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LIMIT THEOREMS FOR NON-CRITICAL BRANCHING PROCESSES WITH CONTINUOUS STATE SPACE

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Abstract. In the paper a modification of the branching stochastic process with immigration and with continuous states introduced by Adke S.R. and Gadag V.G. (1995) is considered. Limit theorems for the non-critical processes with or without non-stationary immigration and finite variance are proved. The subcritical case is illustrated with examples.

1. Introduction. Let $\{W_{i\,n}, i \geq 1, n \geq 1\}$ be a double array of independent and identically distributed non-negative random variables, $\{N_n(t), t \in T, n \geq 1\}$ be a family of nonnegative, integer-valued independent processes with independent stationary increments, with $N_n(0) = 0$ almost surely, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, \dots\}$.

Define a process $X_n, n \geq 0$, as follows. Let the initial state of the process be X_0 which is an arbitrary non-negative random variable and for $n \geq 0$

$$X_{n+1} = \sum_{i=1}^{N_{n+1}(X_n)} W_{i\,n+1} + U_{n+1}, \tag{1}$$

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where $\{U_n, n \geq 1\}$ is a sequence of independent non-negative random variables. Assume that families of random variables $\{W_{i\,n}, i, n \geq 1\}, \{U_n, n \geq 1\}$ of stochastic processes $\{N_n(t), t \in T, n \geq 1\}$ and random variable X_0 are independent. We also assume that $N_n(t), t \in T, n \geq 1$ have common one dimensional distributions.

This is a modification of the branching process which has a continuous space of states. It includes processes with immigration and in varying environments. This process was introduced by Adke and Gadag in [1]. In this paper it was shown that $Z_n = N_n(X_{n-1})$ is a Bienaymé-Galton-Watson (BGW) process with an immigration component, the distributional properties of the processes $\{X_n\}$ and $\{Z_n\}$ are described, a method for obtaining the extinction probabilities of these processes without immigration component is provided and limit theorems for the subcritical and critical cases were obtained in the case when $U_n, n \geq 1$ are independent and identically distributed random variables.

By Rahimov and Sabah [2] were proved certain theorems which establish relationship between processes X_n and Z_n in a sense of asymptotic behavior (see theorems A,B). In other words the problem of obtaining limit distributions in model (1) is connected with similar problem for the discrete-state process with or without immigration. These theorems allow to prove limit theorems for X_n from those of Z_n and vice versa. Applications of these theorems were provided, i.e. asymptotical distributions were obtained for the process (1) with or without stationary immigrations when the process is critical.

In paper [3] one may find further applications of those theorems. By Rahimov it was proved limit theorems for the critical processes X_n with decreasing immigration and also when it satisfies Foster-Williamson condition of weak stability.

In this paper we obtain limit distributions for non-critical processes X_n with or without non-stationary immigration and finite variance as applications of the mentioned theorems of Rahimov and Sabah from paper [2].

2. Limit distributions and examples. We introduce the following Laplace transforms

$$G(\lambda) = Ee^{-\lambda W_{i\,n}}, \quad H_n(\lambda) = Ee^{-\lambda U_n}.$$

It was shown in [1] that the offspring distribution and the distribution of the number of immigrating masses of the process Z_n have Laplace transforms $G(f(\lambda)) = Ee^{-\lambda \xi_{i\,n}}$ and $H_n(f(\lambda)) = Ee^{-\lambda \eta_n}$, respectively. Here $\xi_{i\,n} = N_n(W_{i\,n-1}),\ \eta_n = N_n(U_{n-1})$ for $i, n \geq 1, W_{i\,0} = U_0 = 0$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

If we assume that $N_1(X_0) = 1$, then the offspring mean and second factorial moment can be found by differentiating of the Laplace transform $G(f(\lambda))$ and are equal to [2]

$$m = E\xi_{i\,n} = EW \cdot EN, \quad B = E\xi_{i\,n} \cdot (\xi_{i\,n} - 1) = EW \cdot [varN - EN] + EW^2 \cdot (EN)^2,$$

respectively, where $N = N_1(1),\ W = W_{i\,1},\ i, n \geq 1$.

We consider the case $U_n = 0$ almost surely for each $n \geq 1$ and the process Z_n is subcritical. In this case Z_n is the subcritical BGW process without immigration. Let $\Psi(n, s) = Es^{Z_n}$ be the probability generating function of the process Z_n. We also use the generating function $g(s) = G(f(-\log s)),\ 0 \leq s \leq 1$ of the random variable $\xi_{i\,n},\ i, n \geq 1$.

It is known [4] that if $m < 1$, then there exists a random variable Z with generating function $\Psi(s) = Es^Z$ such that as $n \to \infty$

$$E\{s^{Z_n} | Z_n > 0\} \to \Psi(s), \tag{2}$$

where the generating function $\Psi(s)$ satisfies the functional equation

$$\Psi(g(s)) = m\Psi(s) + 1 - m,\ \Psi(0) = 0,\ \Psi(1) = 1.$$

Besides, $K^{-1} = \Psi'(1) < \infty$ if and only if $E[Z_1 \log Z_1] < \infty$.

Theorem 1. *If $m < 1$ and $E[N_1(X_0) \log N_1(X_0)] < \infty$, then as $n \to \infty$*

$$E[e^{-\lambda X_n} | X_n > 0] \to \Psi(G(\lambda)) \tag{3}$$

for each $\lambda > 0$.

Let $\Phi(s)$ be the inverse to the function $1 - \Psi(1 - s)$. It was shown in [5, page 411]

$$1 - \Psi(n, s) = \Phi(m^n(1 - \Psi(s)) = m^n(1 - \Psi(s)f(m^n(1 - \Psi(s)), \tag{4}$$

where $f(s) = \Phi(s)/s$ is a non-decreasing, slowly varying at zero function and $f(s) \downarrow K$ when $s \downarrow 0$.

We know from paper [3] that $P\{X_n > 0\} \sim 1 - \Psi(n, 0)$ when $n \to \infty$. Then from (4) we obtain a representation for non-extinction probability of the process X_n for the subcritical case.

Lemma. *If $m < 1$ and $E[N_1(X_0) \log N_1(X_0)] < \infty$, then as $n \to \infty$*

$$P\{X_n > 0\} \sim m^n(1 - \Psi(0))f(m^n(1 - \Psi(0))).$$

Example 1. Let the subcritical BGW process without immigration Z_n has the geometrical distribution, i.e. $P\{Z_1 = k | Z_0 = 1\} = bc^{k-1} = p_k,\ k = 1, 2, \ldots;\ 0 < b, c;\ b \leq 1 - c;\ p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Then

$$Es^{Z_1} = 1 - \frac{b}{1 - c} + \frac{bs}{1 - cs}.$$

It is known, that

$$Es^{Z_n} = 1 - m^n \frac{1 - s_0}{m^n - s_0} + \frac{m^n \left(\frac{1 - s_0}{m^n - s_0}\right)^2 s}{1 - \left(\frac{m^n - 1}{m^n - s_0}\right) s}, \tag{5}$$

where $s_0 = (1-b-c)/c(1-c)$. Hence it follows

(6) $$P\{Z_n > 0\} \sim s_0^{-1}(s_0 - 1)m^n, \quad n \to \infty.$$

From (2), (3), (5), (6) and the relation

$$E\{s^{Z_n}|Z_n > 0\} = 1 - \frac{1 - Es^{Z_n}}{P\{Z_n > 0\}}$$

we conclude that

$$\lim_{n\to\infty} E\{e^{-\lambda X_n}|X_n > 0\} = 1 - \frac{s_0(1 - G(\lambda))}{s_0 - G(\lambda)} \ ,$$

where $\lambda > 0$.

Example 2. Let the random variable W has the exponential distribution function in the previous example, i.e. $G(\lambda) = (1+\lambda)^{-1}$ for each $\lambda > 0$. Then

$$\lim_{n\to\infty} E\{e^{-\lambda X_n}|X_n > 0\} = \frac{C}{C+\lambda} \ ,$$

where $C = s_0^{-1}(s_0 - 1), \ \lambda > 0$.

Now we consider a supercritical case with non-homogeneous immigration. Let $\gamma(n) = EU_n < \infty$ for each $n \geq 1$, EW, EN, $\alpha(n) = E\eta_n$ and $\beta(n) = E\eta_n(\eta_n - 1)$ are finite for each $n \geq 1$.

Theorem 2. *If $m > 1, B \in (0, \infty)$ and $\gamma(n) \to 0$ as $n \to \infty$, then there exists a Laplace transform $\phi(\lambda)$ such that as $n \to \infty$*

$$Ee^{-\lambda X_n/m^n} \to \phi(\lambda EW)$$

for each $\lambda > 0$.

3. Proofs of the results. We need the following theorems from [2] to prove our results. We introduce the following notations

$$\Delta(n) = \frac{P\{Z_n > 0\}}{P\{X_n > 0\}}, \quad \delta(n,\lambda) = \frac{1 - H_n(\lambda)}{P\{Z_n > 0\}}.$$

Theorem A. *Let $\Delta(n) \to 1$ and $\delta(n,\lambda) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then as $n \to \infty$*

$$E[e^{-\lambda X_n}|X_n > 0] \to \varphi(-\log(G(\lambda)))$$

for each $\lambda > 0$, if and only if as $n \to \infty$ for each $\lambda > 0$

(7) $$E[e^{-\lambda Z_n}|Z_n > 0] \to \varphi(\lambda).$$

Let the sequences of positive numbers $\{k(n), n \geq 1\}$ and $\{a(n), n \geq 1\}$ be such that $k(n), a(n) \to \infty$ and for each $\lambda > 0$ there exists

$$\lim_{n\to\infty} k(n)\big(1 - G(\frac{\lambda}{a(n)})\big) = b(\lambda) \in (0, \infty). \tag{8}$$

Theorem B. *Let for sequences* $\{a(n), n \geq 1\}$ *and* $\{k(n), n \geq 1\}$ *condition* (9) *be satisfied. Then*

$$Ee^{-\lambda X_n/a(n)} \to \phi(b(\lambda))$$

if and only if for each $\lambda > 0$ *as* $n \to \infty$

$$Ee^{-\lambda Z_n/k(n)} \to \phi(\lambda). \tag{9}$$

Proof of Theorem 1. We apply Theorem A. If we write (2) in terms of Laplace transforms, we obtain the condition (7) with $\varphi(\lambda) = \Psi(e^{-\lambda})$.

It is known [4], that if $m < 1$ and $E[Z_1 \log Z_1] < 1$, then it is fulfilled the asymptotical formula

$$P\{Z_n > 0\} = Km^n(1 + o(1)),\ n \to \infty, \tag{10}$$

where $0 < K < \infty$.

Using the fact that the function $f(s)$ is slowly varying at zero function from Lemma and (10) we have $\Delta(n) \to 1 - \Psi(0)$ as $n \to \infty$, i.e. the condition $\Delta(n) \to 1$, $n \to \infty$ is fulfilled.

On the other hand, since $U_n = 0$ it follows $H_n(\lambda) = 1$ for each $n \geq 1$. Hence we obtain that $\delta(n, \lambda) = 0$ for each $n \geq 1$. Thus we conclude that all conditions of Theorem 1 are fulfilled. □

Proof of Theorem 2. We apply Theorem B. We use the following result proved in [6] for the supercritical BGW processes with decreasing immigration.

Theorem C. *If* $m > 1$, $B \in (0, \infty)$, $\max \beta(n) \leq C < \infty$, $\gamma(n) \to 0$ *as* $n \to \infty$, *then the random variable* $\zeta_n = Z_n/m^n$ *converges to some random variable* ζ *in distribution.*

If we write this convergence in terms of Laplace transforms, we obtain that condition (9) is satisfied with $k(n) = m^n$. If we choose $a(n) = k(n)$, then as $n \to \infty$

$$k(n)\big(1 - G(\frac{\lambda}{a(n)})\big) \to \lambda EW.$$

Thus condition (8) is fulfilled with $b(\lambda) = \lambda EW$. Therefore due to Theorem B as $n \to \infty$

$$Ee^{-\lambda X_n/m^n} \to Ee^{-\lambda\zeta EW},$$

and the proof is completed. □

Concluding remarks. We would like to note as concluding remark that in the subcritical case with stationary or increasing immigration Theorem A is not applicable because of the condition $\delta(n,\lambda) \to 0,\ t \to \infty$ is not fulfilled. Indeed, in these cases $P\{Z_n > 0\} \to 1$ and $1 - H_n(\lambda) \sim EU_n$ as $n \to \infty$, but $EU_n \nrightarrow 0$ as $n \to \infty$.

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