Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **34** (2008), 531–542

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

OSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

E. M. Elabbasy, T. S. Hassan

Communicated by T. Gramchev

Abstract. In this paper, we study the oscillatory behavior of first order nonlinear neutral delay differential equation

$$(x\,(t) - q\,(t)\,x\,(t - \sigma(t)))' + f\,(t, x\,(t - \tau(t))) = 0,$$

where σ, $\tau \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), [0, \infty))$ and $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$. The obtained results extended and improve several of the well known previously results in the literature. Our results are illustrated with an example.

1. Introduction. In recent years the literature on the oscillation of neutral delay differential equations is growing very fast. It is relatively a new field with interesting applications in real world life problems. In fact, the neutral delay differential equations appear in modelling of the networds containing lossless

2000 *Mathematics Subject Classification:* 34K15, 34C10.
Key words: Oscillation, neutral delay, differential equations.

transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect swithching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role, see Hale [20], Driver [5], Brayton and Willoughby [4], Popove [21] and Boe and Chang [3], and reference cited therein.Also this evidenced by the extensive references in the books of Ladde et al. [17] and Gyori and ladas [19]. In a paper [18], Li and Kuang obtained a sufficient codition for the oscillation of a nonlinear delay differential equation of the form

$$x'(t) + p(t)g(x(t-\tau(t))) = 0, \tag{1.1}$$

where

$$p \in C\left([0,\infty),[0,\infty)\right),\ \tau \in C\left([0,\infty),(0,\infty)\right),\ \lim_{t\to\infty}(t-\tau(t)) = \infty, \tag{1.2}$$

$$g \in C(\mathbb{R},\mathbb{R}) \quad \text{and} \quad ug(u) > 0, \quad \text{for } u \neq 0. \tag{1.3}$$

The main results in [18] is the following:

Theorem 1.1. *Assume that* (1.2) *and* (1.3) *hold and that for some* $\varepsilon > 0,\ M \geq 0$ *and* $r > 0$,

$$|g(u) - u| \leq M|u|^{1+r}, \quad \textit{for } |u| < \varepsilon.$$

Furthermore, suppose that

$$\int_{\delta(t)}^{t} p(s)ds \geq e^{-1}, \quad t \geq t_0,$$

and

$$\int_{t_0}^{\infty} p(t)\left[\exp\left(\int_{\delta(t)}^{t} p(s)ds - e^{-1}\right) - 1\right]dt = \infty,$$

where $\delta(t) := \max_{s\in[0,t]}\{s-\tau(s)\}$. *Then every solution of* (1.1) *oscillates.*

The purpose of this paper is to extend the above mentioned oscillation criteria to the nonlinear neutral delay differential equation

$$\left(x(t) - q(t)x(t-\sigma(t))\right)' + f\left(t, x(t-\tau(t))\right) = 0, \quad \text{for} t \geq t_0 > 0, \tag{1.4}$$

where

$$\sigma, \tau \in C\left([t_0, \infty), (0, \infty)\right), \lim_{t\to\infty}(t - \sigma(t)) = \lim_{t\to\infty}(t - \tau(t)) = \infty, \tag{1.5}$$

$$q \in C\left([t_0, \infty), [0, \infty)\right), \ 0 \le q_1 \le q(t) \le q_2 \le 1, \tag{1.6}$$

$$f \in C\left([t_0, \infty) \times \mathbb{R}, \mathbb{R}\right), \ uf(t, u) \ge 0, \tag{1.7}$$

and there exist

$$p \in C\left([t_0, \infty), (0, \infty)\right) \quad \text{and} \quad g \in C\left(\mathbb{R}, \mathbb{R}\right), \tag{1.8}$$

such that

$$g'(u) \ge 0, \quad f(t, u) \ge p(t)\, g(u) \quad \text{and} \quad |g(u) - u| \le M\, |u|^{1+r}, \tag{1.9}$$

for $u \in (-\varepsilon, \varepsilon)$ and, for some $\varepsilon > 0$, $M \ge 0$, $r > 0$ and $ug(u) > 0$, for $u \neq 0$.

Let $\varrho = \max\{\sigma, \tau\}$, and $t_1 \ge t_0$. A function $x(t) \in C\left([t_1 - \varrho, \infty), \mathbb{R}\right)$ is said to be a solution of equation (1.4), for some $t_1 \ge t_0$ if $x(t) - q(t)\, x(t - \sigma)$ is continuously differentiable on $[t_1, \infty)$ and satisfies (1.4), for $t > t_1$. For further research on the oscillation of neutral delay differential equations, see the recent papers by Elabbasy and Saker [10], Tang and Shen [23] and Elabbasy, Hassan and Saker [8] and [9]. As usual a solution $x(t)$ of equation (1.4) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$, otherwise it is nonoscillatory. Equation (1.4) is called oscillatory if every solution of this equation is oscillatory.

2. Main results. In this section, we will establish oscillation criteria for equation (1.4), which improves and extends known results.

Theorem 2.1. *Assume that* (1.5)–(1.9) *hold. Furthermore, suppose that*

$$\int_{\delta(t)}^{t} p(s)\, ds \ge \frac{1}{e}, \quad \text{for } t \ge t_0, \tag{2.1}$$

and

$$\int_{t_0}^{\infty} p(t) \left[\exp\left(\int_{\delta(t)}^{t} p(s)\, ds - \frac{1}{e} \right) - 1 \right] dt = \infty, \tag{2.2}$$

where $\delta(t) := \max_{s\in[t_0,t]}\{s-\tau(s)\}$. *Then every solution of equation* (1.4) *oscillates.*

Proof. Assume (1.4) has a nonoscillatory solution on $[t_0,\infty)$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)$, sufficiently large, so that $x(t)$, $x(t-\sigma(t))$, and $x(t-\tau(t)) > 0$ on $[t_1,\infty)$. *Set*

$$z(t) := x(t) - q(t)\,x(t-\sigma(t)). \tag{2.3}$$

Then, by (1.4) and (1.7), we find

$$z'(t) = -f(t, x(t-\tau(t))) < 0,$$

which implies that $z(t)$ is strictly decreasing. We claim that $z(t) > 0$ on $[t_1,\infty)$. If not, then there exists $t_2 \geq t_1$ such that $z(t_2) =: c < 0$ and hence $z(t) \leq c$, for all $t \geq t_2$. Then, from(2.3), we get

$$x(t) \leq c + q(t)\,x(t-\sigma(t)), \quad \text{for all } t \geq t_2. \tag{2.4}$$

We consider the following two possible cases. Case 1: $x(t)$ is unbounded, i.e., $\limsup_{t\to\infty} x(t) = \infty$. Thus, there is a increasing sequene $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$,

$$x(t_k) = \sup_{t\leq t_k} x(t) \text{ and } \lim_{k\to\infty} x(t_k) = \infty.$$

From (1.6) and (2.4), we find

$$x(t_k) \leq c + q(t_k)\,x(t_k - \sigma(t_k)) \leq c + x(t_k),$$

which is a contradiction. Case 2: $x(t)$ is bounded, i.e., $\limsup_{t\to\infty} x(t) = l < \infty$. Thus, there is a sequene $\{t_k\}$ such that $\lim_{k\to\infty} x(t_k) = l$ and then $\limsup_{k\to\infty} x(t_k - \sigma(t_k)) \leq l$. By (1.6) and (2.4), we have

$$x(t_k) \leq c + q(t_k)\,x(t_k - \sigma(t_k)) \leq c + x(t_k - \sigma(t_k)),$$

and so

$$l \leq c + \lim_{k\to\infty} x(t_k - \sigma(t_k)) \leq c + l,$$

which is also a contradiction. Then, we get $z(t) > 0$ on $[t_1,\infty)$. By using that fact $z(t)$ is strictly decreasing and positive function, we find that $\lim_{t\to\infty} z(t) = \alpha \geq 0$. If $\alpha > 0$, from (1.4), (1.9) and (2.3), we obtain

$$\begin{aligned} z(t) - z(t_1) &= -\int_{t_1}^{t} f(s, x(s-\tau(s)))\,ds \leq -\int_{t_1}^{t} p(s)g(x(s-\tau(s)))ds \\ &\leq -\int_{t_1}^{t} p(s)g(z(s-\tau(s)))ds \leq -\frac{g(\alpha)}{2}\int_{t_1}^{t} p(s)ds. \end{aligned}$$

Hence by (2.1), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts that $z(t)$ being positive function, then $\alpha = 0$. It follows from (1.9), and (2.3) that, for $t \geq t_1$

$$\begin{aligned} f\left(t, x(t-\tau(t))\right) &\geq p\left(t\right) g\left(x\left(t-\tau(t)\right)\right) \geq p\left(t\right) g\left(z\left(t-\tau(t)\right)\right) \\ &\geq p\left(t\right) \left(z\left(t-\tau(t)\right) - M z^{1+r}\left(t-\tau(t)\right)\right) \\ &\geq p\left(t\right) z\left(t-\tau(t)\right)\left(1 - M z^{r}\left(t-\tau(t)\right)\right). \end{aligned}$$

From this and equation (1.4), we find

$$z'\left(t\right) + p\left(t\right) z\left(t-\tau(t)\right)\left(1 - M z^{r}\left(t-\tau(t)\right)\right) \leq 0, \quad \text{for } t \geq t_1. \tag{2.5}$$

The rest of the proof is similar to that of Theorem 1 in [18] and hence is omitted. □

Theorem 2.2. *Assume that* (1.5)–(1.9) *hold. Furthermore, suppose that*

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} P\left(s\right) ds > \frac{1}{e}, \tag{2.6}$$

or

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} P\left(s\right) ds > 1, \tag{2.7}$$

where $\delta(t)$ *is defined as in Theorem* 2.1 *and* $P\left(t\right) := \left(1-\epsilon\right) p\left(t\right)$*, for* $\epsilon > 0$*. Then every solution of* (1.4) *oscillates.*

Proof. Assume (1.4) has a nonoscillatory solution on $[t_0, \infty)$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)$, sufficiently large, so that $x(t)$, $x\left(t-\sigma(t)\right)$, and $x\left(t-\tau(t)\right) > 0$ on $[t_1, \infty)$. As in the proof of Theorem 2.1, from (2.5) there exists $T \geq t_1$, sufficiently large such that

$$z'\left(t\right) + P\left(t\right) z\left(\delta\left(t\right)\right) \leq 0, \quad \text{for all } t \geq T. \tag{2.8}$$

But, then by Corollary 3.2.2 [19] the delay differential equation

$$z'\left(t\right) + P\left(t\right) z\left(\delta\left(t\right)\right) = 0, \tag{2.9}$$

has an eventually positive solution as well. It is also well known that (2.6) or (2.7) implies (2.9) has no eventually positive solution (see, [19] Theorem 3.4.3). This contradiction completes the proof. □

It is clear that there is a gap between (2.6) and (2.7) for the oscillation of all solutions of (1.4). The problem how to fill this gap for the equation (1.4) when the limit

$$\lim_{t\to\infty}\int_{\delta(t)}^{t} P(s)\,ds,$$

does not exist needs to be considered. This problem has been cleared for the linear (2.9). Let the numbers k and l be defined by

$$k := \liminf_{t\to\infty}\int_{\delta(t)}^{t} P(s)ds, \quad l := \limsup_{t\to\infty}\int_{\delta(t)}^{t} P(s)ds,$$

$$0 < k \le \frac{1}{e}, \quad l < 1,$$

and λ is the smallest root of the equation $\lambda = e^{k\lambda}$. Then (2.9) will be oscillatory if either of the following conditions is satisfied:

(A)

$$l > \frac{\ln\lambda + 1}{\lambda}, \qquad [15].$$

(B)

$$l > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \qquad [24].$$

(C)

$$l > \frac{1 + \ln\lambda}{\lambda} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \qquad [12].$$

(D)

$$l > 2k + \frac{2}{\lambda} - 1, \qquad [13].$$

(E)

$$l > \frac{\ln\lambda - 1 + \sqrt{5 - 2\lambda + 2k\lambda}}{\lambda}, \qquad [16].$$

(F)

$$l > \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2}, \qquad [7].$$

Note that Theorem 2.2 implies that (1.4) will also be oscillatory if either of the conditions (A)–(F) is satisfied.

Theorem 2.3. *Assume that* (1.5)–(1.9) *hold with* $\tau(t) = \tau \in (0, \infty)$. *Furthermore, suppose that*

$$\int_{t-\tau}^{t} p(s)\,ds \geq \frac{1}{a} > 0,$$

and

$$\int_{t_0}^{\infty} p(t)\left[\exp\left(a\int_{t-\tau}^{t} p(s)\,ds\right)\right]dt = \infty. \tag{2.10}$$

Then every solution of (1.4) *oscillates.*

Proof. Assume (1.4) has a nonoscillatory solution on $[t_0, \infty)$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)$, sufficiently large, so that $x(t)$, $x(t - \sigma(t))$, and $x(t - \tau) > 0$ on $[t_1, \infty)$. As in the proof of Theorem 2.1, from (2.5) there exists $T \geq t_1$, sufficiently large such that

$$z'(t) + \frac{1}{2}p(t)\,z(t-\tau) \leq 0, \quad \text{for all } t \geq T,$$

since $\lim_{t\to\infty} z(t) = 0$. In view [18, Lemmas 3, 4], we get $\dfrac{z(t-\tau)}{z(t)}$ is bounded and $\int_{t-\tau}^{t} p(s)ds \leq 2$. Set

$$\lambda(t) = -\frac{z'(t)}{z(t)},$$

we have the generalized characteristic equation

$$\lambda(t) \geq \frac{1}{2}p(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)\,ds\right).$$

It is easy to see that

$$e^{\frac{s}{r}} \geq 1 + \frac{s}{r^2}, \quad \text{for } s > 0,\ r \geq 1. \tag{2.11}$$

Let

$$A(t) := \exp\left(a\int_{t-\tau}^{t} p(s)\,ds\right),$$

then

$$\lambda(t) \geq \frac{1}{2}p(t)\exp\left(\frac{1}{A(t)}A(t)\int_{t-\tau}^{t}\lambda(s)\,ds\right).$$

By using (2.11), we have

$$\lambda(t) \geq \frac{1}{2} p(t) \left(1 + \frac{1}{A(t)} \int_{t-\tau}^{t} \lambda(s)\, ds\right).$$

Hence

$$A(t)\lambda(t) - \frac{1}{2} p(t) \int_{t-\tau}^{t} \lambda(s)\, ds \geq \frac{1}{2} p(t) A(t).$$

Integrating from T to N, where $T \leq N$ such that $0 < N - \tau < T$,

$$\int_{T}^{N} A(t)\lambda(t)\, dt - \frac{1}{2} \int_{T}^{N} p(t) \left(\int_{t-\tau}^{t} \lambda(s)\, ds\right) dt \geq \frac{1}{2} \int_{T}^{N} p(t) A(t)\, dt.$$

Interchanging the order of integration, we get

$$\begin{aligned} \int_{T}^{N} p(t) \left(\int_{t-\tau}^{t} \lambda(s)\, ds\right) dt &\geq \int_{T}^{N-\tau} \lambda(s) \left(\int_{s-\tau}^{s} p(t)\, dt\right) ds \\ &= \int_{T}^{N-\tau} \lambda(t) \left(\int_{t-\tau}^{t} p(s)\, ds\right) dt. \end{aligned}$$

Thus, we have

$$\int_{T}^{N} A(t)\lambda(t)\, dt - \frac{1}{2} \int_{T}^{N-\tau} \lambda(t) \left(\int_{t-\tau}^{t} p(s)\, ds\right) dt \geq \frac{1}{2} \int_{T}^{N} p(t) A(t)\, dt,$$

and hence

$$\int_{T}^{N} A(t)\lambda(t)\, dt + \int_{N-\tau}^{T} \lambda(t) \left(\int_{t-\tau}^{t} p(s)\, ds\right) dt \geq \frac{1}{2} \int_{T}^{N} p(t) A(t)\, dt,$$

then,

$$\int_{T}^{N} A(t)\lambda(t)\, dt + \int_{N-\tau}^{T} \lambda(t) A(t)\, dt \geq \frac{1}{2} \int_{T}^{N} p(t) A(t)\, dt,$$

since

$$A(t) \geq \int_{t-\tau}^{t} p(s)\, ds.$$

Then

$$\int_{N-\tau}^{N} \lambda(t) A(t)\, dt \geq \frac{1}{2} \int_{T}^{N} p(t) A(t)\, dt,$$

On the other hand, as $A(t) < \beta,\ \beta = e^{2a}$, we find

$$\int_{N-\tau}^{N} \lambda(t)\, dt \geq \frac{1}{2\beta} \int_{T}^{N} p(t)\, A(t)\, dt,$$

or

$$\ln \frac{z(N-\tau)}{z(N)} \geq \frac{1}{2\beta} \int_{T}^{N} p(t)\, A(t)\, dt.$$

In view of (2.10), we have

$$\lim_{t\to\infty} \frac{z(t-\tau)}{z(t)} = \infty,$$

which contradicts that $\dfrac{z(t-\tau)}{z(t)}$ is bounded for $t \geq T$. The proof of Theorem 2.3 is complete. □

Example 2.1. *Consider the neutral delay differential equation*

(2.12)
$$\left(x(t) - \frac{1}{2}(1+\sin t)\, x\left(\sqrt{t}\right)\right)' + f\left(t, x\left(\frac{t}{\lambda} - \lambda + 1\right)\right) = 0, \quad t \geq \lambda(\lambda-1)+1,$$

where $\lambda > 1$ *and*

$$f\left(t, x\left(\frac{t}{\lambda} - \lambda + 1\right)\right) = \left[\frac{1}{e \ln\lambda\, (t+\lambda)} + \frac{1}{(t+\lambda)\ln(t+\lambda)}\right] |x(t-\tau)|^r,\ r > 1.$$

Let

$$p(t) = \frac{1}{e \ln\lambda\, (t+\lambda)} + \frac{1}{(t+\lambda)\ln(t+\lambda)}, \quad \lambda > 1$$

and

$$g(x(t-\tau)) = |x(t-\tau)|^r, \quad r > 1.$$

It is easy to see that assumptions (1.5)–(1.9) *hold. Clearly, for* $t \geq \lambda(\lambda-1)+1$,

$$\int_{\frac{t}{\lambda}-\lambda+1}^{t} p(s)\, ds = \frac{1}{e} + \ln \frac{\ln(t+\lambda)}{\ln(\frac{t}{\lambda}+1)} = \frac{1}{e} - \ln\left(1 - \frac{\ln\lambda}{\ln(t+\lambda)}\right) > \frac{1}{e},$$

and

$$\lim_{t\to\infty} \int_{\frac{t}{\lambda}-\lambda+1}^{t} p(s)\, ds = \frac{1}{e}.$$

On the other hand

$$\begin{aligned} &\int_{\lambda(\lambda-1)+1}^{\infty} p\,(t)\left[\exp\left[\int_{\frac{t}{\lambda}-\lambda+1}^{t} p\,(s)\,ds-\frac{1}{e}\right]-1\right]dt \\ \geq\ &\int_{\lambda(\lambda-1)+1}^{\infty} p\,(t)\left[\int_{\frac{t}{\lambda}-\lambda+1}^{t} p\,(s)\,ds-\frac{1}{e}\right]dt \\ \geq\ &\frac{-1}{e\ln\lambda}\int_{\lambda(\lambda-1)+1}^{\infty}\frac{1}{t+\lambda}\ln\left(1-\frac{\ln\lambda}{\ln(t+\lambda)}\right)dt=\infty, \end{aligned}$$

since

$$\int_{\lambda(\lambda-1)+1}^{\infty}\frac{dt}{(t+\lambda)\ln(t+\lambda)}=\infty$$

and

$$\lim_{t\to\infty}\left(\ln(t+\lambda)\right)\ln\left(1-\frac{\ln\lambda}{\ln(t+\lambda)}\right)=-\ln\lambda.$$

Then Theorem 2.1 *implies that every solution of* (2.12) *oscillates.*

Additional examples may be readily given. We leave this to interested reader.

REFERENCES

[1] R. P. Agarwal, S. R. Grace, D. O'Regan. Oscillation theory for difference and Functional differential equations, Kluwer, Dordrecht, 2000.

[2] D. D. Bainov, D. P. Mishev. Oscillation Theory for Neutral Differential with Delay, Adam Hilger, NewYork, 1991.

[3] E. Boe, H. C. Chang. Dynamics of delayed systems under feedback control. *Chem. Engg. Sci.* **44** (1989), 1281–1294.

[4] R. K. Brayton, R. A. Willoughby. On the numerical integration of a symmetric system of difference-differential equations of neutral type. *J. Math. Anal. Appl.* **18** (1976), 182–189.

[5] R. D. Driver. A mixed neutral system. *Nonlinear Anal.* (1976), 182–189.

[6] K. Dib, R. M. Mathsen. Oscillation of solutions of neutral delay differential equations. *Math. Comput. Modelling* **32** (2000), 609–619.

[7] E. M. Elabbasy, T. S. Hassan. Oscillation criteria for first order delay differential equations. *Serdica Math. J.* **30** (2004), 71–86.

[8] E. M. Elabbasy, T. S. Hassan, S. H. Saker. Oscillation criteria for first-order nonlinear neutral delay differential equations. *J. Differential Equations* **134** (2005), 1–18.

[9] E. M. Elabbasy, T. S. Hassan, S. H. Saker. Oscillation criteria for first-order nonlinear neutral delay differential equations. *J. Appl. Math. Comput.* **21**, *1–2* (2006), 99–118.

[10] E. M. Elabbasy, S. H. Saker. Oscillation of first order neutral delay differential equations. *Kyungpook. Math. J.* **41** (2001), 311–321.

[11] A. Elbert, I. P. Stavroulakis. Oscillation and non-oscillation criteria for delay differential equations. *Proc. Amer. Math. Soc.* **124** (1995), 1503–1511.

[12] J. Jaros, I. P. Stavroulakis. Oscillation tests for delay equations. *Rocky Mountain J. Math.* **28** (1999), 197–207.

[13] M. Kon, Y. G. Sficas, I. P. Stavroulakis. Oscillation criteria for delay equations. *Proc. Amer. Math. Soc.* **128** (2000), 2989–2997.

[14] I. Kubiaczyk, S. H. Saker, J. Morchalo. New oscillation criteria for first order nonlinear neutral deelay differential equations. *Appl. Math. Comput.* **142** (2003), 225–242.

[15] M. K. Kwong. Oscillation of first order delay equations. *J. Math. Anal. Appl.* **159** (1991), 469–484.

[16] Y. G. Sficas, I. P. Stavroulakis. Oscillation criteria for first order delay equations. *Bull. London Math. Soc.* **35** (2003), 239–246.

[17] G. S. Ladde, V. Lakshmikantham, B. G. Zhang. Oscillation Theory of Differential Equations with Deviating Arguments. Marcel Dekker, New York, 1987.

[18] B. Li, Y. Kuang. Sharp condition for oscillations in some nonlinear nonautonomous delay differential equations. *Nonlinear Anal.* **29** (1997), 1265–1276.

[19] I. Györi, G. Ladas. Oscillation Theory of Delay Differential Equations: with Applications. Oxford Mathematical Monographs, Clarendon Press, 1991, 368 p.

[20] J. K. Hale. Theory of Functional Differential Equations. Springer-Verlag, New York, 1977.

[21] E. P. Popove. Automatic Regulation and Control, Nauka, Moscow, 1966 (in Russian).

[22] C. Qian, M. R. S. Kulenovic, G. Ladas. Oscillation of neutral equations with variable coefficients. *Rad. Mat.* **5** (1989), 321–331.

[23] X. Tang, J. Shen. New oscillation criteria for first order nonlinear delay differential equations. *Colloq. Math.* **83** (2000), 21–41.

[24] J. S. Yu, Z. C. Wang. Some futher results on oscillation of neutral equations. *Bull. Austral. Math. Soc.* **46** (1992), 149–157.

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: emelabbasy@mans.edu.eg
tshassan@mans.edu.eg

Received June 5, 2006
Revised April 8, 2008