arcs $U_1 \subseteq U$ we introduce the set of incidental nodes $I(U_1) = \{i \in I : (i,j) \in U_1 \vee (j,i) \in U_1\}$. We construct for every $k \in K$ a forest from $\widetilde{t}_k$ trees $T^{k,t_k} = \{I^{k,t_k}_T, U^{k,t_k}_T\}, I^{k,t_k}_T = I(U^{k,t_k}_T)$, $U^{k,t_k}_T \subseteq U^k_R$, $t_k = \overline{1, \widetilde{t}_k}$ so that every tree T^{k,t_k} contains exactly one node $u_{t_k} \in {I^*_R}^k$, for $t_k = \overline{1, \widetilde{t}_k}$, $k \in K$. For each $k \in K$ let's form the following sets:

$$U^k_R = \bigcup_{t_k=1}^{\widetilde{t}_k} U^{k,t_k}_T, \quad {I^*_R}^k = \bigcup_{t_k=1}^{\widetilde{t}_k} \{u_{t_k}\}.$$

Theorem 4 ***(Network Support Criterion)*** *An aggregate of sets* $R = \{U^k_R, {I^*_R}^k, k \in K\}, U^k_R \subseteq U^k$*, and* ${I^*_R}^k \subseteq I^*_k$ *is a support of the multi graph G for System (1) if and only if for each* $k \in K$ *the following conditions are carried out:*

- *Each coherence component* $T^{k,t_k} = \{I(U^{k,t_k}_T), U^{k,t_k}_T\}$ *for* $t_k = \overline{1, \widetilde{t}_k}$ *is a tree;*
- *The set of the nodes of the collection* $T^{k,t_k} = \{I(U^{k,t_k}_T), U^{k,t_k}_T\}$ *for* $t_k = \overline{1, \widetilde{t}_k}$ *covers all nodes of the graph* $G^k = (I^k, U^k)$:
$$\bigcup_{t_k=1}^{\widetilde{t}_k} I^{k,t_k}_T = I^k;$$
- $|I^{*k}_R \bigcap I(U^{k,t_k}_T)| = 1$, $t_k = \overline{1, \widetilde{t}_k}$.

After the support $R = \{U^k_R, {I^*_R}^k, k \in K\}$ of system (1) is chosen, we determine what structures can be obtained after adding one non-supporting element to the support R.

Definition 2 *The* ***characteristic vector entailed by an arc*** $(\tau, \rho)^k \in U^k \backslash U^k_R$ *is the vector* $\delta^k(\tau, \rho) = (\delta^k_{ij}(\tau, \rho), (i,j)^k \in U^k; \delta^k_i(\tau, \rho), i \in I^*_k)$ *constructed according to the following rules for fixed k:*

- *If the set* $U^k_R \bigcup \{(\tau, \rho)^k\}$ *has a cycle* $L_k = \{I^k_L, U^k_L\}$*, then the entailed characteristic vector is the characteristic vector of that cycle, and the arc* $(\tau, \rho)^k$ *is chosen to define the detour direction of the cycle.*
- *If the set* $U^k_R \bigcup \{(\tau, \rho)^k\}$ *has a chain* $C_k = \{I^k_C, U^k_C\}$ *that connects nodes* $u, v \in I^{*k}_R$*, then the entailed characteristic vector is the characteristic vector of that chain, and the arc that defines the detour direction is chosen to be* $(\tau, \rho)^k$.

Definition 3 *The* ***characteristic vector entailed by a node*** $\gamma \in I_k^* \backslash I_R^{*k}$ *is the characteristic vector* $\delta^k(\gamma) = (\delta_{ij}^k(\gamma), (i,j)^k \in U^k; \delta_i^k(\gamma), i \in I_k^*)$ *of the chain for fixed* k *that connects nodes* γ *and* $v \in I_R^{*k}$ *with node* γ *being chosen as the beginning of the chain.*

Theorem 5 *The general solution of system (1) for fixed* $k \in K$ *may be uniquely represented in the following way:*

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \backslash U_R^k} x_{\tau,\rho}^k \delta_{ij}^k(\tau,\rho) + \sum_{\gamma \in I_k^* \backslash I_R^{*k}} x_\gamma^k \delta_{ij}^k(\gamma) + \widetilde{x}_{ij}^k, \quad \text{for} \quad (i,j)^k \in U_R^k; \quad (5)$$

$$x_i^k = \sum_{(\tau,\rho)^k \in U^k \backslash U_R^k} x_{\tau,\rho}^k \delta_i^k(\tau,\rho) + \sum_{\gamma \in I_k^* \backslash I_R^{*k}} x_i^k \delta_i^k(\gamma) + \widetilde{x}_i^k, \text{ for } i \in I_R^{*k} \quad (6)$$

where $\widetilde{x}^k = (\widetilde{x}_{ij}^k, (i,j)^k \in U^k, \widetilde{x}_i^k, i \in I_k^*)$ is a partial solution of the nonhomogeneous system.

The proof of Theorem 5 for fixed $k \in K$ is given [1, 3], where the general solution of the nonhomogeneous system (1) is the sum of the general solution of the homogeneous system and a partial solution of the nonhomogeneous system.

Remark. The formulas (5) and (6) are correct, if the partial solution $\widetilde{x}^k = (\widetilde{x}_{ij}^k, (i,j)^k \in U^k, \widetilde{x}_i^k, i \in I_k^*)$ for fixed $k \in K$ is constructed according to the rules:

$$\widetilde{x}_{\tau\rho}^k = 0, (\tau,\rho) \in U^k \setminus U_R^k, \widetilde{x}_\gamma^k = 0, \gamma \in I_k^* \setminus I_R^{*k}$$

and solves system (1).

Further, we shall use formulas (5) and (6) where the partial solution $\widetilde{x}^k, k \in K$ is constructed according to the above rules.

2 Support of the Graph

Let $R = \{U_R^k, I_R^{*k}, k \in K\}, U_R^k \subseteq U^k$, $I_R^{*k} \subseteq I_k^*$ be a support of the multi graph $G = \{I, U\}$ of system (1). In arbitrary order, we choose sets $W = \{U_W^k, I_W^{*k}, k \in K\}, |W| = q$, $U_W^k \subseteq U^k \setminus U_R^k$ and $I_W^{*k} \subseteq I_k^* \setminus I_R^{*k}$. After substituting the general solution of system (1), which has the form (5) – (6), into (2), the system (2) takes the form:

$$\sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \backslash U_R^k} \Lambda_{\tau\rho}^{k,p} x_{\tau\rho}^k + \sum_{k \in K} \sum_{\gamma \in I_k^* \backslash I_R^{*k}} \Lambda_\gamma^{k,p} x_\gamma^k = A_p \quad p = \overline{1,q}, \quad (7)$$

where

$$\Lambda^{k,p}_{\tau\rho} = \lambda^{k,p}_{\tau\rho} + \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \delta^k_{ij}(\tau,\rho) + \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \delta^k_i(\tau,\rho),$$

$$\Lambda^{k,p}_{\gamma} = \lambda^{k,p}_{\gamma} + \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \delta^k_{ij}(\gamma) + \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \delta^k_i(\gamma),$$

$$A_p = \beta_p - \sum_{k \in K} \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \widetilde{x}^k_{ij} - \sum_{k \in K} \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \widetilde{x}^k_i .$$

In system (7), we separate variables that correspond to set W and then we obtain (8).

$$\sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k_W} \Lambda^{k,p}_{\tau\rho} x^k_{\tau\rho} + \sum_{k \in K} \sum_{\gamma \in I^{*k}_W} \Lambda^{k,p}_{\gamma} x^k_{\gamma} = \tag{8}$$

$$= A_p - \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus (U^k_W \cup U^k_R)} \Lambda^{k,p}_{\tau\rho} x^k_{\tau\rho} - \sum_{k \in K} \sum_{\gamma \in I^*_k \setminus (I^{*k}_W \cup I^{*k}_R)} \Lambda^{k,p}_{\gamma} x^k_{\gamma}$$

for $p = \overline{1,q}$.

3 Theoretical-Graphical Properties

Definition 4 *We call a support of multigraph G for system (1) – (2) such an aggregate of sets $Z = \{U^k_Z, {I^*_Z}^k, k \in K\}, U^k_Z \subseteq U^k$ and ${I^*_Z}^k \subseteq I^*_k$, that for a given $\widetilde{Z} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}, \widetilde{U}^k = U^k_Z, \widetilde{I}^*_k = {I^*_Z}^k$ the system*

$$\sum_{j \in I^+_i(\widetilde{U}^k)} x^k_{ij} - \sum_{j \in I^-_i(\widetilde{U}^k)} x^k_{ji} = \begin{cases} 0, & i \in I^k \setminus \widetilde{I}^*_k, \\ x^k_i \cdot \mathrm{sign}[i], & i \in \widetilde{I}^*_k, k \in K; \end{cases} \tag{9}$$

$$\sum_{k \in K} \sum_{(i,j)^k \in \widetilde{U}^k} \lambda^{k,p}_{ij} x^k_{ij} + \sum_{k \in K} \sum_{i \in \widetilde{I}^*_k} \lambda^{k,p}_i x^k_i = 0, \; for \quad p = \overline{1,q}$$

has only a trivial solution. But it has a nontrivial solution for any of the following aggregations of sets:

- $\widetilde{Z} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}, \quad \widetilde{U}^{k_0} = U^{k_0}_Z \bigcup (i_0, j_0)^{k_0}$ *for* $(i_0, j_0)^{k_0} \in U^{k_0} \setminus U^{k_0}_Z$; $\widetilde{U}^k = U^k_Z$ *for* $k \in K \setminus k_0$ *and* $\widetilde{I}^*_k = I^{*k}_Z, \quad k \in K$;

- $\widetilde{Z} = \{\widetilde{U}^k, \widetilde{I}^*_k, k \in K\}$, $\widetilde{U}^k = U^k_Z$, *for* $k \in K$; $\widetilde{I}^*_{k_0} = I^{*k_0}_Z \bigcup \{i_0\}$, $i_0 \in I^*_{k_0} \backslash I^{*k_0}_Z$ *and* $\widetilde{I}^*_k = I^{*k}_Z$, $k \in K \setminus k_0$.

Theorem 6 *The aggregation of sets* $Z = \{U^k_Z, {I^*_Z}^k, k \in K\}, U^k_Z \subseteq U^k$ *and* ${I^*_Z}^k \subseteq I^*_k$ *is a support of multigraph* $G = \{I, U\}$ *for system (1) – (2) if and only if*

- *the aggregation of sets* $Z = \{U^k_Z, {I^*_Z}^k, k \in K\}$ *may be divided into two aggregations:* $R = \{U^k_R, {I^*_R}^k, k \in K\}$ *and* $W = \{U^k_W, {I^*_W}^k, k \in K\}$*, such that* $R \bigcup W = Z$*,* $R \bigcap W = \emptyset$ *and the aggregation of sets* R *is a support of the multigraph* $G = \{I, U\}$ *for system (1);*
- $|W| = q$*, where* q *is the number of independent equations in system (2);*
- *matrix* D *of the system (8), which consists of determinants* $\Lambda^p_{\tau\rho}$*,* Λ^p_γ *of the structures entailed by the arcs and nodes of the aggregation* W*, is nondegenerate.*

We now investigate theoretical-graphical properties of the structure of the support of multigraph $G = \{I, U\}$ for system (1) – (2). According to Theorem 6, the aggregation of sets $Z = \{U^k_Z, {I^*_Z}^k, k \in K\}$ includes the support $R = \{U^k_R, {I^*_R}^k, k \in K\}$ of multigraph G for system (1). Supporting elements that correspond to the aggregate R make up a forest of trees that covers all the nodes of the set I^k, for each $k \in K$ and every tree of the forest has exactly one node from the set ${I^*_R}^k$. We make a cycle or a chain after adding each additional element from $W = \{U^k_W, {I^*_W}^k, k \in K\}$ or $N = \{U^k_N, {I^*_N}^k, k \in K\}$, $U^k_N = U^k \setminus (U^k_R \bigcup U^k_W), {I^*_N}^k = I^*_k \setminus ({I^*_R}^k \bigcup {I^*_W}^k)$ to the elements of the set $R = \{U^k_R, {I^*_R}^k, k \in K\}$.

4 Sensor Location Problem

Let's consider the finite connected directed multigraph $G = (I, U)$. We assume, that for a fixed $k \in K$ the graph $G^k = (I^k, U^k)$ is symmetric – that is: if $(i, j)^k \in U^k$, then $(j, i)^k \in U^k$. We note that the graph G^k is not undirected: the flow on arc $(i, j)^k$, in general, will not be the same as the flow on arc $(j, i)^k$. To designate this distinction, we refer to the graph $G^k = (I^k, U^k)$ as a two way directed graph.

We represent the traffic flow by a network flow function $x : U \to \mathbb{R}$ that satisfies the following system:

$$\sum_{j \in I^+_i(U^k)} x^k_{i,j} - \sum_{j \in I^-_i(U^k)} x^k_{j,i} = \begin{cases} x^k_i, & i \in I^*_k, \\ 0, & i \in I^k \backslash I^*_k, \quad k \in K, \end{cases} \tag{10}$$

where $I_k^* \subseteq I^k, k \in K$ is the set of nodes with variable intensities, x_i^k is the variable intensity of node $i \in I_k^*$. If the variable intensity x_i^k of node i is positive, the node i is a source; if it is negative, this node i is a sink. For system (10) the following condition is true: $\sum\limits_{i \in I^k} x_i^k = 0$ for fixed $k \in K$. According to Theorem 1 if $I_k^* \neq \emptyset$, then the rank of the matrix of system (10) for a connectivity graph $G^k = (I^k,\, U^k)$ for fixed $k \in K$ is equal to $|I^k|$.

In order to obtain information about the variables x_{ij}^k for the arcs $(i,j)^k \in U^k$ and variable intensities x_i^k of nodes $i \in I_k^*$, $k \in K$ sensors are placed at the nodes. The nodes in the graph with sensors we call monitored ones and denote the set of monitored nodes M where $M = \bigcup_{k \in K} M_k, M_k \subseteq I^k, k \in K$. We assume that if a node i^k (further i) is monitored, we know the values of flows on all outgoing and all incoming arcs for the node $i \in M_k$:

$$x_{ij}^k = f_{ij}^k,\, j \in I_i^+(U^k), \quad x_{ji}^k = f_{ji}^k,\, j \in I_i^-(U^k), \quad i \in M_k, \quad k \in K.$$

If the set M includes the nodes from the set I_k^*, then we know the values of flows for all incoming and outgoing arcs for the nodes of the set M and we know also the values

$$x_i^k = f_i^k,\ i \in M_k \bigcap I_k^*, \quad k \in K.$$

Consider any node $i \in I^k$ of the network $G^k = (I^k, U^k)$. For every outgoing arc $(i,j)^k \in U^k$ for this node i let's determine a real number $p_{ij}^k \in (0,1]$ which denotes the corresponding part of the total outgoing flow $\sum\limits_{j \in I_i^+(U^k)} x_{ij}^k$ from i which leaves along this arc $(i,j)^k$. That is, $x_{ij}^k = p_{ij}^k \sum\limits_{j \in I_i^+(U^k)} x_{ij}^k$.

If $|I_i^+(U^k)| \geq 2$ for node $i \in I^k$ then we can write the flow along all outgoing arcs from node i in terms of a single outgoing arc, for example, $(i, v_i)^k, v_i \in I_i^+(U^k)$:

$$x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} x_{i,v_i}^k, \quad j \in I_i^+(U^k) \setminus v_i. \tag{11}$$

This process continues for each node $i \in I^k$, if $|I_i^+(U^k)| \geq 2$, $k \in K$.

So we shall formulate the sensor location problem: **what is the minimum number of monitored nodes**

$$M = \bigcup_{k \in K} M_k$$

such that system (10) has an unique solution?

For the case when the system of the linear equations for the traffic is homogeneous, combinatory properties of algorithm for the solution of the sensor location problem are considered in [5].

Let's substitute the flows on all outgoing and all incoming arcs for the nodes $M = \bigcup_{k \in K} M_k$:

$$x_{ij}^k = f_{ij}^k,\ j \in I_i^+(U^k), \quad x_{ji}^k = f_{ji}^k,\ j \in I_i^-(U^k), \quad i \in M_k, \quad k \in K$$

to the equations of system (10).

If $|I_i^+(U^k)| \geq 2$ for the node $i \in I^k$ then we can write the flow along all outgoing arcs from node i in terms of a single known outgoing flow arc f_{i,v_i}^k, for example, for the arc $(i, v_i)^k, v_i \in I_i^+(U^k)$, where x_{i,v_i}^k known $x_{i,v_i}^k = f_{i,v_i}^k$:

$$x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} f_{i,v_i}^k, \quad j \in I_i^+(U^k) \setminus v_i. \tag{12}$$

This process continues for each node $i \in I^k$, if $|I_i^+(U^k)| \geq 2$, $k \in K$.

Also, we substitute (12) to the equations of system (10), $i \in I^k, k \in K$. Let's delete from graphs $G^k = (I^k,\, U^k), k \in K$ the set of the arcs and nodes on which the arc flows and values x_i^k are known. Then we have a new multigraph $\overline{G} = (\overline{I},\, \overline{U})$, which consists of the set of graphs $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$, $k \in K$, where $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$ is, in general, a disconnected graph $\overline{G}^k$, corresponding to a certain type of flow $k \in K$. We denote for each multiarc $(i,j) \in \overline{U}$ of multigraph $\overline{G}$ the set $\overline{K}(i,j) = \{k \in K : (i,j)^k \in \overline{U}^k\}$ of types of flow transported through a multiarc (i,j).

The new multigraph $\overline{G}$ consists of components of connectivity. Some components of connectivity could not contain nodes of the set $\overline{I}_k^*$, where $\overline{I}_k^*$ is the set of nodes with variable intensities of graph $\overline{G}^k, k \in K$.

If $|I_i^+(\overline{U}^k)| \geq 2$ for the node $i \in \overline{I}^k$ then we can write the flow along all outgoing arcs from node i in terms of a single unknown outgoing flow along arc x_{i,v_i}^k, for example, for the arc $(i, v_i)^k$, where x_{i,v_i}^k is unknown flow:

$$x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} x_{i,v_i}^k, \quad j \in I_i^+(\overline{U}^k) \setminus v_i, \quad i \in \overline{I}^k, k \in K. \tag{13}$$

The system (10) and (13) for multigraph $\overline{G} = (\overline{I}, \overline{U})$ will be the following one:

$$\sum_{j \in I_i^+(\overline{U}^k)} x_{i,j}^k - \sum_{j \in I_i^-(\overline{U}^k)} x_{j,i}^k = \begin{cases} x_i^k + b_i, & i \in \overline{I}_k^*, \\ a_i^k, & i \in \overline{I}^k \setminus \overline{I}_k^*, k \in K \end{cases} \tag{14}$$

$$\sum_{(i,j) \in \overline{U}} \sum_{k \in \overline{K}(i,j)} \overline{\lambda}_{ij}^{k,p} x_{ij}^k = 0, \ p = \overline{1, \ q}, \tag{15}$$

where a_i, b_i, $\overline{\lambda}_{ij}^{k,p}$ – are constants.

Let's state the steps of the algorithm for modelling of the set $\bigcup_{k \in K}(I^k \setminus M_k^*)$ for the given set $M = \bigcup_{k \in K} M_k$ for the new multigraph $\overline{G}$.

Step 1. Construct cuts $\bigcup_{k \in K} CC(M_k)$ for the set of monitored nodes $M = \bigcup_{k \in K} M_k$.

Step 2. Find the nodes of the set $I(CC(M)) = I(\bigcup_{k \in K} CC(M_k))$.

Step 3. Construct the set $M_k^+ = I(\bigcup_{k \in K} CC(M_k)) \setminus M_k, k \in K$.

Step 4. Form sets $M_k^* = M_k \bigcup M_k^+$ and $I^k \setminus M_k^*, k \in K$.

The part of the unknowns of the system (14) – (15) are the flows for outgoing arcs from the nodes of the set $I^k \setminus M_k^*, k \in K$. Also the unknowns in the system (14) – (15) are the variable intensities x_i^k, where $i \in \overline{I}_k^*, k \in K$ for the new multigraph $\overline{G} = (\overline{I}, \overline{U})$. The new multigraph $\overline{G}$ consists of components of connectivity. If the fixed component of connectivity of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ contains nodes of the set $\overline{I}_k^*$, then according to Theorem 1, Theorem 4 computes the rank of the matrix of system (14) since system (14) – (15) is a private case of the system (1) – (2) for that component of connectivity. If a fixed component of connectivity of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ doesn't include the nodes of the set $\overline{I}_k^*$, we use the theory of decomposition [2] for that component of connectivity.

The system (14) – (15) has an unique solution for the given set M if and only if the rank of the matrix of system (14) – (15) is equal to the number of unknowns of the system (14) – (15). For computing the rank of the matrix of system (14) – (15) we use theoretical-graphical properties of the structure of the support according to Theorem 6.

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