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## PREDEGREE POLYNOMIALS OF PLANE CONFIGURATIONS IN PROJECTIVE SPACE

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*Communicated by V. Kanev*

ABSTRACT. We work over an algebraically closed field of characteristic zero. The group  $PGL(4)$  acts naturally on  $\mathbb{P}^N$  which parameterizes surfaces of a given degree in  $\mathbb{P}^3$ . The orbit of a surface under this action is the image of a rational map  $PGL(4) \subset \mathbb{P}^{15} \dashrightarrow \mathbb{P}^N$ . The closure of the orbit is a natural and interesting object to study. Its *predegree* is defined as the degree of the orbit closure multiplied by the *degree* of the above map restricted to a general  $\mathbb{P}^j$ ,  $j$  being the dimension of the orbit. We find the predegrees and other invariants for all surfaces supported on unions of planes. The information is encoded in the so-called *predegree polynomials*, which possess nice multiplicative properties allowing us to compute the predegree (polynomials) of various special plane configurations.

The predegree has both combinatorial and geometric significance. The results obtained in this paper would be a necessary step in the solution of the problem of computing predegrees for all surfaces.

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2000 *Mathematics Subject Classification*: 14N10, 14C17.

*Key words*: Planes, hyperplanes, arrangements, configurations.

**1. Introduction.** The group  $PGL(3)$  of projective linear transformations of  $\mathbb{P}^2$  acts naturally on the space  $\mathbb{P}^{d(d+3)/2}$  parameterizing plane curves of degree  $d$ . Aluffi and Faber [1] have computed the degree of the closure in this space of the orbit of an arbitrary plane curve. The orbit closure of a curve is a natural object of study and its degree has a simple enumerative meaning: for a reduced curve with finite stabilizer, it counts the translates of the curve that contain eight given general points. The case of curves supported on unions of lines is of special separate interest [2, 3]. The constructions in this case were essential for the computations in the general case.

We have a similar situation for surfaces in  $\mathbb{P}^3$ . We use a projective space  $\mathbb{P}^N$  to parameterize degree  $d$  surfaces in  $\mathbb{P}^3$ . The group  $PGL(4)$  acts naturally on  $\mathbb{P}^N$  and we can ask the same question of computing the degree of the closure in this space of the orbit of a surface. It still has a similar geometric meaning: for a reduced surface with finite stabilizer, it counts the translates of the surface that contain fifteen given general points. More generally, if  $j$  is the dimension of the closure of the orbit of a reduced surface, then the degree of the orbit is the number of translates containing  $j$  points in general position. In very special cases (for example when the surface consists of 3 planes in general position) this number can be computed by naive combinatorial considerations. In general this is not possible.

We consider the case of surfaces supported on unions of planes, i.e. plane configurations.

The main problem of finding the *predegree polynomial* of a plane configuration in  $\mathbb{P}^3$  evolves from the original one of finding the degree of the closure of the linear orbit of the configuration in  $\mathbb{P}^N$ .

The action of  $PGL(4)$  defines a regular map  $s : PGL(4) \rightarrow \mathbb{P}^N$  for a fixed plane configuration  $S$  in  $\mathbb{P}^3$ . The orbit of  $S$  is  $Im(s)$ . This map extends to a rational map  $s$  from the compactification  $\mathbb{P}^{15}$  of  $PGL(4)$  to  $\mathbb{P}^N$ .

The predegree polynomial of  $S$  is  $\sum_{i \geq 0} (f_i d_i) t^i / i!$ , where  $f_i = \text{degs}_{|\mathbb{P}^i} =$  the number of points in a general fiber of  $s|_{\mathbb{P}^i}$  and  $d_i$  is the intersection number of  $\overline{s(\mathbb{P}^i)}$  and a codimension  $i$  linear subspace of  $\mathbb{P}^N$  for a general  $\mathbb{P}^i$  in  $\mathbb{P}^{15}$ .

The leading coefficient  $f_j d_j$  is called the *predegree* of (the orbit of)  $S$ . This information records the degree  $d_j$  of the orbit closure of  $S$ , corrected by the “size” of the stabilizer of the configuration. As such, the predegree retains an enumerative interpretation similar to the one of the degree of the orbit closure of a reduced  $S$ : it counts the *ordered* (as opposed to non-ordered) plane configurations in the orbit of  $S$  which pass through  $j$  general points.

The main tool in our computation of the predegree polynomial of a plane

configuration  $S$  is a series of blow-ups over  $\mathbb{P}^{15}$ . We construct a regular map  $\tilde{s} : \tilde{V} \rightarrow \mathbb{P}^N$  which lifts the rational map  $s$  (to the non-singular variety  $\tilde{V}$  dominating the orbit closure of  $S$ ) and resolves its indeterminacies. An intersection-theoretic analysis of this map allows us to compute the coefficients of the predegree polynomial for  $S$ .

We are interested in the whole predegree polynomial rather than the individual *predegrees*, the coefficients of the polynomial. As it carries more invariants of the orbit closure, the predegree polynomial allows us to solve a more refined degree problem and deal at once with all configurations of planes.

We find some non-obvious properties of the predegree polynomials. They endow the whole theory with a non-trivial structure, which would be interesting to study further. As an application, simple formulae will be found for specific arrangements of planes.

Most of the facts we use can be found in [6, 7]. For a detailed exposition of the material of this paper refer to [10].

## 2. Geometry.

**2.1. Orbit.** Let  $\mathbb{P}^3$  be the three dimensional projective space over a fixed algebraically closed field  $K$  of characteristic zero with projective coordinates  $(x : y : z : w)$ . We use  $\mathbb{P}^N = \mathbb{P}^{d(d^2+6d+11)/6}$  to parameterize degree  $d$  surfaces in  $\mathbb{P}^3$ .

The group  $PGL(4)$  of  $4 \times 4$  matrices over  $K$  acts on  $\mathbb{P}^N$  in a natural way.

Let  $\phi = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_4 & \phi_5 & \phi_6 & \phi_7 \\ \phi_8 & \phi_9 & \phi_{10} & \phi_{11} \\ \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} \end{pmatrix} \in PGL(4)$  and a surface  $S$  (as an element of

$\mathbb{P}^N$ ) be defined by  $F(x : y : z : w) = 0$ . Then we define the action of  $\phi$  on  $S$ , denoted  $S \circ \phi$ , to be the surface with equation  $F(\phi(x : y : z : w)) = 0$ .

Let us fix a surface  $S$  and define a regular map  $s$  from  $PGL(4)$  to  $\mathbb{P}^N$  which is induced by the  $PGL(4)$  action on  $S$ . More precisely,

$$s : PGL(4) \longrightarrow \mathbb{P}^N, \phi \mapsto S \circ \phi$$

The image  $Im(s)$  of the map  $s$  is called the *linear orbit* of the surface  $S$ . It is denoted by  $\mathcal{O}_S$  and consists of all *translates* of  $S$ .

We embed  $PGL(4)$  as an open set in  $\mathbb{P}^{15}$  with coordinates  $(\phi_0 : \dots : \phi_{15})$ .

This embedding  $PGL(4) \subset \mathbb{P}^{15}$  extends the regular map  $s$  to a rational map  $s : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^N$ . Then the orbit closure  $\overline{\mathcal{O}_S}$  of  $S$  is the closure of the image of this rational map  $s$ .

Note that if  $Stab_S$  is the stabilizer of  $S$  under the  $PGL(4)$  action, then  $\dim \mathcal{O}_S + \dim Stab_S = 15$ .

From now on the surface  $S$  will be supported on unions of planes, hence  $F(x : y : z : w) = \prod_{i=1}^n L_i^{r_i}(x : y : z : w)$  where  $r_i$  is the multiplicity of the  $i$ -th plane  $P_i$ , the homogeneous polynomial  $L_i$  is the equation of  $P_i$  and  $n$  is the number of planes. Also, the degree of  $S$  will be  $\sum_{i=1}^n r_i = d$ .

One of the most important characteristics of the closure of the linear orbit of  $S$  is its degree  $\deg \overline{\mathcal{O}_S}$ . It possesses the following clear enumerative geometric interpretation:

**Lemma 2.1.1.** *If the dimension of the orbit of a reduced plane configuration is  $j$ , then the degree of the orbit closure equals the number of translates of the plane configuration which pass through  $j$  general points.*

As an application of this lemma, we see that in very special cases this number can be computed by naive combinatorial considerations. These are the cases in which the orbit consists of all configurations with the same “incidence data” as the original configuration. In general this is not possible.

**Example 2.1.2.** Let  $S$  be a configuration consisting of two distinct reduced planes. The orbit consists of all pairs of planes and hence its dimension is 6 (3 “degrees of freedom” for each plane). Thus we need to count how many pairs of planes contain 6 points in general position in  $\mathbb{P}^3$ . If 3 of the points determine a plane, then the other 3 points automatically determine the other plane. Hence the number of pairs of planes going through 6 general points is  $\binom{6}{3} / 2 = 10$ , because  $\binom{6}{3}$  counts each pair twice. Therefore, the degree of the orbit closure  $\deg \overline{\mathcal{O}_S}$  is 10.

**Example 2.1.3.** If  $S$  is a configuration consisting of three general (not going through a line) reduced planes, then the orbit consists of all configurations like the original one, i.e., of configurations of 3 general reduced planes. Then the dimension of the orbit is 9 and like in the previous example we count the number of all such configurations through 9 general points to be  $\binom{9}{3} \binom{6}{3} / 6 = \deg \overline{\mathcal{O}_S} = 280$ .

**Example 2.1.4.** Our first non-combinatorial example is the configuration of four planes going through a line. Indeed, the dimension of the set of all such arrangements is clearly 8, whereas, as we will see in Theorem 5.4, the

dimension of the orbit is 7. In other words, the orbit does not consist of all configurations consisting of four planes with a common line.

**2.2. Point Conditions. Base Locus.** All surfaces in  $\mathbb{P}^3$  going through a fixed point clearly form a hyperplane in  $\mathbb{P}^N$ . Such hyperplanes are called *point conditions*. Each point condition corresponds to a point in  $\check{\mathbb{P}}^N$ , the dual space of  $\mathbb{P}^N$ .

If the set of point conditions were degenerate in  $\check{\mathbb{P}}^N$ , then the point conditions in  $\mathbb{P}^N$ , which are hyperplanes, should all go through one point, which is not true, as this point would correspond to a surface containing all points of  $\mathbb{P}^3$ . So the set of point conditions is non-degenerate. Since the span of a non-degenerate set is a non-degenerate linear subspace of  $\check{\mathbb{P}}^N$ , then it must be the whole space, i.e., we proved

**Lemma 2.2.1.** *The point conditions span  $\check{\mathbb{P}}^N$ .*

We define the *point conditions* in  $\mathbb{P}^{15}$  to be the pull-backs of the point conditions from  $\mathbb{P}^N$ . In these terms then, we state two more useful observations:

**Remark 2.2.2.** The linear system of  $s : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^N$  is spanned by the point conditions in  $\mathbb{P}^{15}$ .

**Remark 2.2.3.** The base locus of  $s$  is the intersection over all points in  $\mathbb{P}^3$  of the point conditions in  $\mathbb{P}^{15}$ .

Now we are ready to describe the base locus of  $s$ .

**Proposition 2.2.4.** *The base locus of  $s$  is a union of  $\mathbb{P}^{11}$ 's which are in a one-to-one correspondence with the planes in the plane configuration  $S$ . Furthermore, all possible intersections of such  $\mathbb{P}^{11}$ 's are  $\mathbb{P}^7$ 's and  $\mathbb{P}^3$ 's which are also in one-to-one correspondences with the geometric intersections of the corresponding planes: lines and points of intersection in  $S$  respectively.*

**Proof.** Since a point condition corresponding to a point  $p$  has an equation  $F(\phi(p)) = 0$  in  $\mathbb{P}^{15}$ , the base locus is  $\{\phi \in \mathbb{P}^{15} / F(\phi(x : y : z : w)) \equiv 0\}$ . More explicitly, it is  $\bigcup_{i=1}^n \{\phi \in \mathbb{P}^{15} / L_i(\phi(x : y : z : w)) \equiv 0\}$ .

Consider any component of this union. It consists of all  $\phi = (\phi_0 : \phi_1 : \dots : \phi_{15}) = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_4 & \phi_5 & \phi_6 & \phi_7 \\ \phi_8 & \phi_9 & \phi_{10} & \phi_{11} \\ \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} \end{pmatrix}$  for which  $\phi(\mathbb{P}^3)$  is in  $P_i$ . In particular, it

depends on the plane  $P_i$  only (and not the rest of  $S$ ). Without loss of generality we may assume that the equation  $L_i = 0$  of  $P_i$  is  $w = 0$ . Then the corresponding component of the base locus consists of all  $\phi$ 's for which  $\phi_{12}x + \phi_{13}y + \phi_{14}z + \phi_{15}w$

is zero for all points  $(x : y : z : w) \in \mathbb{P}^3$ , i.e., it is  $\{\phi \in \mathbb{P}^{15} / \phi_{12} = \phi_{13} = \phi_{14} = \phi_{15} = 0\}$  which, as an intersection of four distinct hyperplanes, is a  $\mathbb{P}^{11}$  in  $\mathbb{P}^{15}$ . This  $\mathbb{P}^{11}$  is the set of all elements of  $\mathbb{P}^{15}$  which map  $\mathbb{P}^3$  into a linear subspace of the component  $P_i$ .

Clearly, the intersection of any two  $\mathbb{P}^{11}$ 's consists of matrices sending  $\mathbb{P}^3$  into a linear subspace of the line which is the intersection of the two planes corresponding to the two  $\mathbb{P}^{11}$ 's. If we, again without loss of generality, assume that these are the planes  $z = 0$  and  $w = 0$ , we can show that any two  $\mathbb{P}^{11}$ 's intersect in a  $\mathbb{P}^7$ . Indeed, the intersection set is characterized by  $\phi_8x + \phi_9y + \phi_{10}z + \phi_{11}w \equiv \phi_{12}x + \phi_{13}y + \phi_{14}z + \phi_{15}w \equiv 0$ , hence it is  $\{\phi \in \mathbb{P}^{15} / \phi_8 = \phi_9 = \phi_{10} = \phi_{11} = \phi_{12} = \phi_{13} = \phi_{14} = \phi_{15} = 0\} \simeq \mathbb{P}^7$ .

Similarly, the intersection of three general distinguished  $\mathbb{P}^{11}$ 's (i.e. the intersection of any distinguished  $\mathbb{P}^{11}$  and a general distinguished  $\mathbb{P}^7$ ) is a  $\mathbb{P}^3$  which corresponds to the point of intersection of the three planes associated with the three  $\mathbb{P}^{11}$ 's. To check this we can safely assume again that these three planes are the planes  $y = 0, z = 0, w = 0$  and get that the  $\mathbb{P}^3$  is the intersection of the hyperplanes  $\phi_4 = \phi_5 = \phi_6 = \phi_7 = \phi_8 = \phi_9 = \phi_{10} = \phi_{11} = \phi_{12} = \phi_{13} = \phi_{14} = \phi_{15} = 0$ . Also, any such distinguished  $\mathbb{P}^3$  is the set of matrices in  $\mathbb{P}^{15}$  which map  $\mathbb{P}^3$  to the point corresponding to the  $\mathbb{P}^3$ .  $\square$

From the proof, we notice also that the incidence relations among all these  $\mathbb{P}^{11}$ 's,  $\mathbb{P}^7$ 's and  $\mathbb{P}^3$ 's precisely reflect those of the corresponding geometric objects: planes, lines and points of intersection in  $S$ . In particular, any two  $\mathbb{P}^7$ 's either intersect along a  $\mathbb{P}^3$  or are disjoint; a  $\mathbb{P}^3$  is either completely inside or outside a  $\mathbb{P}^{11}$  or a  $\mathbb{P}^7$ ; and all  $\mathbb{P}^3$ 's are disjoint.

**2.3. Blow-Ups.** We resolve the indeterminacies of the rational map  $s$ . We do this by the means of a sequence of blow-ups of  $\mathbb{P}^{15}$  along the distinguished sets in the base locus of  $s$ . More precisely, there are three stages of blow-ups and one theorem is devoted to each of them. After a coordinate change in  $\mathbb{P}^3$  the point  $(1 : 0 : 0 : 0)$  will be a point of intersection of some of the planes in the configuration  $S$ , i.e., we can assume that the equation of  $S$  is

$$F(x : y : z : w) = \prod_i (\beta_i y + \gamma_i z + \delta_i w)^{r_i} \prod_j (\alpha_j x + \beta_j y + \gamma_j z + \delta_j w)^{r_j} \text{ where } \alpha_j \neq 0.$$

The point condition in  $\mathbb{P}^{15}(\phi_0 : \phi_1 : \dots : \phi_{15})$  corresponding to a point  $(x_0 : y_0 : z_0 : w_0)$  is

$$\prod_i (\beta_i(\phi_4 x_0 + \phi_5 y_0 + \phi_6 z_0 + \phi_7 w_0) + \gamma_i(\phi_8 x_0 + \phi_9 y_0 + \phi_{10} z_0 + \phi_{11} w_0))$$

$$\begin{aligned}
 & +\delta_i(\phi_{12}x_0 + \phi_{13}y_0 + \phi_{14}z_0 + \phi_{15}w_0))^{r_i} \prod_j (\alpha_j(\phi_0x_0 + \phi_1y_0 + \phi_2z_0 + \phi_3w_0) \\
 & +\beta_j(\phi_4x_0 + \phi_5y_0 + \phi_6z_0 + \phi_7w_0) + \gamma_j(\phi_8x_0 + \phi_9y_0 + \phi_{10}z_0 + \phi_{11}w_0) \\
 & +\delta_j(\phi_{12}x_0 + \phi_{13}y_0 + \phi_{14}z_0 + \phi_{15}w_0))^{r_j} = 0
 \end{aligned}$$

where  $\alpha_j \neq 0$ .

We consider all blow-ups over the representative affine chart  $\mathbb{A}^{15}(1 : p_1 : p_2 : \dots : p_{15})$  in  $\mathbb{P}^{15}$ .

**Theorem 2.3.1.** *Let  $V'$  be the variety obtained from  $V = \mathbb{P}^{15}$  after all blow-ups centered at the distinguished  $\mathbb{P}^3$ 's (if any) in the base locus of  $s$ . Then*

- (i) *The proper transforms of the distinguished  $\mathbb{P}^7$ 's in  $V'$  are disjoint.*
- (ii) *The multiplicity of a point condition along each  $\mathbb{P}^3$  corresponding to the intersection of planes of the configuration  $S$  equals the sum of the multiplicities of those planes.*
- (iii) *The intersection of the proper transforms of the point conditions in  $V'$  consists of the proper transforms of the distinguished  $\mathbb{P}^{11}$ 's.*

*Proof.* (i) is clear from the structure of the base locus discussed in the previous section and the fact that transversal intersections are separated by blow-ups.

In  $\mathbb{A}^{15}(1 : p_1 : p_2 : \dots : p_{15}) \subset \mathbb{P}^{15}$  the point condition corresponding to  $(x_0 : y_0 : z_0 : w_0)$  has the equation

$$\begin{aligned}
 & \prod_i (\beta_i(p_4x_0 + p_5y_0 + p_6z_0 + p_7w_0) + \gamma_i(p_8x_0 + p_9y_0 + p_{10}z_0 + p_{11}w_0) \\
 & +\delta_i(p_{12}x_0 + p_{13}y_0 + p_{14}z_0 + p_{15}w_0))^{r_i} \prod_j (\alpha_j(x_0 + p_1y_0 + p_2z_0 + p_3w_0) \\
 & +\beta_j(p_4x_0 + p_5y_0 + p_6z_0 + p_7w_0) + \gamma_j(p_8x_0 + p_9y_0 + p_{10}z_0 + p_{11}w_0) \\
 & +\delta_j(p_{12}x_0 + p_{13}y_0 + p_{14}z_0 + p_{15}w_0))^{r_j} = 0
 \end{aligned}$$

where  $\alpha_j \neq 0$ .

The first time we blow-up  $V$  along a  $\mathbb{P}^3$  we can assume that this is the  $\mathbb{P}^3$  which corresponds to the point  $(1 : 0 : 0 : 0)$ , i.e., the  $\mathbb{P}^3$  with equations  $p_4 = p_5 = \dots = p_{15} = 0$ .

The map from a representative chart  $\mathbb{A}^{15}(q_1, q_2, \dots, q_{15})$  of the blow-up to

$$\begin{aligned}
 & \mathbb{A}^{15}(p_1, p_2, \dots, p_{15}) \text{ in } V \text{ is given by the following equations:} \\
 & p_1 = q_1, p_2 = q_2, \dots, p_4 = q_4, p_5 = q_4q_5, p_6 = q_4q_6, \dots, p_{15} = q_4q_{15}.
 \end{aligned}$$



In this  $\mathbb{A}^{15}(q_1, q_2, \dots, q_{15})$  the exceptional divisor is  $q_4 = 0$ .  
 The (full) transform of the above point condition then is

$$\begin{aligned} & \prod_i q_4^{r_i} (\beta_i(x_0 + q_5y_0 + q_6z_0 + q_7w_0) + \gamma_i(q_8x_0 + q_9y_0 + q_{10}z_0 + q_{11}w_0) \\ & + \delta_i(q_{12}x_0 + q_{13}y_0 + q_{14}z_0 + q_{15}w_0))^{r_i} \prod_j (\alpha_j(x_0 + q_1y_0 + q_2z_0 + q_3w_0) \\ & + \beta_jq_4(x_0 + q_5y_0 + q_6z_0 + q_7w_0) + \gamma_jq_4(q_8x_0 + q_9y_0 + q_{10}z_0 + q_{11}w_0) \\ & + \delta_jq_4(q_{12}x_0 + q_{13}y_0 + q_{14}z_0 + q_{15}w_0))^{r_j} = 0. \end{aligned}$$

Since  $\alpha_j \neq 0$ , this shows that the multiplicity of the exceptional divisor in this transform is  $\sum_i r_i$ , which proves (ii) for this particular  $\mathbb{P}^3$  whence for all other disjoint  $\mathbb{P}^3$ 's.

(iii) is true on the complement of the exceptional divisors, so we only need to verify it along each exceptional divisor. We find the intersection of the proper transform of the point condition with the exceptional divisor:

$$\begin{aligned} & \prod_i (\beta_i(x_0 + q_5y_0 + q_6z_0 + q_7w_0) + \gamma_i(q_8x_0 + q_9y_0 + q_{10}z_0 + q_{11}w_0) \\ & + \delta_i(q_{12}x_0 + q_{13}y_0 + q_{14}z_0 + q_{15}w_0))^{r_i} \prod_j \alpha_j(x_0 + q_1y_0 + q_2z_0 + q_3w_0)^{r_j} = 0, \\ & q_4 = 0. \end{aligned}$$

This is identically zero for all  $(x_0 : y_0 : z_0 : w_0)$  precisely in the intersection of the proper transforms of all point conditions and the exceptional divisor. Clearly this happens when a factor of the first product is zero for each of the points  $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1)$ . So we get that this intersection is the union (over  $i$ ) of the following sets

$$\left\{ \begin{array}{l} q_4 = 0 \\ \beta_i + \gamma_iq_8 + \delta_iq_{12} = 0 \\ \beta_iq_5 + \gamma_iq_9 + \delta_iq_{13} = 0 \\ \beta_iq_6 + \gamma_iq_{10} + \delta_iq_{14} = 0 \\ \beta_iq_7 + \gamma_iq_{11} + \delta_iq_{15} = 0 \end{array} \right.$$

Each of these, however, we recognize as the intersection of the proper transform of a distinguished  $\mathbb{P}^{11}$  containing the blow-up center  $\mathbb{P}^3$  with the exceptional divisor. Indeed, a  $\mathbb{P}^{11}$  containing the  $\mathbb{P}^3$  is the  $\mathbb{P}^{11}$  which corresponds to a plane from the first product  $(\prod_i)$  in the equation of  $S$ . So it consists of all

$\phi$ 's for which  $\beta_i(\phi_4x_0 + \phi_5y_0 + \phi_6z_0 + \phi_7w_0) + \gamma_i(\phi_8x_0 + \phi_9y_0 + \phi_{10}z_0 + \phi_{11}w_0) + \delta_i(\phi_{12}x_0 + \phi_{13}y_0 + \phi_{14}z_0 + \phi_{15}w_0) \equiv 0$ , i.e., it is the set in  $\mathbb{A}^{15}(p_1, p_2, \dots, p_{15})$  given by

$$\begin{cases} \beta_i p_4 + \gamma_i p_8 + \delta_i p_{12} = 0 \\ \beta_i p_5 + \gamma_i p_9 + \delta_i p_{13} = 0 \\ \beta_i p_6 + \gamma_i p_{10} + \delta_i p_{14} = 0 \\ \beta_i p_7 + \gamma_i p_{11} + \delta_i p_{15} = 0 \end{cases}$$

Now we observe that the intersection of the proper transforms of these four hyperplanes and the exceptional divisor is given precisely by the previous set of equations. This is clearly an irreducible subset of dimension 10, so it must be the intersection of the exceptional divisor and the proper transform of the  $\mathbb{P}^{11}$  defined by the latter set of equations.

Thus the intersection of the proper transforms of the point conditions in each of the disjoint exceptional divisors in  $V'$  is the union of the proper transforms of the distinguished  $\mathbb{P}^{11}$ 's containing the blow-up center yielding this exceptional divisor. This implies (iii).  $\square$

The proofs of the other two theorems use similar straightforward coordinate computations and we leave them to the reader.

**Theorem 2.3.2.** *Let  $V''$  be the variety obtained from  $V'$  after all blow-ups centered at the proper transforms of the distinguished  $\mathbb{P}^7$ 's under the first set of blow-ups from Theorem 2.3.1 Then*

- (i) *The proper transforms of the distinguished  $\mathbb{P}^{11}$ 's in  $V''$  are disjoint.*
- (ii) *The multiplicity of a point condition in  $V'$  along the proper transform of each  $\mathbb{P}^7$  corresponding to a line of the configuration  $S$  equals the multiplicity of  $S$  along this line.*
- (iii) *The intersection of the proper transforms of the point conditions in  $V''$  is the disjoint union of the proper transforms of the distinguished  $\mathbb{P}^{11}$ 's.*

**Theorem 2.3.3.** *Let  $V''' = \tilde{V}$  be the variety obtained from  $V''$  after all blow-ups centered at the proper transforms of the distinguished  $\mathbb{P}^{11}$ 's under the first and second sets of blow-ups from Theorem 2.3.1 and Theorem 2.3.2. Then*

- (i) *The intersection of the proper transforms of the point conditions in  $\tilde{V}$  is empty.*
- (ii) *The multiplicity of a point condition in  $V''$  along the proper transform of each  $\mathbb{P}^{11}$  corresponding to a plane of the configuration  $S$  equals the multiplicity of  $S$  along this plane.*

Let  $\pi$  be the composition of all blowing-ups from the theorems, i.e.,

$$\tilde{V} \rightarrow V'' \rightarrow V' \rightarrow V = \mathbb{P}^{15}$$

Define the rational map  $\tilde{s} := s \circ \pi$  from  $\tilde{V}$  to  $\mathbb{P}^N$ .

We saw that the linear system of  $s$  is spanned by the point conditions in  $V$  and we also proved that the proper transforms of the latter have an empty intersection in  $V$ . Hence the rational map  $\tilde{s}$  is in fact regular because its linear system is spanned by the proper transforms of the point conditions.

**Theorem 2.3.4.** *The variety  $\tilde{V}$  resolves the indeterminacies of  $s$ , i.e., there is a commutative diagram:*

$$\begin{array}{ccc} \tilde{V} & & \\ \pi \downarrow & \searrow \tilde{s} & \\ V - \frac{\overline{S}}{s} & \xrightarrow{\quad} & \mathbb{P}^N \end{array}$$

with  $\tilde{s}$  a regular map and  $\overline{\mathcal{O}_S} = \text{Im}(\tilde{s})$ .

**2.4. Predegrees.** For  $0 \leq i \leq 15$ , consider a general  $\mathbb{P}^i$  in  $\mathbb{P}^{15}$ , and let  $f_i, d_i$  denote respectively the number of points in the general fiber of  $s|_{\mathbb{P}^i}$  (i.e. the degree of  $s|_{\mathbb{P}^i}$ ), and the intersection number of  $\overline{s(\mathbb{P}^i)}$  and a codimension  $i$  linear subspace of  $\mathbb{P}^N$ . The products  $f_i \cdot d_i$  are called the *predegrees* of  $S$  and we use them to define the following (*adjusted*) *predegree polynomial* of  $S$  as a function of  $t$  in the same fashion as done in [3] for line configurations.

$$\sum_{i \geq 0} (f_i \cdot d_i) \frac{t^i}{i!}$$

From this polynomial we could extract information about the degrees of the closures of the images of general linear subspaces of  $\mathbb{P}^{15}$  of all dimensions. These subsets of the orbit closure of  $S$  are determined by imposing linear conditions on the transformations applied to  $S$ .

We distinguish the planes of  $S$  by labelling them. This induces a labelling on each translate of  $S$ . We will refer to the differently labelled translates as to *ordered* ones. Let, as before,  $j$  be the dimension of the orbit closure of  $S$ .

**Remarks 2.4.1.**

- $d_j$  is the degree of the orbit closure,  $d_i$  is the degree of  $\overline{s(\mathbb{P}^i)}$  for  $i < j$  and  $d_i = 0$  for  $i > j$ ;
- $f_j$  is the degree of the closure of  $\text{Stab}_S$ ,  $f_i = 1$  for  $i < j$  and  $f_i = \infty$  for  $i > j$ ;

- Since  $S$  consists of planes, every irreducible component of the closure of  $Stab_S$  is a linear subspace of  $\mathbb{P}^{15}$ , hence  $f_j$  is the number of irreducible components of  $\overline{Stab_S}$ . These components clearly account for the different automorphisms of  $S$  as a set of planes (not points) and  $f_j = 1$  if the planes have all different multiplicities. Hence:
- For reduced configurations the predegrees count the ordered translates going through certain numbers of general points, while the degree is the number of the corresponding non-ordered configurations (see Example 2.4.2 below).
- Since  $d_i = 0$  iff  $f_i = \infty$  iff  $i > j$ , we make the convention that  $f_i d_i$  is zero in this case, so that the (adjusted) predegree polynomial is really a polynomial. We refer to the last nonzero predegree  $f_j d_j$  simply as to *the predegree* of  $S$ . It can be recovered from the leading coefficient of the predegree polynomial and in turn it recovers the degree of the orbit closure.
- The denominators are introduced in the definition of the adjusted predegree polynomial because they endow certain multiplicative structure (Theorem 5.2) to the polynomial allowing for a convenient shortcut in computing predegrees of special plane configurations (Theorem 5.4). Computing adjusted predegree polynomials, rather than the individual predegrees allows us to deal at once with all configurations, regardless of the dimension of their stabilizer.

**Example 2.4.2.** The predegree polynomial of a pair of reduced planes will be shown to be  $(1 + t + t^2/2 + t^3/3!)^2$ , so the predegree is 20. From Example 2.1.2 the degree is 10. Hence  $f_j = f_6 = 2$ , i.e., each pair produces two ordered pairs of planes.

**Remark 2.4.3.** Note that in general the ordered translates are not all possible orderings (permutations) of the (non-ordered) translates as it happens to be the case in the example above. One nice example illustrating this fact is Example 5.6.

**Example 2.4.4.** The geometric interpretations of degrees and predegrees extend to the case of non-reduced plane configurations if we count translates “with multiplicities”.

Let  $S$  consist of three general planes with distinct multiplicities, for example 1, 2 and 3. The number of the reduced ordered triples of planes containing

$j = 9$  general points is  $\binom{9}{3} \binom{6}{3} = 1680$ . Now thinking of a double plane as of two copies of coinciding single planes and of a triple plane as of three coinciding copies of single planes, we see that there are  $3^3$  ways in which a triple plane can contain 3 points (and be determined by them) and  $2^3$  possibilities for a double plane to go through 3 general points. So the number of (non-reduced) translates of  $S$  (counted with multiplicities) is  $1680 \times 216 = 9!$  which is the degree of the orbit closure of  $S$ .

On the other hand, the predegree polynomial of this non-reduced configuration is  $(1 + t + t^2/2 + t^3/6)(1 + 2t + 4t^2/2 + 8t^3/6)(1 + 3t + 9t^2/2 + 27t^3/6)$ . Therefore the predegree is  $9!$ , the same as the degree, in accordance with  $f_9 = 1$ : differently labelled (by the plane multiplicities) translates in this example represent different actual configurations.

Tracing definitions, we prove

**Theorem 2.4.5.** *The  $i$ -th predegree is the intersection degree  $\int_{\tilde{V}} \tilde{W}^i \tilde{H}^{15-i}$ ,  $i \geq 0$  where  $\tilde{W}$  and  $\tilde{H}$  are respectively the pull-backs to the blow-up  $\tilde{V}$  of the hyperplane classes  $W$  in  $\mathbb{P}^N$  and  $H$  in  $\mathbb{P}^{15}$ .*

**3. Intersection Theory.** There is a convenient lemma, due to Aluffi, controlling intersection numbers through blow-ups [4].

**Lemma 3.1.** *Let  $B \hookrightarrow V$  be nonsingular varieties;  $X, Y$  hypersurfaces in  $V$ ;  $\tilde{X}, \tilde{Y}$  their proper transforms in the blow-up  $\tilde{V}$  of  $V$  along  $B$ . Then*

$$\int_{\tilde{V}} \tilde{X}^{\dim V - j} \cdot \tilde{Y}^j = \int_V X^{\dim V - j} \cdot Y^j - \int_B \frac{(m_{B,X}[B] + X \cdot B)^{\dim V - j} (m_{B,Y}[B] + Y \cdot B)^j}{c(N_B V)},$$

where  $m_{B,X}, m_{B,Y}$  denote the multiplicities of  $X, Y$  along  $B$  and  $c(N_B V)$  is the Chern class of the normal bundle to  $B$  in  $V$ .

We apply the lemma to the intersection numbers  $\int_{\tilde{V}} \tilde{W}^i \tilde{H}^{15-i}$  and the blow-up sequence  $\tilde{V} = V''' \rightarrow V'' \rightarrow V' \rightarrow V = \mathbb{P}^{15}$ .

Let us denote by  $\dot{H}, \ddot{H}, \tilde{H}$  the pull-backs of the hyperplane class  $H$  in  $\mathbb{P}^{15}$  to the blow-ups  $V', V'', \tilde{V}$  respectively. Similarly,  $\dot{W}, \ddot{W}, \tilde{W}$  will be the proper transforms of a point condition  $W$  in  $\mathbb{P}^{15}$  to the same blow-ups respectively. We use dot notation, in unison with the above, to denote also the proper transforms of the distinguished  $\mathbb{P}^{11}$ 's,  $\mathbb{P}^7$ 's and  $\mathbb{P}^3$ 's in  $\mathbb{P}^{15}$  at the corresponding stages of

the blow-up sequence. In these notations, three iterations of the lemma yield the following result.

$$\begin{aligned} \int_{\tilde{V}} \widetilde{W}^j \widetilde{H}^{15-j} &= \int_{\mathbb{P}^{15}} W^j H^{15-j} \\ &- \sum_i \int_{\mathbb{P}_i^3} \frac{\left(m_{\mathbb{P}_i^3, H}[\mathbb{P}_i^3] + H \cdot [\mathbb{P}_i^3]\right)^{15-j} \left(m_{\mathbb{P}_i^3, W}[\mathbb{P}_i^3] + W \cdot [\mathbb{P}_i^3]\right)^j}{c\left(N_{\mathbb{P}_i^3} \mathbb{P}_i^{15}\right)} \\ &- \sum_i \int_{\dot{\mathbb{P}}_i^7} \frac{\left(m_{\dot{\mathbb{P}}_i^7, \dot{H}}[\dot{\mathbb{P}}_i^7] + \dot{H} \cdot [\dot{\mathbb{P}}_i^7]\right)^{15-j} \left(m_{\dot{\mathbb{P}}_i^7, \dot{W}}[\dot{\mathbb{P}}_i^7] + \dot{W} \cdot [\dot{\mathbb{P}}_i^7]\right)^j}{c\left(N_{\dot{\mathbb{P}}_i^7} V'\right)} \\ &- \sum_i \int_{\ddot{\mathbb{P}}_i^{11}} \frac{\left(m_{\ddot{\mathbb{P}}_i^{11}, \ddot{H}}[\ddot{\mathbb{P}}_i^{11}] + \ddot{H} \cdot [\ddot{\mathbb{P}}_i^{11}]\right)^{15-j} \left(m_{\ddot{\mathbb{P}}_i^{11}, \ddot{W}}[\ddot{\mathbb{P}}_i^{11}] + \ddot{W} \cdot [\ddot{\mathbb{P}}_i^{11}]\right)^j}{c\left(N_{\ddot{\mathbb{P}}_i^{11}} V''\right)}, \end{aligned}$$

where the first summation is over all points of intersection, the second summation over all lines of intersection and the third one over all planes of the plane configuration  $S$ .

Clearly,  $\int_{\mathbb{P}^{15}} W^j H^{15-j} = d^j$  and to compute the other three intersection numbers on the right hand side, which we call first, second and third contributions (to the predegree) respectively, we need the following claims which can be checked with standard intersection theory:

- Claim 3.2.** (i)  $c(N_{\mathbb{P}_i^3} \mathbb{P}_i^{15}) = (1 + H)^{12}$ ;  
(ii)  $c(N_{\dot{\mathbb{P}}_i^7} V') = \left(1 + \dot{H} - \sum_k E_k\right)^8$ , where  $E_k$  is the exceptional divisor in  $V'$  corresponding to the  $k$ -th blow-up center  $\mathbb{P}_i^3$ ;  
(iii)  $c(N_{\ddot{\mathbb{P}}_i^{11}} V'') = \left(1 + \ddot{H} - \sum_k \dot{E}_k - \sum_s e_s\right)^4$ , where  $e_s$  is the exceptional divisor in  $V''$  corresponding to the  $s$ -th blow-up center  $\dot{\mathbb{P}}_i^7$  and  $\dot{E}_k$  is the full transform of  $E_k$  in  $V''$ ;  
(iv)  $\dot{W} = d\dot{H} - \sum_k \pi_k E_k$ , where  $\pi_k$  is the multiplicity of  $S$  along its component corresponding to  $\mathbb{P}_i^3$  (the  $k$ -th  $\mathbb{P}_i^3$ ) and the sum is over all points on a line of intersection (corresponding to  $\mathbb{P}_i^7$ );  
(v)  $\ddot{W} = d\ddot{H} - \sum_k \pi_k \dot{E}_k - \sum_s \lambda_s e_s$ , where  $\lambda_s$  is the multiplicity of  $S$  along the  $s$ -th line of intersection,  $s$  runs over the lines and  $k$  over the points in the ( $i$ -th) plane.

**Claim 3.3.** *If we write  $\rightarrow$  to denote push-forwards under the appropriate set of blow-ups, then*

(i)  $E_m E_n = 0$  for  $m \neq n$ ;  $E_i \rightarrow 0, E_i^2 \rightarrow 0, E_i^3 \rightarrow 0, E_i^4 \rightarrow -H^4, E_i^5 \rightarrow -4H^5, E_i^6 \rightarrow -10H^6, E_i^7 \rightarrow -20H^7$ ;

(ii)  $e_m e_n = 0, \dot{E}_m \dot{E}_n = 0$  for  $m \neq n$ ;  $\dot{E}_k \cdot e_s \neq 0$  if and only if the corresponding ( $k$ -th)  $\mathbb{P}^3$  is in the corresponding ( $s$ -th)  $\mathbb{P}^7$ ;  $E_k^n \rightarrow 0, 1 \leq n \leq 7, E_k^8 \rightarrow -H^8, E_k^9 \rightarrow -8H^9, E_k^{10} \rightarrow -36H^{10}, E_k^{11} \rightarrow -120H^{11}$ ;  $e_s^n \rightarrow 0, 1 \leq n \leq 3, e_s^4 \rightarrow -D^4, e_s^5 \rightarrow -4D^5, e_s^6 \rightarrow -10D^6, e_s^7 \rightarrow -20D^7, e_s^8 \rightarrow -35D^8, e_s^9 \rightarrow -56D^9, e_s^{10} \rightarrow -84D^{10}, e_s^{11} \rightarrow -120D^{11}$ , where  $D$  is  $\dot{H} - \sum_k E_k$ , the sum being over the points in the line corresponding to (the  $s$ -th)  $\mathbb{P}^7$ .

Sketch of Proof of (i)  $E_m E_n = 0$  since  $E_m$  and  $E_n$  are disjoint for  $m \neq n$ . We use the birational invariance of Segre classes [6], according to which, the push-forward of the Segre class  $s(E_i, \mathbb{P}_i^7)$ , under the first blow-up, is the Segre class  $s(\mathbb{P}_k^3, \mathbb{P}_i^7)$ , i.e. the push-forward of  $\frac{1}{c(N_{E_i \mathbb{P}_i^7})} = \frac{1}{1 + E_i} \cap [E_i]$  is  $\frac{1}{c(N_{\mathbb{P}_k^3 \mathbb{P}_i^7})} = \frac{1}{(1 + H)^4} \cap [\mathbb{P}_k^3]$ . Since the nonzero push-forwards keep the same dimensions, we find the different-dimension components of the above two and match them.

Clearly  $\frac{1}{1 + E_i} \cap [E_i] = E_i - E_i^2 + E_i^3 - \dots$  and  $\frac{1}{(1 + H)^4} \cap [\mathbb{P}_k^3] = (1 - 4H + 10H^2 - 20H^3 + \dots)[\mathbb{P}_k^3]$ . Thus:

$E_i \rightarrow 0, E_i^2 \rightarrow 0, E_i^3 \rightarrow 0; E_i^4 \rightarrow -H^4, E_i^5 \rightarrow -4H^5; E_i^6 \rightarrow -10H^6, E_i^7 \rightarrow -20H^7$ , where by  $H$  we mean the restriction  $H[\mathbb{P}_k^3]$ .  $\square$

Using Claim 3.2, Theorem 2.3.1 (ii), Theorem 2.3.2 (ii) and Theorem 2.3.3 (ii), we write the  $j$ -th predegree as

$$\begin{aligned} d^j &= \sum_i \int_{\mathbb{P}_i^3} \frac{H^{15-j}(\pi_i + dH)^j}{(1 + H)^{12}} \\ &\quad - \sum_i \int_{\dot{\mathbb{P}}_i^7} \frac{\dot{H}^{15-j}(\lambda_i + y)^j}{(1 + \dot{H} - \sum_k E_k)^8} \\ &\quad - \sum_i \int_{\ddot{\mathbb{P}}_i^{11}} \frac{\ddot{H}^{15-j}(r_i + \ddot{W})^j}{(1 + \ddot{H} - \sum_k \dot{E}_k - \sum_s e_s)^4} \end{aligned}$$

and then using Claim 3.3, we push-forward the second and the third contributions and finally compute all predegrees.

The expressions we get for the predegrees are quite lengthy. For example,

for the coefficient of  $t^{12}/12!$  we have

$$\begin{aligned}
 & d^{12} - \sum_k \pi_k^{12} \\
 & - \sum_s \lambda_s^8 \left[ 330\lambda_s^4 \left( 1 - \sum_k 1 \right) - 1440\lambda_s^3 \left( d - \sum_k \pi_k \right) + 2376\lambda_s^2 \left( d^2 - \sum_k \pi_k^2 \right) \right. \\
 & \quad \left. - 1760\lambda_s \left( d^3 - \sum_k \pi_k^3 \right) + 495 \left( d^4 - \sum_k \pi_k^4 \right) \right] \\
 & - \sum_i \left[ 17325 r_i^4 \sum_s \lambda_s^8 \sum_k \pi_k^0 - 79200 r_i^4 \sum_s \lambda_s^7 \sum_k \pi_k^1 - 1440 r_i^{11} \sum_s \lambda_s^0 \sum_k \pi_k^1 \right. \\
 & \quad + 79200 r_i^7 \sum_s \lambda_s^4 \sum_k \pi_k^1 - 221760 r_i^6 \sum_s \lambda_s^5 \sum_k \pi_k^1 + 221760 r_i^5 \sum_s \lambda_s^6 \sum_k \pi_k^1 \\
 & \quad - 495 r_i^4 \sum_k \pi_k^8 + 12320 r_i^9 \sum_k \pi_k^3 + 3168 r_i^5 \sum_k \pi_k^7 - 17325 r_i^8 \sum_k \pi_k^4 \\
 & \quad - 165 r_i^{12} \sum_k \pi_k^0 - 12320 r_i^9 \sum_s \lambda_s^0 \sum_k \pi_k^3 - 5544 r_i^{10} \sum_k \pi_k^2 \\
 & \quad + 15840 r_i^7 \sum_k \pi_k^5 + 332640 r_i^5 \sum_s \lambda_s^4 \sum_k \pi_k^3 + 47520 r_i^5 \sum_s \lambda_s^7 \\
 & \quad - 9240 r_i^6 \sum_k \pi_k^6 - 110880 r_i^4 \sum_s \lambda_s^5 \sum_k \pi_k^3 + 495 r_i^4 d^8 - 1440 r_i^{11} d \\
 & \quad - 15840 r_i^7 d^5 - 12320 r_i^9 d^3 + 17325 r_i^8 d^4 - 3168 r_i^5 d^7 + 9240 r_i^6 d^6 + 5544 r_i^{10} d^2 \\
 & \quad + 1440 r_i^{11} \sum_k \pi_k^1 - 369600 r_i^6 \sum_s \lambda_s^3 \sum_k \pi_k^3 + 158400 r_i^7 \sum_s \lambda_s^2 \sum_k \pi_k^3 \\
 & \quad + 221760 r_i^6 d \sum_s \lambda_s^5 + 1440 r_i^{11} d \sum_s \lambda_s^0 + 79200 r_i^4 d \sum_s \lambda_s^7 - 5544 r_i^{10} d^2 \sum_s \lambda_s^0 \\
 & \quad + 369600 r_i^6 d^3 \sum_s \lambda_s^3 - 34650 r_i^4 d^4 \sum_s \lambda_s^4 + 158400 r_i^7 d^2 \sum_s \lambda_s^3 \\
 & \quad + 110880 r_i^5 d^4 \sum_s \lambda_s^3 + 79200 r_i^7 d^4 \sum_s \lambda_s^1 - 158400 r_i^7 d^3 \sum_s \lambda_s^2 \\
 & \quad \left. - 138600 r_i^6 d^4 \sum_s \lambda_s^2 - 17325 r_i^8 d^4 \sum_s \lambda_s^0 + 12320 r_i^9 d^3 \sum_s \lambda_s^0 \right]
 \end{aligned}$$



$$\begin{aligned}
& -138600 r_i^4 d^2 \sum_s \lambda_s^6 - 221760 r_i^5 d \sum_s \lambda_s^6 - 79200 r_i^7 d \sum_s \lambda_s^4 \\
& -332640 r_i^5 d^3 \sum_s \lambda_s^4 - 415800 r_i^6 d^2 \sum_s \lambda_s^4 + 110880 r_i^4 d^3 \sum_s \lambda_s^5 \\
& +399168 r_i^5 d^2 \sum_s \lambda_s^5 + 165 r_i^{12} \sum_s \lambda_s^0 \sum_k \pi_k^0 - 47520 r_i^5 \sum_s \lambda_s^7 \sum_k \pi_k^0 \\
& +46200 r_i^6 \sum_s \lambda_s^6 \sum_k \pi_k^0 - 79200 r_i^7 \sum_s \lambda_s^1 \sum_k \pi_k^4 - 15840 r_i^7 \sum_s \lambda_s^5 \sum_k \pi_k^0 \\
& +34650 r_i^4 \sum_s \lambda_s^4 \sum_k \pi_k^4 + 17325 r_i^8 \sum_s \lambda_s^0 \sum_k \pi_k^4 + 138600 r_i^6 \sum_s \lambda_s^2 \sum_k \pi_k^4 \\
& -110880 r_i^5 \sum_s \lambda_s^3 \sum_k \pi_k^4 - 158400 r_i^7 \sum_s \lambda_s^3 \sum_k \pi_k^2 + 5544 r_i^{10} \sum_s \lambda_s^0 \sum_k \pi_k^2 \\
& -399168 r_i^5 \sum_s \lambda_s^5 \sum_k \pi_k^2 + 415800 r_i^6 \sum_s \lambda_s^4 \sum_k \pi_k^2 + 138600 r_i^4 \sum_s \lambda_s^6 \sum_k \pi_k^2 \\
& +165 r_i^{12} + 15840 r_i^7 \sum_s \lambda_s^5 - 46200 r_i^6 \sum_s \lambda_s^6 - 165 r_i^{12} \sum_s \lambda_s^0 \\
& -17325 r_i^4 \sum_s \lambda_s^8 \Big],
\end{aligned}$$

where all  $k$ -indexed summations inside  $s$ -summations are over the points on the line having multiplicity  $\lambda_s$  and all  $s$ -summations inside  $i$ -summations range over the lines in the plane with multiplicity  $r_i$ . The outer-most summations are over all points, lines and planes.

The rest of the contributions and the computations leading to them may be found in [10].

#### 4. Predegree polynomial.

**4.1. First compact form.** We express the original raw forms of the coefficients of the predegree polynomial (like the one above) in terms of the coefficients of the lower degree terms (see [10] for the computational techniques used). This allows us to write the whole polynomial in a much more compact and convenient form revealing certain useful multiplicative properties of predegree polynomials of specific plane configurations:

The predegree polynomial of a plane configuration  $S$  with the usual notations is the truncation to the fifteenth degree of

$$\prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) + \sum_{i=1}^n \left( \sum_{j=0}^7 \frac{(d - r_i)^j t^j}{j!} \right) \left( \sum_{k=8}^{15} C_{k,i} \frac{t^k}{k!} \right) + \sum_{i=1}^n \left( \sum_{j=0}^3 \frac{(d - r_i)^j t^j}{j!} \right) \left( \sum_{k=12}^{15} K_{k,i} \frac{t^k}{k!} \right),$$

where the  $C$ 's are polynomials of the multiplicities of  $(S$  along) the planes in the configuration and their lines of intersection while, in addition to these, the  $K$ 's also depend on the multiplicities of the points of intersection of the planes.

As in the previous section, it would be difficult to include all computations (done with the help of Maple) proving the results in this section. However, we list the final formulae we find for the  $C$ -,  $K$ - functions in the appendix.

**4.2. Transversal configurations.** We notice that  $C_{n,i}$  and  $K_{n,i}$  vanish if  $S$  is transversal (every plane intersects the rest of  $S$  transversally) and that  $C_{n,i}$  and  $K_{n,i}$  keep their values (under “small perturbations”) as long as the incidence data of  $S$  do not change. If we denote by  $C_{n,i}^{\text{tr}}$  and  $K_{n,i}^{\text{tr}}$  those values for transversal configurations, we next claim that we can use  $S_{n,i} := C_{n,i} - C_{n,i}^{\text{tr}}$  and  $Q_{n,i} := K_{n,i} - K_{n,i}^{\text{tr}}$  for  $C_{n,i}$  and  $K_{n,i}$  respectively. In fact, since  $C_{n,i}$  do not depend on the points of intersection in  $S$ , then  $C_{n,i} = C_{n,i}^{\text{tr}}$  is constant whenever the plane  $P_i$  does not contain lines of intersection of the remaining planes. So we have the following:

**Theorem 4.2.1.** *The predegree polynomial of a plane configuration  $S$  is the truncation to the fifteenth degree of*

$$\prod_{i=1}^n \left( \sum_{j=0}^3 \frac{r_i^j t^j}{j!} \right) + \sum_i \left( \sum_{j=0}^7 \frac{(d - r_i)^j t^j}{j!} \right) \left( \sum_{k=8}^{15} S_{k,i} \frac{t^k}{k!} \right) + \sum_i \left( \sum_{j=0}^3 \frac{(d - r_i)^j t^j}{j!} \right) \left( \sum_{k=12}^{15} Q_{k,i} \frac{t^k}{k!} \right),$$

where  $Q_{m,i}$ ,  $12 \leq m \leq 15$  are zero if the  $i$ -th plane  $P_i$  intersects the rest of  $S$  transversally and  $S_{n,i} = 0$ ,  $8 \leq n \leq 15$  when  $P_i$  does not contain a line of intersection of the rest of the planes in  $S$ .

We illustrate the idea of constructing  $C_{n,i}^{\text{tr}}$  and  $K_{n,i}^{\text{tr}}$  (hence also  $S_{n,i}$  and  $Q_{n,i}$ ) from  $C_{n,i}$  and  $K_{n,i}$  by giving

$$C_{8,i}^{\text{tr}} = \sum_{k \neq i} r_k (35r_i^8 - 160r_i^7(r_i + r_k) + 280r_i^6(r_i + r_k)^2 - 224r_i^5(r_i + r_k)^3 + 70r_i^4(r_i + r_k)^4 - r_i(r_i + r_k)^7 - 35r_i^4r_k^4)$$

and

$$\begin{aligned} K_{13,i}^{\text{tr}} = & \sum_{j \neq i} \sum_{k \neq i, k \neq j} [-1925r_i^{13}/2 + 18720r_i^{12}(r_i + r_j + r_k)/2 \\ & - 81432r_i^{11}(r_i + r_j + r_k)^2/2 + 208208r_i^{10}(r_i + r_j + r_k)^3/2 \\ & - 345345r_i^9(r_i + r_j + r_k)^4/2 - r_i^8(2059200(r_i + r_j)^2(r_i + r_j + r_k)^3 \\ & - 2059200(r_i + r_j)^3(r_i + r_j + r_k)^2 + 205920(r_i + r_j + r_k)^5/2 \\ & - 205920(r_i + r_j)^5 + 1029600(r_i + r_j)^4(r_i + r_j + r_k) \\ & - 1029600(r_i + r_j)(r_i + r_j + r_k)^4 - 180180(r_i + r_j + r_k)^5/2 \\ & - r_i^7(-3432000(r_i + r_j)^3(r_i + r_j + r_k)^3 - 291720(r_i + r_j + r_k)^6/2 \\ & + 257400(r_i + r_j)^2(r_i + r_j + r_k)^4 + 634920(r_i + r_j)^6 \\ & + 4890600(r_i + r_j)^4(r_i + r_j + r_k)^2 + 823680(r_i + r_j)(r_i + r_j + r_k)^5 \\ & - 2882880(r_i + r_j)^5(r_i + r_j + r_k)) - r_i^6(-3747744(r_i + r_j)^5(r_i + r_j + r_k)^2 \\ & + 2162160(r_i + r_j)^3(r_i + r_j + r_k)^4 + 720720(r_i + r_j)^4(r_i + r_j + r_k)^3 \\ & - 720720(r_i + r_j)^7 - 1441440(r_i + r_j)^2(r_i + r_j + r_k)^5 \\ & + 2882880(r_i + r_j)^6(r_i + r_j + r_k) + 144144(r_i + r_j + r_k)^7/2) \\ & - r_i^5(-42471(r_i + r_j + r_k)^8/2 - 2792790(r_i + r_j)^4(r_i + r_j + r_k)^4 \\ & + 360360(r_i + r_j)^6(r_i + r_j + r_k)^2 + 333333(r_i + r_j)^8 \\ & + 2018016(r_i + r_j)^5(r_i + r_j + r_k)^3 + 1153152(r_i + r_j)^3(r_i + r_j + r_k)^5 \\ & - 1029600(r_i + r_j)^7(r_i + r_j + r_k) + 105105r_i^4r_k^4) \\ & - r_i^4(-1201200(r_i + r_j)^6(r_i + r_j + r_k)^3 - 360360(r_i + r_j)^4(r_i + r_j + r_k)^5 \\ & + 1081080(r_i + r_j)^5(r_i + r_j + r_k)^4 + 5720(r_i + r_j + r_k)^9/2 \\ & - 40040(r_i + r_j)^9 + 514800(r_i + r_j)^7(r_i + r_j + r_k)^2) \\ & - r_i^2(-6435(r_i + r_j)^7(r_i + r_j + r_k)^4 + 22880(r_i + r_j)^8(r_i + r_j + r_k)^3 \\ & - 30888(r_i + r_j)^9(r_i + r_j + r_k)^2 + 18720(r_i + r_j)^{10}(r_i + r_j + r_k) \end{aligned}$$

$$\begin{aligned}
 & - 4290(r_i + r_j)^{11} + 13(r_i + r_j + r_k)^{11}/2 - r_i(5148(r_i + r_j)^7(r_i + r_j + r_k)^5 \\
 & - 17160(r_i + r_j)^8(r_i + r_j + r_k)^4 + 20592(r_i + r_j)^9(r_i + r_j + r_k)^3 \\
 & - 9360(r_i + r_j)^{10}(r_i + r_j + r_k)^2 + 792(r_i + r_j)^{12} - 12(r_i + r_j + r_k)^{12}/2)].
 \end{aligned}$$

**5. Conclusions.** The following theorem is a corollary of Theorem 4.2.1.

**Theorem 5.1.** *Let  $S$  be a configuration of planes  $P_i$  respectively with multiplicities  $r_i, 1 \leq i \leq n$  and  $\sum_i r_i = d$ .*

(i) *The predegree polynomial of  $S$  is the truncation to the fifteenth degree of*

$$\prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) + O(t^8),$$

where  $O(t^8) = 0$  if  $S$  is transversal (consisting of general planes).

(ii) *If no three planes of  $S$  contain a common line, then the predegree polynomial of  $S$  is the truncation to the fifteenth degree of*

$$\begin{aligned}
 & \prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \\
 & + \sum_{i=1}^n \left( 1 + (d - r_i)t + \frac{(d - r_i)^2 t^2}{2!} + \frac{(d - r_i)^3 t^3}{3!} \right) \left( Q_{12,i} \frac{t^{12}}{12!} + \cdots + Q_{15,i} \frac{t^{15}}{15!} \right),
 \end{aligned}$$

where  $Q_{k,i}, 12 \leq k \leq 15$  vanish if  $P_i$  intersects the rest of  $S$  is transversally.

We consider some special configurations.

An  $n$ -plane configuration consisting of planes with a common line will be called an  $n$ -book.

An  $n$ -plane configuration whose planes have a common point and no three share a line will be named an  $n$ -star.

A union of two plane configurations will be called *transversal* if each plane of each configuration intersects the other configuration transversally. We also say that the two configurations meet *transversally*.

A transversal union of a star and a plane will be referred to as a *fan*.

We will write  $\{G(t)\}_n$  to denote the truncation of any polynomial  $G(t)$  to degree  $n$ . Our main result is

**Theorem 5.2.** *If  $S$  and  $S'$  are plane configurations in which no three planes pass through a common line and the two configurations meet transversally,*

then the predegree polynomial  $\mathcal{P}(t)$  of the union  $\mathcal{S}$  of  $S$  and  $S'$  is the truncation to degree fifteen of the product of the predegree polynomials  $P(t)$  and  $P'(t)$  of  $S$  and  $S'$  respectively.

Proof. Let  $S$  consist of planes with multiplicities  $r_i$  and  $r'_j$  be the multiplicities of the planes in  $S'$ . Let also  $\sum_i r_i = d$  and  $\sum_j r'_j = d'$ . According to Theorem 5.1(ii),

$$P(t) = \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) + \sum_i \left( 1 + (d - r_i)t + \frac{(d - r_i)^2 t^2}{2!} + \frac{(d - r_i)^3 t^3}{3!} \right) \left( Q_{12,i} \frac{t^{12}}{12!} + \dots + Q_{15,i} \frac{t^{15}}{15!} \right),$$

truncated to degree fifteen, and

$$P'(t) = \prod_j \left( 1 + r'_j t + \frac{(r'_j)^2 t^2}{2} + \frac{(r'_j)^3 t^3}{3!} \right) + \sum_j \left( 1 + (d' - r'_j)t + \frac{(d' - r'_j)^2 t^2}{2!} + \frac{(d' - r'_j)^3 t^3}{3!} \right) \left( Q'_{12,j} \frac{t^{12}}{12!} + \dots + Q'_{15,j} \frac{t^{15}}{15!} \right),$$

truncated to degree fifteen.

Notice that

$$\left\{ \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \left( 1 + (d' - r'_j)t + \frac{(d' - r'_j)^2 t^2}{2!} + \frac{(d' - r'_j)^3 t^3}{3!} \right) \right\}_3 = \left\{ \left( 1 + (d + d' - r'_j)t + \frac{(d + d' - r'_j)^2 t^2}{2!} + \frac{(d + d' - r'_j)^3 t^3}{3!} \right) \right\}_3$$

and

$$\left\{ \prod_j \left( 1 + r'_j t + \frac{(r'_j)^2 t^2}{2} + \frac{(r'_j)^3 t^3}{3!} \right) \left( 1 + (d - r_i)t + \frac{(d - r_i)^2 t^2}{2!} + \frac{(d - r_i)^3 t^3}{3!} \right) \right\}_3 = \left\{ \left( 1 + (d + d' - r_i)t + \frac{(d + d' - r_i)^2 t^2}{2!} + \frac{(d + d' - r_i)^3 t^3}{3!} \right) \right\}_3.$$

Thus for  $PP'$  we now have the truncation up to the fifteenth degree of

$$\begin{aligned}
 &= \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \prod_j \left( 1 + r'_j t + \frac{(r'_j)^2 t^2}{2} + \frac{(r'_j)^3 t^3}{3!} \right) + \\
 &\quad + \sum_j \left( 1 + (d + d' - r'_j) t + \frac{(d + d' - r'_j)^2 t^2}{2!} + \frac{(d + d' - r'_j)^3 t^3}{3!} \right) \\
 &\hspace{25em} \times \left( Q'_{12,j} \frac{t^{12}}{12!} + \cdots + Q'_{15,j} \frac{t^{15}}{15!} \right) \\
 &\quad + \sum_i \left( 1 + (d + d' - r_i) t + \frac{(d + d' - r_i)^2 t^2}{2!} + \frac{(d + d' - r_i)^3 t^3}{3!} \right) \\
 &\hspace{25em} \times \left( Q_{12,i} \frac{t^{12}}{12!} + \cdots + Q_{15,i} \frac{t^{15}}{15!} \right).
 \end{aligned}$$

To see that this is in fact  $\mathcal{P}$ , we use the fact that clearly  $\mathcal{S}$  is a configuration of degree  $d + d'$  which falls in the same case (ii) of Theorem 5.1 and also observe that the “ $Q$ -functions” of  $\mathcal{S}$  agree with  $Q_{k,i}$  and  $Q'_{k,i}$ ,  $12 \leq k \leq 15$  on  $S$  and  $S'$  respectively because of the transversal union and the properties of  $Q_{k,i}$  and  $Q'_{k,i}$  from Theorem 5.1(ii).  $\square$

**Remark 5.3.** We compute that the adjusted predegree polynomial of the transversal union of two reduced 4-books is not the (truncation of the) product of the predegree polynomials of the 4-books. So the requirement of no three planes containing a common line is essential and the statement of Theorem 5.2 cannot be improved in this sense.

Next we notice that the dimension of the stabilizer of any book is at least eight and that the dimension of the stabilizer of any star is at least four.

Everything so far in this section can be used to summarize the adjusted predegree polynomials of the special plane configurations.

**Theorem 5.4.** *If a plane configuration  $S$  consists of  $n$  planes with multiplicities  $r_i, 1 \leq i \leq n$ , then the predegree polynomial of  $S$  is*

$$(1) \left\{ \prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \right\}_{15}, \text{ if } S \text{ is transversal;}$$

$$(2) \left\{ \prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \right\}_7, \text{ if } S \text{ is a book;}$$

$$(3) \left\{ \prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \right\}_{11}, \text{ if } S \text{ is a star;}$$

$$(4) \left\{ \prod_{i=2}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \right\}_{11} \left( 1 + r_1 t + \frac{r_1^2 t^2}{2} + \frac{r_1^3 t^3}{3!} \right), \text{ if } S \text{ is a fan;}$$

$$(5) \left\{ \prod_{i=1}^n \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) + O(t^{12}) \right\}_{15}, \text{ if no three planes of } S \text{ have a common line. This could also be written as}$$

$$\left\{ \prod_j \left[ \left\{ \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} + \frac{r_i^3 t^3}{3!} \right) \right\}_{11} \right] \right\}_{15}, \text{ where the product over } i \text{ has } n_j \text{ factors and } \sum_j n_j = n.$$

**Example 5.5.** Consider a reduced 3-book. Its orbit consists of all 3-books in  $\mathbb{P}^3$  and has dimension 7 (3 degrees of freedom for each of 2 planes, 1 degree of freedom for the plane that completes the 3-book and contains the intersection of the first two planes). There are two ways in which such a configuration can contain 7 general points. First, if each of two of the planes contains 3 points, then the third plane (hence the whole configuration) is determined by the remaining point.

The number of such configurations is  $\binom{7}{3} \binom{4}{3} / 2 = 70$ . We divide by 2 because otherwise the pair of the first two planes would be counted twice.

The other possibility occurs when one plane is determined by 3 of the points and of the remaining 4 points we choose 2 pairs of points. Each pair determines a line intersecting the plane in a point. The line in the plane which connects these two points is the line (spine) of the 3-book, i.e., the other two planes are uniquely determined if we require that each of them contain a pair of points and that all three planes form a 3-book.

Clearly the computation in this case is  $\binom{7}{2} \binom{5}{2} / 2 = 105$  (the division by 2 is because of the same reason as above).

So the total number of configurations is  $70 + 105 = 175 = \deg \overline{\mathcal{O}}_S$ .

According to Theorem 5.4, the predegree polynomial is the truncation of  $(1 + t + t^2/2 + t^3/6)^3$  to degree 7 and this gives us the predegree of 1050. Since the degree is 175, this implies that  $f_j = f_7$  is the expected  $3!$ .

**Example 5.6.** Similarly, we compute combinatorially the degree of a fan consisting of a 3-star and a general plane to be 21000. From Theorem 5.4 (4) we find that the predegree is 126000, so  $f_j = f_{10} = 6$ . Our intuition behind this fact is that, although all planes are reduced, the general plane cannot be exchanged with any of the others under an automorphism of the configuration.

**6. Future work.** A natural component of an enumerative analysis of orbit closures of more general surfaces would be the computation of the predegree polynomial for a larger class of surfaces than the one considered here: for example cones over smooth plane curves (where the result is likely to relate to the formulae by Aluffi and Faber for predegree polynomials of smooth plane curves). The strategy for such a computation would be a natural generalization of the one employed in this paper: i.e. a sequence of blow-ups over  $\mathbb{P}^{15}$  modelled after the geometry of the cone. The natural expectation is that one blow-up would be needed to account for the vertex of the cone and two additional ones accounting for the curve, one of which due to features of the curve, such as inflection points (cf. Example 1.1 in [1]).

The most obvious and naturally arising problem, however, is the one about predegree polynomials of arrangements of hyperplanes in a projective space of any dimension. The techniques used in this article should in principle be applicable to this more general case. The aim would be to resolve indeterminacies of a basic rational map analogous to the one considered here; in dimension  $n$ , it is natural to expect that  $n$  blow-ups over a projective space of dimension  $n^2 + 2n$  should achieve this resolution. These blow-ups are possibly related to those leading to “matroid varieties”, considered by Aluffi [5] and D. Jones [8]. However, the higher dimensional case is bound to be very challenging from the computational point of view.

The predegree polynomial of an  $r$ -fold hyperplane in  $\mathbb{P}^M$  could be shown to be  $P_M(t) = \sum_{k=0}^M \frac{r^k t^k}{k!}$  (which approaches  $\exp(rt)$  as  $M \rightarrow \infty$ ) using the fact that the orbit of the corresponding rational map from the projective space, which is the compactification of  $PGL(M + 1)$ , to the projective space  $\mathbb{P}^N$ , parameterizing degree  $r$  hypersurfaces, is the  $r$ -fold Veronese embedding of  $\mathbb{P}^M$  in  $\mathbb{P}^N$ .



Concerning predegree polynomials for hyperplane arrangements in higher dimension, the multiplicativity properties observed in the case of lines and planes suggest a rather precise conjecture for the predegree polynomial of the union of two arrangements, provided that they meet transversally and satisfy certain incidence requirements – one for each dimension between points and hyperplanes. It may be possible to attack this conjecture independently of the general solution to the problem for hyperplane arrangements; this would suffice for the computation of many classes of examples.

Another natural, possibly more ambitious, general class of problems amounts to establishing a clear connection between the invariants encoded in the predegree polynomial of a plane arrangement in space (or of a hyperplane arrangement in higher dimension) and the invariants arising in the well-developed field of hyperplane arrangements (see e.g. [9]). Previous work of P. Aluffi suggests a connection between Segre classes such as the ones used in the computations of the contributions to the predegrees in this paper and the geometry of the hyperplane arrangement. It is therefore natural to expect that formulae such as the ones obtained in this work may relate rather directly to (for example) Poincare polynomials of hyperplane arrangements.

## 7. Appendix

$$C_{8,i} = 35r_i^8 \sum_s 1 - 160r_i^7 \sum_s \lambda_s + 280r_i^6 \sum_s \lambda_s^2 - 224r_i^5 \sum_s \lambda_s^3 \\ + 70r_i^4 \sum_s \lambda_s^4 - r_i \sum_s \lambda_s^7 - 35r_i^4 \sum_{k \neq i} r_k^4;$$

$$C_{9,i} = 259r_i^9 \sum_s 1 - 1440r_i^8 \sum_s \lambda_s + 3240r_i^7 \sum_s \lambda_s^2 - 3696r_i^6 \sum_s \lambda_s^3 + 2142r_i^5 \sum_s \lambda_s^4 \\ - 504r_i^4 \sum_s \lambda_s^5 - 9r_i^2 \sum_s \lambda_s^7 + 8r_i \sum_s \lambda_s^8 + 189r_i^4 \sum_{k \neq i} r_k^5;$$

$$C_{10,i} = 1099r_i^{10} \sum_s \lambda_s^0 - 7200r_i^9 \sum_s \lambda_s + 19800r_i^8 \sum_s \lambda_s^2 - 29280r_i^7 \sum_s \lambda_s^3 \\ + 24570r_i^6 \sum_s \lambda_s^4 - 11088r_i^5 \sum_s \lambda_s^5 + 2100r_i^4 \sum_s \lambda_s^6 - 45r_i^3 \sum_s \lambda_s^7 \\ + 80r_i^2 \sum_s \lambda_s^8 - 36r_i \sum_s \lambda_s^9 - 525r_i^4 \sum_{k \neq i} r_k^6 - 126r_i^5 \sum_{k \neq i} r_k^5;$$

$$\begin{aligned}
 C_{11,i} = & 3499r_i^{11} \sum_s \lambda_s^0 - 26400r_i^{10} \sum_s \lambda_s + 85800r_i^9 \sum_s \lambda_s^2 - 155760r_i^8 \sum_s \lambda_s^3 \\
 & + 170610r_i^7 \sum_s \lambda_s^4 - 112728r_i^6 \sum_s \lambda_s^5 + 41580r_i^5 \sum_s \lambda_s^6 - 6765r_i^4 \sum_s \lambda_s^7 \\
 & + 440r_i^3 \sum_s \lambda_s^8 - 396r_i^2 \sum_s \lambda_s^9 + 120r_i \sum_s \lambda_s^{10} + 825r_i^4 \sum_{k \neq i} r_k^7 + 924r_i^5 \sum_{k \neq i} r_k^6;
 \end{aligned}$$

$$\begin{aligned}
 C_{12,i} = & 9274r_i^{12} \sum_s \lambda_s^0 - 79200r_i^{11} \sum_s \lambda_s + 297000r_i^{10} \sum_s \lambda_s^2 - 638880r_i^9 \sum_s \lambda_s^3 \\
 & + 862290r_i^8 \sum_s \lambda_s^4 - 747648r_i^7 \sum_s \lambda_s^5 + 406560r_i^6 \sum_s \lambda_s^6 \\
 & - 127215r_i^5 \sum_s \lambda_s^7 + 19085r_i^4 \sum_s \lambda_s^8 - 2376r_i^3 \sum_s \lambda_s^9 \\
 & + 1440r_i^2 \sum_s \lambda_s^{10} - 330r_i \sum_s \lambda_s^{11} \\
 & - 3564r_i^5 \sum_{k \neq i} r_k^7 - 462r_i^6 \sum_{k \neq i} r_k^6;
 \end{aligned}$$

$$\begin{aligned}
 C_{13,i} = & 21594r_i^{13} \sum_s \lambda_s^0 - 205920r_i^{12} \sum_s \lambda_s + 875160r_i^{11} \sum_s \lambda_s^2 - 2175888r_i^{10} \sum_s \lambda_s^3 \\
 & + 3487770r_i^9 \sum_s \lambda_s^4 - 3737448r_i^8 \sum_s \lambda_s^5 + 2676960r_i^7 \sum_s \lambda_s^6 \\
 & - 1236807r_i^6 \sum_s \lambda_s^7 + 339053r_i^5 \sum_s \lambda_s^8 - 50336r_i^4 \sum_s \lambda_s^9 \\
 & + 9360r_i^3 \sum_s \lambda_s^{10} - 4290r_i^2 \sum_s \lambda_s^{11} + 792r_i \sum_s \lambda_s^{12} \\
 & + 9009r_i^5 \sum_{k \neq i} r_k^8 + 4290r_i^6 \sum_{k \neq i} r_k^7 - 5005r_i^4 \sum_{k \neq i} r_k^9;
 \end{aligned}$$

$$\begin{aligned}
C_{14,i} = & 45618r_i^{14} \sum_s \lambda_s^0 - 480480r_i^{13} \sum_s \lambda_s + 2282280r_i^{12} \sum_s \lambda_s^2 - 6438432r_i^{11} \sum_s \lambda_s^3 \\
& + 11945934r_i^{10} \sum_s \lambda_s^4 - 15231216r_i^9 \sum_s \lambda_s^5 + 13513500r_i^8 \sum_s \lambda_s^6 \\
& - 8239803r_i^7 \sum_s \lambda_s^7 + 3316313r_i^6 \sum_s \lambda_s^8 - 820820r_i^5 \sum_s \lambda_s^9 \\
& + 127764r_i^4 \sum_s \lambda_s^{10} - 30030r_i^3 \sum_s \lambda_s^{11} + 11088r_i^2 \sum_s \lambda_s^{12} \\
& - 1716r_i \sum_s \lambda_s^{13} + 21021r_i^4 \sum_{k \neq i} r_k^{10} - 14014r_i^5 \sum_{k \neq i} r_k^9 \\
& - 21021r_i^6 \sum_{k \neq i} r_k^8 - 1716r_i^7 \sum_{k \neq i} r_k^7;
\end{aligned}$$

$$\begin{aligned}
C_{15,i} = & 89298r_i^{15} \sum_s \lambda_s^0 - 1029600r_i^{14} \sum_s \lambda_s + 5405400r_i^{13} \sum_s \lambda_s^2 \\
& - 17057040r_i^{12} \sum_s \lambda_s^3 + 35945910r_i^{11} \sum_s \lambda_s^4 - 53117064r_i^{10} \sum_s \lambda_s^5 \\
& + 56156100r_i^9 \sum_s \lambda_s^6 - 42477435r_i^8 \sum_s \lambda_s^7 + 22556105r_i^7 \sum_s \lambda_s^8 \\
& - 8076068r_i^6 \sum_s \lambda_s^9 + 1857492r_i^5 \sum_s \lambda_s^{10} - 313950r_i^4 \sum_s \lambda_s^{11} \\
& + 83160r_i^3 \sum_s \lambda_s^{12} - 25740r_i^2 \sum_s \lambda_s^{13} + 3432r_i \sum_s \lambda_s^{14} \\
& + 70070r_i^6 \sum_{k \neq i} r_k^9 + 19305r_i^7 \sum_{k \neq i} r_k^8 - 61425r_i^4 \sum_{k \neq i} r_k^{11};
\end{aligned}$$

$$\begin{aligned}
K_{12,i} = & 165r_i^{12} \left( \sum_k \pi_k^0 - \sum_s \lambda_s^0 \sum_k \pi_k^0 \right) + 1440r_i^{11} \left( - \sum_k \pi_k + \sum_s \lambda_s^0 \sum_k \pi_k \right) \\
& + 5544r_i^{10} \left( \sum_k \pi_k^2 - \sum_s \lambda_s^0 \sum_k \pi_k^2 \right) + 12320r_i^9 \left( - \sum_k \pi_k^3 + \sum_s \lambda_s^0 \sum_k \pi_k^3 \right) \\
& + 17325r_i^8 \left( \sum_k \pi_k^4 - \sum_s \lambda_s^0 \sum_k \pi_k^4 \right)
\end{aligned}$$

$$\begin{aligned}
 & +r_i^7 \left( -158400 \sum_s \lambda_s^2 \sum_k \pi_k^3 + 158400 \sum_s \lambda_s^3 \sum_k \pi_k^2 - 15840 \sum_k \pi_k^5 \right. \\
 & \quad \left. + 15840 \sum_s \lambda_s^5 \sum_k \pi_k^0 - 79200 \sum_s \lambda_s^4 \sum_k \pi_k + 79200 \sum_s \lambda_s \sum_k \pi_k^4 \right) \\
 & \quad + r_i^6 \left( -415800 \sum_s \lambda_s^4 \sum_k \pi_k^2 + 9240 \sum_k \pi_k^6 - 46200 \sum_s \lambda_s^6 \sum_k \pi_k^0 \right. \\
 & \quad \left. + 221760 \sum_s \lambda_s^5 \sum_k \pi_k + 369600 \sum_s \lambda_s^3 \sum_k \pi_k^3 - 138600 \sum_s \lambda_s^2 \sum_k \pi_k^4 \right) \\
 & \quad + r_i^5 \left( 399168 \sum_s \lambda_s^5 \sum_k \pi_k^2 - 221760 \sum_s \lambda_s^6 \sum_k \pi_k + 110880 \sum_s \lambda_s^3 \sum_k \pi_k^4 \right. \\
 & \quad \quad \left. - 3168 \pi_k^7 + 47520 \sum_s \lambda_s^7 \sum_k \pi_k^0 - 332640 \sum_s \lambda_s^4 \sum_k \pi_k^3 \right) \\
 & \quad + r_i^4 \left( 79200 \sum_s \lambda_s^7 \sum_k \pi_k + 495 \sum_k \pi_k^8 - 17325 \sum_s \lambda_s^8 \sum_k \pi_k^0 \right. \\
 & \quad \quad + 110880 \sum_s \lambda_s^5 \sum_k \pi_k^3 - 34650 \sum_s \lambda_s^4 \sum_k \pi_k^4 \\
 & \quad \quad \left. - 138600 \sum_s \lambda_s^6 \sum_k \pi_k^2 + 5775 \sum_{j \neq i} r_j^4 \sum_{k \neq i, j} r_k^4 \right) \\
 & \quad + r_i \left( 330 \sum_s \lambda_s^{11} \sum_k \pi_k^0 - 1440 \sum_s \lambda_s^{10} \sum_k \pi_k + 2376 \sum_s \lambda_s^9 \sum_k \pi_k^2 \right. \\
 & \quad \quad \left. - 1760 \sum_s \lambda_s^8 \sum_k \pi_k^3 + 495 \sum_s \lambda_s^7 \sum_k \pi_k^4 - \sum_k \pi_k^{11} \right); \\
 K_{13,i} & = 1925r_i^{13} \left( \sum_k \pi_k^0 - \sum_s \lambda_s^0 \sum_k \pi_k^0 \right) - 18720r_i^{12} \left( \sum_k \pi_k - \sum_s \lambda_s^0 \sum_k \pi_k \right) \\
 & \quad + 81432r_i^{11} \left( \sum_k \pi_k^2 - \sum_s \lambda_s^0 \sum_k \pi_k^2 \right) - 208208r_i^{10} \left( \sum_k \pi_k^3 - \sum_s \lambda_s^0 \sum_k \pi_k^3 \right)
 \end{aligned}$$

$$\begin{aligned}
& +345345r_i^9 \left( \sum_k \pi_k^4 - \sum_s \lambda_s^0 \sum_k \pi_k^4 \right) \\
& -r_i^8 \left( 2059200 \sum_s \lambda_s^2 \sum_k \pi_k^3 - 2059200 \sum_s \lambda_s^3 \sum_k \pi_k^2 + 386100 \sum_k \pi_k^5 \right. \\
& \quad - 205920 \sum_s \lambda_s^5 \sum_k \pi_k^0 + 1029600 \sum_s \lambda_s^4 \sum_k \pi_k \\
& \quad \left. - 1029600 \sum_s \lambda_s \sum_k \pi_k^4 - 180180 \sum_s \lambda_s^0 \sum_k \pi_k^5 \right) \\
& -r_i^7 \left( -3432000 \sum_s \lambda_s^3 \sum_k \pi_k^3 - 291720 \sum_k \pi_k^6 + 257400 \sum_s \lambda_s^2 \sum_k \pi_k^4 \right. \\
& \quad + 634920 \sum_s \lambda_s^6 \sum_k \pi_k^0 + 4890600 \sum_s \lambda_s^4 \sum_k \pi_k^2 \\
& \quad \left. + 823680 \sum_s \lambda_s \sum_k \pi_k^5 - 2882880 \sum_s \lambda_s^5 \sum_k \pi_k \right) \\
& -r_i^6 \left( -3747744 \sum_s \lambda_s^5 \sum_k \pi_k^2 + 2162160 \sum_s \lambda_s^3 \sum_k \pi_k^4 + 720720 \sum_s \lambda_s^4 \sum_k \pi_k^3 \right. \\
& \quad - 720720 \sum_s \lambda_s^7 \sum_k \pi_k^0 - 1441440 \sum_s \lambda_s^2 \sum_k \pi_k^5 \\
& \quad \left. + 2882880 \sum_s \lambda_s^6 \sum_k \pi_k + 144144 \sum_k \pi_k^7 \right) \\
& -r_i^5 \left( -42471 \sum_k \pi_k^8 - 2792790 \sum_s \lambda_s^4 \sum_k \pi_k^4 + 360360 \sum_s \lambda_s^6 \sum_k \pi_k^2 \right. \\
& \quad + 333333 \sum_s \lambda_s^8 \sum_k \pi_k^0 + 2018016 \sum_s \lambda_s^5 \sum_k \pi_k^3 + 1153152 \sum_s \lambda_s^3 \sum_k \pi_k^5 \\
& \quad \left. - 1029600 \sum_s \lambda_s^7 \sum_k \pi_k + 105105 \sum_{j \neq i} r_j^4 \sum_{k \neq i, j} r_k^4 \right) \\
& -r_i^4 \left( -1201200 \sum_s \lambda_s^6 \sum_k \pi_k^3 - 360360 \sum_s \lambda_s^4 \sum_k \pi_k^5 + 1081080 \sum_s \lambda_s^5 \sum_k \pi_k^4 \right)
\end{aligned}$$

$$\begin{aligned}
 & +5720 \sum_k \pi_k^9 - 40040 \sum_s \lambda_s^9 \sum_k \pi_k^0 + 514800 \sum_s \lambda_s^7 \sum_k \pi_k^2 \Big) \\
 -r_i^2 & \left( -6435 \sum_s \lambda_s^7 \sum_k \pi_k^4 + 22880 \sum_s \lambda_s^8 \sum_k \pi_k^3 - 30888 \sum_s \lambda_s^9 \sum_k \pi_k^2 \right. \\
 & \left. +18720 \sum_s \lambda_s^{10} \sum_k \pi_k - 4290 \sum_s \lambda_s^{11} \sum_k \pi_k^0 + 13 \sum_k \pi_k^{11} \right) \\
 -r_i & \left( 5148 \sum_s \lambda_s^7 \sum_k \pi_k^5 - 17160 \sum_s \lambda_s^8 \sum_k \pi_k^4 + 20592 \sum_s \lambda_s^9 \sum_k \pi_k^3 \right. \\
 & \left. -9360 \sum_s \lambda_s^{10} \sum_k \pi_k^2 + 792 \sum_s \lambda_s^{12} \sum_k \pi_k^0 - 12 \sum_k \pi_k^{12} \right);
 \end{aligned}$$

$$\begin{aligned}
 K_{14,i} = & 12221r_i^{14} \left( \sum_k \pi_k^0 - \sum_s \lambda_s^0 \sum_k \pi_k^0 \right) + 131040r_i^{13} \left( -\sum_k \pi_k + \sum_s \lambda_s^0 \sum_k \pi_k \right) \\
 + & 635544r_i^{12} \left( \sum_k \pi_k^2 - \sum_s \lambda_s^0 \sum_k \pi_k^2 \right) + 1837472r_i^{11} \left( -\sum_k \pi_k^3 + \sum_s \lambda_s^0 \sum_k \pi_k^3 \right) \\
 & + 3510507r_i^{10} \left( \sum_k \pi_k^4 - \sum_s \lambda_s^0 \sum_k \pi_k^4 \right) \\
 + & r_i^9 \left( -4636632 \sum_k \pi_k^5 + 3195192 \sum_s \lambda_s^0 \sum_k \pi_k^5 + 14414400 \sum_s \lambda_s^3 \sum_k \pi_k^2 \right. \\
 & \left. + 7207200 \sum_s \lambda_s \sum_k \pi_k^4 - 14414400 \sum_s \lambda_s^2 \sum_k \pi_k^3 \right. \\
 & \left. + 1441440 \sum_s \lambda_s^5 \sum_k \pi_k^0 - 7207200 \sum_s \lambda_s^4 \sum_k \pi_k \right) \\
 + & r_i^8 \left( -1051050 \sum_s \lambda_s^0 \sum_k \pi_k^6 + 4294290 \sum_k \pi_k^6 - 4684680 \sum_s \lambda_s^6 \sum_k \pi_k^0 \right)
 \end{aligned}$$

$$\begin{aligned}
& +14414400 \sum_s \lambda_s^3 \sum_k \pi_k^3 + 9009000 \sum_s \lambda_s^2 \sum_k \pi_k^4 - 30630600 \sum_s \lambda_s^4 \sum_k \pi_k^2 \\
& \quad - 11531520 \sum_s \lambda_s \sum_k \pi_k^5 + 20180160 \sum_s \lambda_s^5 \sum_k \pi_k \Big) \\
& + r_i^7 \left( -2759328 \sum_k \pi_k^7 + 11531520 \sum_s \lambda_s^2 \sum_k \pi_k^5 - 33153120 \sum_s \lambda_s^3 \sum_k \pi_k^4 \right. \\
& \quad + 4804800 \sum_s \lambda_s \sum_k \pi_k^6 + 17777760 \sum_s \lambda_s^4 \sum_k \pi_k^3 + 5834400 \sum_s \lambda_s^7 \sum_k \pi_k^0 \\
& \quad \left. + 16144128 \sum_s \lambda_s^5 \sum_k \pi_k^2 - 20180160 \sum_s \lambda_s^6 \sum_k \pi_k \right) \\
& + r_i^6 \left( -31615584 \sum_s \lambda_s^5 \sum_k \pi_k^3 + 23333310 \sum_s \lambda_s^4 \sum_k \pi_k^4 + 1180179 \sum_k \pi_k^8 \right. \\
& \quad - 8408400 \sum_s \lambda_s^2 \sum_k \pi_k^6 - 3300297 \sum_s \lambda_s^8 \sum_k \pi_k^0 + 4036032 \sum_s \lambda_s^3 \sum_k \pi_k^5 \\
& \quad \left. + 7567560 \sum_s \lambda_s^6 \sum_k \pi_k^2 + 7207200 \sum_s \lambda_s^7 \sum_k \pi_k - 945945 \sum_{j \neq i} r_j^4 \sum_{k \neq i, j} r_k^4 \right) \\
& + r_i^5 \left( 3027024 \sum_s \lambda_s^5 \sum_k \pi_k^4 + 784784 \sum_s \lambda_s^9 \sum_k \pi_k^0 + 10090080 \sum_s \lambda_s^6 \sum_k \pi_k^3 \right. \\
& \quad + 6726720 \sum_s \lambda_s^3 \sum_k \pi_k^6 - 13117104 \sum_s \lambda_s^4 \sum_k \pi_k^5 - 304304 \sum_k \pi_k^9 \\
& \quad \left. - 7207200 \sum_s \lambda_s^7 \sum_k \pi_k^2 + 2018016 \sum_{j \neq i} r_j^5 \sum_{k \neq i, j} r_k^4 \right) \\
& + r_i^4 \left( -84084 \sum_s \lambda_s^{10} \sum_k \pi_k^0 + 36036 \sum_k \pi_k^{10} + 2402400 \sum_s \lambda_s^7 \sum_k \pi_k^3 \right. \\
& \quad \left. + 6054048 \sum_s \lambda_s^5 \sum_k \pi_k^5 - 6306300 \sum_s \lambda_s^6 \sum_k \pi_k^4 - 2102100 \sum_s \lambda_s^4 \sum_k \pi_k^6 \right)
\end{aligned}$$

$$\begin{aligned}
 & +r_i^3 \left( 30030 \sum_s \lambda_s^{11} \sum_k \pi_k^0 - 131040 \sum_s \lambda_s^{10} \sum_k \pi_k + 216216 \sum_s \lambda_s^9 \sum_k \pi_k^2 \right. \\
 & \qquad \qquad \qquad \left. - 160160 \sum_s \lambda_s^8 \sum_k \pi_k^3 + 45045 \sum_s \lambda_s^7 \sum_k \pi_k^4 - 91 \sum_k \pi_k^{11} \right) \\
 & +r_i^2 \left( -11088 \sum_s \lambda_s^{12} \sum_k \pi_k^0 + 131040 \sum_s \lambda_s^{10} \sum_k \pi_k^2 - 288288 \sum_s \lambda_s^9 \sum_k \pi_k^3 \right. \\
 & \qquad \qquad \qquad \left. + 240240 \sum_s \lambda_s^8 \sum_k \pi_k^4 - 72072 \sum_s \lambda_s^7 \sum_k \pi_k^5 + 168 \sum_k \pi_k^{12} \right) \\
 & +r_i \left( 1716 \sum_s \lambda_s^{13} \sum_k \pi_k^0 - 43680 \sum_s \lambda_s^{10} \sum_k \pi_k^3 + 108108 \sum_s \lambda_s^9 \sum_k \pi_k^4 \right. \\
 & \qquad \qquad \qquad \left. - 96096 \sum_s \lambda_s^8 \sum_k \pi_k^5 + 30030 \sum_s \lambda_s^7 \sum_k \pi_k^6 - 78 \sum_k \pi_k^{13} \right); \\
 \\
 K_{15,i} = & 55901r_i^{15} \left( \sum_k \pi_k^0 - \sum_s \lambda_s^0 \sum_k \pi_k^0 \right) + 655200r_i^{14} \left( -\sum_k \pi_k + \sum_s \lambda_s^0 \sum_k \pi_k \right) \\
 + & 3505320r_i^{13} \left( \sum_k \pi_k^2 - \sum_s \lambda_s^0 \sum_k \pi_k^2 \right) + 11305840r_i^{12} \left( -\sum_k \pi_k^3 + \sum_s \lambda_s^0 \sum_k \pi_k^3 \right) \\
 & + 24443055r_i^{11} \left( \sum_k \pi_k^4 - \sum_s \lambda_s^0 \sum_k \pi_k^4 \right) \\
 & + r_i^{10} \left( 30017988 \left( -\sum_k \pi_k^5 + \sum_s \lambda_s^0 \sum_k \pi_k^5 \right) + 7207200 \left( -\sum_k \pi_k^5 + \sum_s \lambda_s^5 \sum_k \pi_k^0 \right) \right) \\
 + & 72072000 \left( \sum_s \lambda_s^3 \sum_k \pi_k^2 - \sum_s \lambda_s^2 \sum_k \pi_k^3 \right) + 36036000 \left( \sum_s \lambda_s \sum_k \pi_k^4 - \sum_s \lambda_s^4 \sum_k \pi_k \right) \\
 & + r_i^9 \left( -18568550 \lambda_s^0 \sum_k \pi_k^6 + 40790750 \sum_k \pi_k^6 - 24624600 \sum_s \lambda_s^6 \sum_k \pi_k^0 \right)
 \end{aligned}$$



$$\begin{aligned}
& +99099000 \sum_s \lambda_s^2 \sum_k \pi_k^4 - 86486400 \sum_s \lambda_s \sum_k \pi_k^5 - 135135000 \sum_s \lambda_s^4 \sum_k \pi_k^2 \\
& \quad + 24024000 \sum_s \lambda_s^3 \sum_k \pi_k^3 + 100900800 \sum_s \lambda_s^5 \sum_k \pi_k \Big) \\
& + r_i^8 \left( -32200740 \sum_k \pi_k^7 + 4504500 \sum_s \lambda_s^0 \sum_k \pi_k^7 + 72072000 \sum_s \lambda_s \sum_k \pi_k^6 \right. \\
& + 33462000 \sum_s \lambda_s^7 \sum_k \pi_k^0 + 190990800 \sum_s \lambda_s^4 \sum_k \pi_k^3 - 100900800 \sum_s \lambda_s^6 \sum_k \pi_k \\
& \left. - 436035600 \sum_s \lambda_s^3 \sum_k \pi_k^4 + 21621600 \sum_s \lambda_s^2 \sum_k \pi_k^5 + 30270240 \sum_s \lambda_s^5 \sum_k \pi_k^2 \right) \\
& + r_i^7 \left( -20592000 \sum_s \lambda_s \sum_k \pi_k^7 + 49999950 \sum_s \lambda_s^4 \sum_k \pi_k^4 - 90090000 \sum_s \lambda_s^2 \sum_k \pi_k^6 \right. \\
& + 152792640 \sum_s \lambda_s^3 \sum_k \pi_k^5 - 22516065 \sum_s \lambda_s^8 \sum_k \pi_k^0 + 17972955 \sum_k \pi_k^8 \\
& - 211891680 \sum_s \lambda_s^5 \sum_k \pi_k^3 + 88288200 \sum_s \lambda_s^6 \sum_k \pi_k^2 \\
& \left. + 36036000 \sum_s \lambda_s^7 \sum_k \pi_k - 5030025 \sum_{j \neq i} r_j^7 \sum_{k \neq i, j} r_k^4 \right) \\
& + r_i^6 \left( -158918760 \sum_s \lambda_s^4 \sum_k \pi_k^5 + 16816800 \sum_s \lambda_s^3 \sum_k \pi_k^6 + 7967960 \sum_s \lambda_s^9 \sum_k \pi_k^0 \right. \\
& + 133693560 \sum_s \lambda_s^5 \sum_k \pi_k^4 - 6766760 \sum_k \pi_k^9 + 36036000 \sum_s \lambda_s^2 \sum_k \pi_k^7 \\
& \left. + 25225200 \sum_s \lambda_s^6 \sum_k \pi_k^3 - 54054000 \sum_s \lambda_s^7 \sum_k \pi_k^2 + 14144130 \sum_{j \neq i} r_j^5 \sum_{k \neq i, j} r_k^4 \right) \\
& + r_i^5 \left( -28828800 \sum_s \lambda_s^3 \sum_k \pi_k^7 + 44144100 \sum_s \lambda_s^4 \sum_k \pi_k^6 + 18162144 \sum_s \lambda_s^5 \sum_k \pi_k^5 \right)
\end{aligned}$$

$$\begin{aligned}
 & -69369300 \sum_s \lambda_s^6 \sum_k \pi_k^4 - 1693692 \sum_s \lambda_s^{10} \sum_k \pi_k^0 + 1549548 \sum_k \pi_k^{10} \\
 & \quad + 36036000 \sum_s \lambda_s^7 \sum_k \pi_k^3 - 8072064 \sum_{j \neq i} r_j^5 \sum_{k \neq i, j} r_k^5 \Big) \\
 & + r_i^4 \left( -164255 \sum_k \pi_k^{11} + 9009000 \sum_s \lambda_s^4 \sum_k \pi_k^7 + 25225200 \sum_s \lambda_s^6 \sum_k \pi_k^5 \right. \\
 & - 25225200 \sum_s \lambda_s^5 \sum_k \pi_k^6 - 8783775 \sum_s \lambda_s^7 \sum_k \pi_k^4 + 313950 \sum_s \lambda_s^{11} \sum_k \pi_k^0 \\
 & \left. - 655200 \sum_s \lambda_s^{10} \sum_k \pi_k + 1081080 \sum_s \lambda_s^9 \sum_k \pi_k^2 - 800800 \sum_s \lambda_s^8 \sum_k \pi_k^3 \right) \\
 & \quad + r_i^3 \left( 1260 \sum_k \pi_k^{12} - 83160 \sum_s \lambda_s^{12} \sum_k \pi_k^0 + 982800 \sum_s \lambda_s^{10} \sum_k \pi_k^2 \right. \\
 & \left. - 2162160 \sum_s \lambda_s^9 \sum_k \pi_k^3 + 1801800 \sum_s \lambda_s^8 \sum_k \pi_k^4 - 540540 \sum_s \lambda_s^7 \sum_k \pi_k^5 \right) \\
 & \quad + r_i^2 \left( -1170 \sum_k \pi_k^{13} + 25740 \sum_s \lambda_s^{13} \sum_k \pi_k^0 - 655200 \sum_s \lambda_s^{10} \sum_k \pi_k^3 \right. \\
 & \left. + 1621620 \sum_s \lambda_s^9 \sum_k \pi_k^4 - 1441440 \sum_s \lambda_s^8 \sum_k \pi_k^5 + 450450 \sum_s \lambda_s^7 \sum_k \pi_k^6 \right) \\
 & \quad + r_i \left( 364 \sum_k \pi_k^{14} - 3432 \sum_s \lambda_s^{14} \sum_k \pi_k^0 + 163800 \sum_s \lambda_s^{10} \sum_k \pi_k^4 \right. \\
 & \left. - 432432 \sum_s \lambda_s^9 \sum_k \pi_k^5 + 400400 \sum_s \lambda_s^8 \sum_k \pi_k^6 - 128700 \sum_s \lambda_s^7 \sum_k \pi_k^7 \right).
 \end{aligned}$$

**Acknowledgements.** The author thanks Paolo Aluffi for his moral support and professional help.

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*Received May 9, 2007*