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SUPERSYMMETRY AND GHOSTS IN QUANTUM MECHANICS

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Abstract. A standard supersymmetric quantum system is defined by a Hamiltonian $\hat{H} = \frac{1}{2}(\hat{Q}^*\hat{Q}+\hat{Q}\hat{Q}^*)$, where the super-charge $\hat{Q}$ satisfies $\hat{Q}^2 = 0$, $\hat{Q}$ commutes with $\hat{H}$. So we have $\hat{H} \geq 0$ and the quantum spectrum of $\hat{H}$ is non negative.
On the other hand Pais-Ulhenbeck proposed in 1950 a model in quantum-field theory where the d'Alembert operator $\Box = \frac{\partial^2}{\partial t^2} - \triangle_x$ is replaced by fourth order operator $\Box(\Box + m^2)$, in order to eliminate the divergences occuring in quantum field theory.
But then the Hamiltonian of the system, obtained by second quantization, has large negative energies called "ghosts" by physicists. We report here on a joint work with A. Smilga (SUBATECH, Nantes) where we consider a similar problem for some models in quantum mechanics which are invariant under supersymmetric transformations. We show in particular that "ghosts" are still present.

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1. Introduction and Preliminaries. In this paper our aim is to discuss some recent results obtained in a joint work [16] with A. Smilga (SUBATECH, Nantes) and to give a pedagogical presention more suitable for mathematicians (it is our hope!). To explain our supersymmetric models we shall first present the Pais-Ulhenbeck oscillators [15] leading directly to the presence of ghosts.

To eliminate the divergences appearing in quantum field theory, Pais and Ulhenbeck (1950) proposed to start with a differential operator of order 4 in time instead of order 2. Usually the classical field theory starts with the box operator $\Box = \dfrac{\partial^2}{\partial t^2} - \triangle_x$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$. Pais and Ulhenbeck have introduced the following fourth order operator $\Box(\Box + m^2)$. After a canonical second quantization procedure, they get a sum of quantum oscillators.

But doing this we easily see that quantum fields with (large) negative energy appear. Physicists do not like solutions with negative energy, they call them "ghosts" because they have no physical meaning. In quantum field theory also exist ghosts, introduced by Faddev-Popov in their quantization theory of fields with gauge symmetry.

In [16] we consider systems with (super)-time derivatives of order greater than one in their Lagrangians and invariant under supersymmetric transformations. These transformations are defined using super-analysis calculus with bosonic and fermionic classical degree of freedom. Quantum models are obtained after canonical quantization.

A standard quantum supersymmetric system is determined by a quantum Hamiltonian $\hat{H}$, H is usually a classical[1] Hamiltonian, and supercharges $\hat{Q}, \hat{\tilde{Q}}$ such that

$$\hat{H} = \frac{1}{2}[\hat{\tilde{Q}}, \hat{Q}]_+ \tag{1}$$

$$[\hat{Q}, \hat{H}] = [\hat{\tilde{Q}}, \hat{H}] = 0 \tag{2}$$

$$\hat{\tilde{Q}}^2 = \hat{Q}^2 = 0 \tag{3}$$

$$\hat{\tilde{Q}} = \hat{Q}^*, \tag{4}$$

where $\tilde{Q} = \bar{Q}$ (complex conjugate, so $\hat{\tilde{Q}}$ is the Hermitian conjugate of $\hat{Q}$)), $[.,.]_+$

[1]In all of this paper linear operators, in some Hilbert space, will be denoted with a hat above. In concrete case $\hat{A}$ means some quantization of a "classical observable" A, for example the Weyl-Wigner quantization of smooth functions A defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, in the Hilbert space $L^2(\mathbb{R}^d)$.

is the anticommutator, $[\hat{A},\hat{B}]_+ := \hat{A}\hat{B} + \hat{B}\hat{A}$ (the commutator will be denoted indifferently $[\hat{A},\hat{B}] = [\hat{A},\hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$).

In particular we have $\hat{H} \geq 0$ and $\mathrm{spect}(\hat{H}) \subseteq [0,+\infty[$.

In our examples we shall see that the systems satisfy the supersymmetry equations (1), (2), (3) but $\hat{\bar{Q}} \neq \hat{Q}^*$, so it will be no more true that the energy spectrum of $\hat{H}$ is non negative. Our aim is to analyze as far as possible the spectrum of $\hat{H}$.

These Preliminaries will be rather long because we shall recall for non specialists (like the author of this paper!) what is supersymmetry, as well from the quantum and classical point of view. We shall also explain what is the Pais-Ulhenbeck oscillator because it is one of the first model with higher order derivatives leading to ghosts.

1.1. The Pais-Ulhenbeck oscillator. Here we follow the presentation of Mannheim [14]. More technical details are given in Appendix A.

Let us consider the classical field equation

$$\Box(\Box + m^2)\Phi(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3,$$

were $\Box = \dfrac{\partial^2}{\partial t^2} - \triangle_x$ is the d'Alembert operator.

As usual to quantize this equation, we first compute harmonic solutions $\Phi_k(t,x) = q(t)\mathrm{e}^{ikx}$, $k \in \mathbb{R}^3$. So q satisfies the 4th order differential equation

$$\frac{d^4}{dt^4}q + (2k^2+m^2)\frac{d^2}{dt^2}q + (k^4 + k^2m^2)q = 0. \tag{5}$$

Remark that for the Klein-Gordon equation $(\Box+m^2)\Phi(t,x) = 0$ we should get instead of (5) the harmonic oscillator equation $\dfrac{d^2}{dt^2}q + (k^2+m^2)q = 0$.

It is convenient to introduce the notations $\omega_1^2 + \omega_2^2 = 2k^2 + m^2$, $\omega_1^2\omega_2^2 = k^4 + k^2m^2$.

Following [13], Equation (5) is the Euler-Lagrange equation for the following Lagrangian

$$\mathcal{L}(\ddot{q},\dot{q},q) = \ddot{q}^2 - (\omega_1^2+\omega_2^2)\dot{q}^2 + \omega_1^2\omega_2^2q^2.$$

So, equation (5) is equivalent to the equation

$$\frac{\partial\mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right).$$

Let us compute the classical Hamiltonian H defined as the Legendre transform of $\mathcal{L}$ in $\ddot{q}$. We introduce new independent variable $x = \dot{q}$ and a Lagrange multiplier λ to get a more familiar Lagrangian with 3 degrees of freedom (q, x, λ)

$$\mathcal{L}_1 = \dot{x}^2 - (\omega_1^2 + \omega_2^2)x^2 + \omega_1^2\omega_2^2 q^2 + \lambda(\dot{q} - x)$$

Let us remark here that considering classical mechanics with high order time derivatives in the Lagrangian is rather old: Ostrogradski (1801–1861) already did that in 1850.

The Legendre transform for $\mathcal{L}_1$ is not well defined : we have the constraints $p_q := \dfrac{\partial \mathcal{L}_1}{\partial \dot{q}} = \lambda$, $p_\lambda = 0$. Dirac [7] found a method to overcome this difficulty. Following Dirac let us first compute the Hamiltonian

$$H_1 = p_x\dot{x} + p_q\dot{q} + p_\lambda\dot{\lambda} - \mathcal{L}_1, \text{ where } \dot{x} = \frac{p_x}{2}. \tag{6}$$

Here the Hamiltonian H_1 is defined on the phase space $\mathbb{R}^6$ with coordinates $(q, x, \lambda, p_q, p_x, p_\lambda)$ and we consider that $\dot{q}$ and $\dot{\lambda}$ are unknown functions defined on the phase which will be determined using all the constraint equations.

If we denote $\phi_1 = p_q - \lambda$ and $\phi_2 = p_\lambda$, that means that we want

$$\{H_1, \phi_1\} = \{H_1, \phi_2\} = 0, \text{for } \phi_1 = \phi_2 = 0.$$

So we get $\dot{q} = x$ and $\dot{\lambda} = 2\omega_1^2\omega_2^2 q$, hence

$$H_1 = \frac{p_x^2}{4} + p_q x + (\omega_1^2 + \omega_2^2)x^2 - \omega_1^2\omega_2^2 q^2 + 2\omega_1^2\omega_2^2 p_\lambda$$

We can forget the λ degree of freedom and finally we find an effective Hamiltonian with 2 degrees of freedom $(q, x) \in \mathbb{R}^2$,

$$\boxed{H = \frac{p_x^2}{4} + p_q x + (\omega_1^2 + \omega_2^2)x^2 - \omega_1^2\omega_2^2 q^2}. \tag{7}$$

We can easily see that the equation of motion (5) can be deduced from the Hamilton flow of H.

Remark 1.1. For constrainted Hamiltonian systems Dirac [7] introduced a modified braket $\{.,;\}_{DB}$ with better properties on the constraint set.

Our Hamiltonian H can be canonically quantized in $L^2(\mathbb{R}^2)$ and gives a second quantization procedure for the classical field equation (5).

Assume that $m > 0$, or equivalently, $\omega_1 \neq \omega_2$. Using Fock quantization (see Appendix A for more details) , it is not difficult to get the quantized Hamiltonian $\hat{H}$. In suitable complex symplectic coordinates it can be writen as

$$\hat{H} = \omega_1 \hat{a}_1^* a_1 - \omega_2 \hat{a}_2^* a_2$$

with the commutation relations $[\hat{a}_1, \hat{a}_1^*] = [\hat{a}_2, \hat{a}_2^*] = 1$, $[\hat{a}_1, \hat{a}_2] = 0$. But $a_1^* a_1 + a_2^* a_2$ is a positive-definite oscillator, so we get all the spectrum of $\hat{H}$: it is the family of eigenvalues $E_{n,m} = \left(n + \frac{1}{2}\right)\omega_1 - \left(m + \frac{1}{2}\right)\omega_2$, with $n, m \in \mathbb{N}$. So we have many ghosts due to higher order time derivatives in the Lagrangian. But these ghost are not dangerous (see more detail in Appendix A and in [1]).

For $\omega_1 = \omega_2 > 0$ $(m = 0)$ physicists have some arguments to claim that ghosts disappear because they go out of the physical space ([14, 8, 18]). It seems to us that this claim has to be supplied with more mathematical details as suggested by the paper [2].

1.2. Supersymmetry. Let us first recall that supersymmetry is a symmetry between fermions and bosons. During the thirty past years many efforts has been devoted by physicists to understand its dynamical and phenemonological consequences for elementary particles. Even if Supersymmetry is a beautiful and elegant theory there is no experimental evidence to valid it. The main reason is that at low energy supersymmetry is broken but it is expected that supersymmetry may be detected soon in high-energy collider experiments in CERN. Nevertheless the mathematical aspects of supersymmetry are very exciting and bring a lot of new ideas in mathematics, in particular after Witten.

Let us now recall some elementary and basic facts concerning supersymmetry. A standard supersymmetric model is an Hamiltonian $\hat{H} = \frac{1}{2}(\hat{Q}^*\hat{Q} + \hat{Q}\hat{Q}^*)$, where $\hat{H}, \hat{Q}$ are operators on some Hilbert space $\mathcal{H}$ defined on a dense domain and $\hat{Q}^*$ is the Hermitian adjoint of $\hat{Q}$. $\hat{H}$ is supposed to have a self-adjoint extension in $\mathcal{H}$.

If E is an eigenvalue of $\hat{H}$ then $\ker(\hat{H} - E)$ is an invariant subspace for $\hat{Q}$ and $\hat{Q}^*$. If $E > 0$ we can define

$$\hat{A} = \frac{1}{\sqrt{2E}} \hat{Q} \mid_{\ker(\hat{H}-E)} \quad \hat{A}^* = \frac{1}{\sqrt{2E}} \hat{Q}^* \mid_{\ker(\hat{H}-E)} .$$

We see that $\hat{A}$, $\hat{A}^*$ satisfy the relations of two generators for a Clifford algebra, $[\hat{a}^*, \hat{a}]_+ = 1$, $\hat{a}^2 = 0$. But this Clifford algebra has only one irrreducible representation, which is of dimension 2, realized with the Pauli matrices

$$\hat{A} = \sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{A}^* = \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The fermionic charge is usually defined by

$$F_c = 2\hat{A}^*\hat{A} - 1 = \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is well defined in $(\ker(\hat{H})^\perp$ and commutes with $\hat{H}$. Hence we can define the bosonic space and the fermionic space: $\mathcal{H}_b := \ker(F_c - 1)$ and $\mathcal{H}_f := \ker(F_c + 1)$. $\mathcal{H}_{b,f}$ are invariant for $\hat{H}$ and $\hat{Q}$ exchanges $\mathcal{H}_b$ and $\mathcal{H}_f$. Then for $E > 0$ the states with energy E appear by pairs: one fermion and one boson which are connected by the super-charge $\hat{Q}$.

Let us now consider the energy $E = 0$. Two cases can be considered.

(i) 0 is a groud state then $\hat{H}\psi = 0$ if and only if $\hat{Q}\psi = \hat{Q}^*\psi = 0$ hence supersymmetry is preserved.

(ii) there exits ψ such that $\hat{Q}\psi = 0$ and $\hat{Q}^*\psi \neq 0$ then supersymmetry is broken.

The main problem when considering supersymmetric problems is to decide to which case belongs the system.

The answer is easy to get for the famous Witten example. Let be V a C^2 real potential on $\mathbb{R}$ such that $\lim_{|x|\to+\infty} |V'(x)| = +\infty$. The supercharge $\hat{Q}$ is defined by

$$\hat{Q} := \hat{Q}_W = \sigma_-(D_x - iV'(x)) \tag{8}$$

where $D_x = \dfrac{d}{idx}$. The Hamiltonian introduced by E. Witten is the following :

$$\hat{H}_W = \frac{1}{2}(\hat{Q}^*\hat{Q} + \hat{Q}\hat{Q}^*) = \frac{1}{2}\left(D_x^2 + V'(x)^2 - \sigma_3 V''(x)\right). \tag{9}$$

Under suitable technical assumptions on V, $\hat{H}$ is essentially self-adjoint in $L^2(\mathbb{R})$ its resolvent is compact and its spectrum is discrete.

Solving the first order equation $(D_x \pm iV'(x))\psi = 0$, it is not difficult to prove that supersymmetry is broken if and only if the following condition is satisfied :

$$\lim_{x\to+\infty} V(x) = -\lim_{x\to-\infty} V(x).$$

In supersymmetric quantum systems bosons and fermions are put at the same level. The classical analogue of bosons is classical mechanics and bosons systems are obtained by canonical quantization of functions of positions and momenta, defined on a phase space with real coordinates. A natural question to ask is: what is the classical analogue of fermions?

Let us remember that fermions follows the exclusion Pauli principle. This principle is implemented by antisymetric wave functions (Fock space) and families of creation/annihilation operators on the antisymmetric Fock space satisfying the anticommutation relations (see [3] for more details)

$$[\hat{a}_j, \hat{a}_k^\dagger]_+ = \delta_{j,k}. \tag{10}$$

By analogy with bosons, it is natural to imagine some classical fermionic variables θ_j, anticommuting ($\theta_j\theta_k + \theta_k\theta_j = 0$), such that after a suitable quantization procedure we can get (10). These fermionic variables are generators of a Grassmann algebra. A mathematician can see a Grassmann algebra as an exterior algebra and every algebraic formula with fermionic variables can be translated into the language of differential forms (for example see [17]).

In several fondamental works, starting almost 40 years ago, physicists and mathematicians have succeeded to construct a beautiful theory putting at the same level bosons and fermions. This theory is valid as well on the quantum side and classical side, not only for quantum mechanics but more important in field theory ([5, 20]).

In the following subsection we shall give a brief approach of these ideas, with enough details to understand the motivations for analysis of the models found in [16].

1.3. Superfields and Hélein's formula. As already said, it is natural to represent fermionic states using Grassmann algebras. Recall that the Grassmann algebra $\mathcal{G}_n$, with n generators $\theta_1, \cdots, \theta_n$ is the algebra with unit (over $\mathbb{R}$ or $\mathbb{C}$ spanned by the θ_j, where $[\theta_k, \theta_k]_+ = 0$, for all j, k. It is a linear space of dimension 2^n (isomorphic the the exterior algebra of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

More explicitly we have the following interpretation. Recall that fermionic states are usually represented in the antisymmetric Fock space $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, where $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_n = \wedge^n\mathcal{K}$ where $\mathcal{K} := L^2(\mathbb{R}^3)$ is the one fermion state space, $\wedge$ is the notation for the antisymmetric tensor product. So that $\mathcal{H}_n$ is the n-particles space). Let us fix an orthonormal basis of $\mathcal{K}$ and consider a state $\psi \in \mathcal{H}$. We have

$$\psi = \sum_{i_1<i_2<\cdots<i_n} \psi_{i_1 i_2\cdots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_n}.$$

The components $\psi_{i_1 i_2 \cdots i_n}$ can be determined from their generating function in the Grasmann algebra $\mathcal{G}_n$ defined by

$$\Psi(\theta_1, \cdots, \theta_n) = \sum_{i_1 < i_2 < \cdots < i_n} \psi_{i_1 i_2 \cdots i_n} \theta_{i_1} \cdots \theta_{i_n}.$$

We remark that in the transformation $\psi \mapsto \Psi$, ψ takes its values in $\mathbb{C}$ but Ψ takes its values in the Grassmann algebra $\mathcal{G}_n$.

Another important remark is the following. In canonical quantification of bosons a fundamental property is that the commutator of two observables is the quantification of the Poisson bracket (at least for the leading terms). For fermions commutators are replaced by anticommutators: $[A, B]_+ := AB + BA$ (A, B are in some associative algebra). A classical analogue of fermions with usual real or complex numbers is not possible because that theory needs to have a lot of nilpotent terms (for example the Witten charge satisfies $Q^2 = 0$). The starting point of this theory appeared in several papers of Golfand-Likhtman (1971), Volkov-Akulov (1972), Wess-Zumino (1973), Salam-Stradee (1974). The mathematical apparatus was later developped by several people, in particular Berezin, Leites, Manin, Rogers, Dewitt, around 1980. Using Grassmann variables, they defined superspaces, superfields, supermanifolds, super Lie groups.

Let us recall briefly the definition of superspaces and superfields. The magic formula very often used in physicist's papers is the following

$$\lll \Phi = \varphi + \theta_1\psi_1 + \theta_2\psi_2 + F\theta_1\theta_2 \ggg \tag{11}$$

where Φ is a superfield depending on real coordinates $x \in \mathbb{R}^d$, ψ_1, ψ_2, F are " complex fermionic coordinates", θ_1, θ_2 are the generators of a Grassmann algebra: $[\theta_j, \theta_k]_+ = 0$. For simplicity, we assume that we have only two fermions in our system. By convention real variables are even, Grassmann variables θ_j are odd.

The meaning of formula (11) is not obvious because Φ is even, θ_j are odd, so ψ_j have to be odd so they cannot be complex numbers.

A possibility to give a mathematical meaning to (11), is to define Φ as a morphism from a "mysterious" space $\mathbb{R}^{d|2}$ into $\mathbb{R}$ ($d|2$ means d bosons and 2 fermions). Intuitively $\mathbb{R}^{d|2}$ is a space defined implicitely by the system of coordinates $(x_1, \cdots, x_d; \theta_1, \theta_2)$. More precisely, Φ is defined as a superalgebra morphism Φ^* ($\Phi^*(f)$ is a fermionic generalisation for the usual pullback: $\Phi^*(f) = f \circ \Phi$ for 0 fermion) from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R}^{d|2})$, where $C^\infty(\mathbb{R}^{d|2}) = C^\infty(\mathbb{R}) \otimes \mathbb{C}[\theta_1, \theta_2]$ and $\mathbb{C}[\theta_1, \theta_2]$ is the Grassmann algebra $\mathcal{G}_2$ with 2 generators.

Let us recall some basic facts about super-structure. In general, a super-structure is defined as a $\mathbb{Z}/2\mathbb{Z}$-structure. For example a super-linear space is a direct sum of two linear spaces: $V = V_0 + V_1$, V_0 is the even part, V_1 is the odd part. In other words, we define a parity map

$$\pi : V_0\backslash\{0\} \cup V_1\backslash\{0\} \to \mathbb{Z}/2\mathbb{Z}, \pi(a) = 0 \text{ if } a \in V_0\backslash\{0\} \text{ and } \pi(a) = 1 \text{ if } a \in V_1\backslash\{0\}.$$

A super-algebra is a super-linear space and an algebra such that $\pi(ab) = \pi(a) + \pi(b)$. This super-algebra is commutative if $ab = ba(-1)^{\pi(a)\pi(b)}$.

A morphism from a super-algebra A into the super-algebra B is a linear map $\Phi : A \to B$, such that $\Phi(a)$ has the same parity as a when a has a parity. A basic example is the space of forms where the graduation is given by the parity of the degree.

Defining spaces by morphisms (ring spaces, sheafs) is a well known approach in algebraic geometry. This method was used to define "classical bosonic-fermionic spaces (Berezin, Leites, see the books [20, 5]). So, in this approach the superspace $\mathbb{R}^{d|2}$ is a kind of virtual space. There exists another approach, coming from differential geometry [6], [19], where superspaces are defined more intuitively as set of points.

Let us come back to formula (11). A trick for ψ_j becomes odd is to add extra fermionic coordinates θ_3, θ_4. Now, a morphism Φ^* from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R}^{1|4})$ has the following expression

$$\Phi^*(f)(x) = \sum_{|J|\leq 4} a_J(f)(x)\theta^J, \quad J = (j_1, \cdots, j_4)$$

In this sum, $|J|$ is even. In particular $f \mapsto a_{0000}(f)(x)$ is multiplicative so there exists a smooth function $\varphi(x)$ such that $a_{0000}(f)(x) = f(\varphi(x))$.

For $|J| = 2$, $f \mapsto a_J(f)(x)$ is a derivation at $t = \varphi(x)$. Elaborating on this approach, Hélein get the following nice formula

$$\boxed{\Phi^*(f)(x) = \exp(Z_x)f \mid_{t=\varphi(x)}} \tag{12}$$

where Z_x is a vector field on $\mathbb{R}$, depending on x and on fermionic coordinates. We have: $Z_x = T_0 + \theta_1 T_1 + \theta_2 T_2 + \theta_1\theta_2 T_{1,2}$, the $T's$ depend only on the extra fermionic variables, T_1, T_2 are odd, $T_0, T_{1,2}$ are even.

An explicit computation gives the following formula

$$\Phi^*(f)(x) = f + (\theta_1\psi_1 + \theta_2\psi_2)f' + \theta_1\theta_2(f'F - f''\psi_1\psi_2) \mid_{t=\varphi(x)} . \tag{13}$$

The interpretation of this formula is the following: the superfield Φ is obtained as a deformation of the classical (bosonic) field φ by the vector field $\theta_1\psi_1 +$

$\theta_2\psi_2 + F\theta_1\theta_2$. This can be seen as a non commutative version of the usual Taylor formula, which can be written as

$$\boxed{e^{h\partial_x} f(x) = f(x+h)}. \tag{14}$$

We urge the reader to read the original Hélein's paper [11].

1.4. Classical interpretation of the Witten model. Classical means here that we want to understand the Witten supersymmetric model as the quantization of a classical system invariant under some supersymmetric transformation with conserved charges.

Let us first recall some rules to compute with fermionic variables. Here analysis is reduced to elementary algebra : any function of the θ_j is a polynomial in θ_j. Derivatives are defined by :

$$\frac{\partial}{\partial\theta_1}(\theta_1\theta_2) = \theta_2, \quad \left[\frac{\partial}{\partial\theta_j}, \frac{\partial}{\partial\theta_k}\right] = 0, \quad \left[\frac{\partial}{\partial\theta_j}, \theta_k\right] = \delta_{j,k}.$$

Integral is defined by:

$$\int \theta_j d\theta_j = 1, \quad \int 1 d\theta_j = 0.$$

It is sometimes convenient to define an involution in $\mathbb{C}[\theta_1, \theta_2]$ as follows: $\theta = \theta_1$, $\bar\theta = \theta_2$.

Now we want to represent the supersymmetric algebra (1) to (4) with $\hat H = i\partial_t$, as a Lie algebra with anticommuting paramaters θ, $\bar\theta$ (see [21]). Here we forget the hat superscripts.

The multiplicative law group, as translation on parameters $(t, \theta, \bar\theta)$, is given by

$$(t, \theta, \bar\theta) \mapsto ((t + s + i(\theta\bar\epsilon - \epsilon\bar\theta), \theta + \epsilon, \bar\theta + \bar\epsilon).$$

So we find that these translations on superfields are generate by the differential operators Q, $\bar Q$,

$$Q = \frac{\partial}{\partial\theta} - i\bar\theta\frac{\partial}{\partial t}, \quad \bar Q = \frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}. \tag{15}$$

We also have two covariant derivatives, defined as above, with right multiplication instead of left mulptiplication,

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}, \quad \bar{\mathcal{D}} = -\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}. \tag{16}$$

We can remark that $\mathcal{D}$ and $\bar{\mathcal{D}}$ anticommute with Q and $\bar{Q}$ and satisfy the following algebra rules of a supersymmetric system :

$$[\mathcal{D}, \bar{\mathcal{D}}]_+ = -2i\frac{\partial}{\partial t} = -2H, \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0.$$

Let be a superpotential $V(X)$ depending on the supervariable (defined in the previous subsection, with the new notation $y = F$),

$$X = x + \theta\bar{\psi} + \psi\bar{\theta} + y\theta\bar{\theta}.$$

Let us consider the super-Lagrangian $L = \frac{1}{2}\bar{\mathcal{D}}X\mathcal{D}X + V(X)$ and the pseudoclassical Lagrangian $L_{psc} = \int L d\bar{\theta} d\theta$. We get

$$L_{psc} = \frac{\dot{x}^2 + y^2}{2} + \frac{i}{2}(\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}) + \frac{1}{2}V''(x)\bar{\psi}\psi + V'(x)y.$$

Let us compute the conjugate variables for this Lagrangian.

$$\begin{aligned}
&(17) \qquad & \frac{\partial L_{psc}}{\partial \dot{x}} &= \dot{x} \Rightarrow p_x = \dot{x} \\
&(18) \qquad & \frac{\partial L_{psc}}{\partial \dot{y}} &= 0 \Rightarrow \frac{\partial L}{\partial y} = 0 \Rightarrow y = -V'(x) \\
&(19) \qquad & \frac{\partial L_{psc}}{\partial \dot{\psi}} &= \frac{i}{2}\bar{\psi} \Rightarrow p_\psi = \frac{i}{2}\bar{\psi} \\
&(20) \qquad & \frac{\partial L_{psc}}{\partial \dot{\bar{\psi}}} &= \frac{i}{2}\bar{\psi} \Rightarrow p_{\bar{\psi}} = \frac{i}{2}\psi
\end{aligned}$$

Here we can eliminate the variable y. But we have constraints. To solve this constrained problem we can use Dirac method (see 1.1 and [12]). Hence we can get the following Witten Hamiltonian:

$$\boxed{H_w = \frac{1}{2}p_x^2 + \frac{V'(x)^2}{2} - \frac{i}{2}V''(x)(p_\psi\psi - p_{\bar{\psi}}\bar{\psi})} \tag{21}$$

and the Witten Lagrangian

$$L_w = \frac{\dot{x}^2}{2} - \frac{V'(x)^2}{2} + \frac{i}{2}(\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}) + \frac{1}{2}V''(x)\bar{\psi}\psi.$$

Now we can see that the Lagrangian L_w is invariant (up to a total time derivative) under the supersymmetric transformation

$$\begin{aligned} \delta x &= \bar{\epsilon}\psi + \bar{\psi}\epsilon, \\ \delta\psi &= -(i\dot{x} + V'(x))\epsilon, \\ \delta\bar{\psi} &= (i\dot{x} - V'(x))\bar{\epsilon}. \end{aligned} \tag{22}$$

By Nœther Theorem we get the conserved charges

$$\boxed{Q = (\dot{x} - iV'(x))\psi}, \quad \boxed{\bar{Q} = (\dot{x} + iV'(x))\bar{\psi}}.$$

But our system has the constraints $p_\psi = \frac{i}{2}\bar{\psi}$, $p_{\bar{\psi}} = \frac{i}{2}\psi$, so in the time evolution of the system the Poisson bracket is replaced by the Dirac bracket ($\{.,.\}_D$) [12]. Explicit computations give here

$$\{x, p_x\}_D = 1,\ \{\psi, p_\psi\}_D = -\frac{1}{2},\ \{\bar{\psi}, p_{\bar{\psi}}\}_D = -\frac{1}{2}, \tag{23}$$

$$\{\psi, \bar{\psi}\}_D = -i,\ \{p_\psi, p_{\bar{\psi}}\}_D = \frac{i}{4}. \tag{24}$$

As usual, the quantization of Dirac bracket is $i^{-1}[.,.]_\pm$ where we have the commutator $[.,.]_-$ for bosonic variables and the anticommutator $[.,.]_+$ for fermionic variables. A simple way to realize the commutation relations (23) is, in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$, to consider the usual Heisenberg representation on the variable x where $\hat{p}_x = \frac{\partial}{i\partial x}$ and the Pauli representation on the fermionic variables where

$$\hat{\psi} = \sigma_-,\ \hat{\bar{\psi}} = \sigma_+,\ \hat{p}_\psi = -\frac{i}{2}\sigma_+,\ \hat{p}_{\bar{\psi}} = -\frac{i}{2}\sigma_-. \tag{25}$$

So we get the Witten supercharges and the Witten supersymmetric Hamiltonian given in (8), (9).

2. Our first model. We consider now the following higher-derivative supersymmetric quantum mechanical system, built upon the super Lagrangian

$$L = \frac{i}{2}(\bar{\mathcal{D}}X)\frac{d}{dt}(\mathcal{D}X) + V(X) \tag{26}$$

V is a smooth real potential on $\mathbb{R}$, X is here a real supervariable, with the meaning explained by the Hélein's formula (13),

$$X = x + \theta\bar{\psi} + \psi\bar{\theta} + y\theta\bar{\theta}\ ,$$

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{\mathcal{D}} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}.$$

Our Lagrangian L is real (up to a total time derivative) for the involution $L \mapsto \bar{L}$.

After integration over $d\bar{\theta}d\theta$, as we did for the Witten model, we obtain the effective Lagrangian :

$$L_{eff} = \dot{x}\dot{y} + V'(x)y + V''(x)\bar{\psi}\psi + \dot{\bar{\psi}}\dot{\psi}. \tag{27}$$

Let us compute the Hamiltonian corresponding to L_{eff}. We shall see that in this example it is not difficult to compute the Legendre transform in the variables $\dot{x}$, $\dot{y}$, $\dot{\psi}$, $\dot{\bar{\psi}}$ (we have no constraints). In the following it is more convenient to denote $\psi = \psi_1$ and $\bar{\psi} = \psi_2$. So we have for the conjugate momenta:

$$p_{\psi_2} = \dot{\psi}_1, \qquad p_{\psi_1} = -\dot{\psi}_2.$$

and

$$p_x = \dot{y}; \qquad p_y = \dot{x}.$$

We can derive by Legendre transform on the Lagrangian the canonical Hamiltonian

$$\boxed{H = p_x p_y - yV'(x) - p_{\psi_1}p_{\psi_2} - V''(x)\psi_2\psi_1}. \tag{28}$$

The Lagrangian (27) (with $(\psi, \bar{\psi}) = (\psi_1, \psi_2)$) is invariant (up to a total derivative) with respect to the supersymmetry transformations, where $\epsilon, \bar{\epsilon}$ are independent fermionic parameters,

$$\begin{aligned} \delta_\epsilon x &= \epsilon\psi_2 + \psi_1\bar{\epsilon} , \\ \delta_\epsilon \psi_1 &= \epsilon(y - i\dot{x}) , \\ \delta_\epsilon \psi_2 &= \bar{\epsilon}(y + i\dot{x}) , \\ \delta_\epsilon y &= i(\epsilon\dot{\psi}_2 - \dot{\psi}_1\bar{\epsilon}) . \end{aligned} \tag{29}$$

The corresponding Nœther supercharges are computed as in subsection (1.4).

$$\begin{aligned} &\boxed{Q \;=\; \psi_1[p_x + iV'(x)] - ip_{\psi_2}(p_y - iy)}, \\ &\boxed{\tilde{Q} \;=\; ip_{\psi_1}(p_y + iy) - \psi_2[p_x - iV'(x)]}. \end{aligned} \tag{30}$$

We can compute easily:

$$Q^2 = \tilde{Q}^2 = 0, \;\; [Q, \tilde{Q}]_+ = 2H \;, \tag{31}$$

It is important here to remark that the bracket $\{\bullet, \bullet\}$ is an extension of the usual Poisson bracket to the super-Lie algebra $C^\infty(\mathbb{R}^{4|4})^2$. Here $\mathbb{R}^{4|4}$ is identified to our classical phase space.

We define an involution on $\mathbb{R}^{4|4}$ with the convention $\bar{\psi_1} = ip_{\psi_1}$, $\bar{\psi_2} = ip_{\psi_2}$. Then, our Hamiltonian H is

$$H = p_x p_y - yV'(x) + \bar{\psi_1}\bar{\psi_2} - V''(x)\psi_2\psi_1.$$

Unfortunately H is not real (because the fermionic component is not invariant under the involution $H \mapsto \bar{H}$).

Let us now proceed to a canonical quantization of H. The idea is to mimic the holomorphic quantization for bosons. Let us double the fermionic generators by introducing $\bar{\theta}_1$ and $\bar{\theta}_2$ such that $\{\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2\}$ define a Grassmann algebra on $\mathbb{C}$, $\mathcal{G}_4 := \mathbb{C}[\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2]$ with involution $f \mapsto \bar{f}$. f is holomorphic means here that $\frac{\partial f}{\partial \bar{\theta}} = 0$. Let us denote by $\mathcal{H}_4$ the space of holomorphic elements of $\mathcal{G}_4$. It is an Hermitian space (dim$[\mathcal{H}_4]$=4) for the inner product

$$< f, g >= \int f(\theta)\overline{f(\theta)} e^{\theta\bar{\theta}} d\bar{\theta} d\theta.$$

So our total Hilbert space will be $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathcal{H}_4$.

In the Hilbert space $\mathcal{H}$, the real (bosonic) variables (x, y) are quantized as usual (Weyl-Wigner quantization or other) and the quantization rule for the Grassmann (fermionic) variables is the holomorphic quantization, given for $f \in \mathcal{H}_4$ by

$$\psi_j \longleftrightarrow \hat{\psi}_j : \{f \mapsto \theta_j f\} \tag{32}$$

$$\bar{\psi_j} \longleftrightarrow \hat{\bar{\psi_j}} : \{f \mapsto \frac{\partial f}{\partial \theta_j}\} \tag{33}$$

We have the expected canonical anticommutative relations for fermions

$$[\hat{p}_{\psi_j}, \hat{\psi}_k]_+ = [\hat{\psi}_j, \hat{\psi}_k]_+ = 0, \quad [\hat{\psi}_j, \hat{p}_{\psi_j}]_+ = 1. \tag{34}$$

and we get the supersymmetry quantum system $(\hat{Q}, \hat{\bar{Q}}, \hat{H})$,

$$\boxed{\hat{H} = \frac{1}{2}[\hat{Q}, \hat{\bar{Q}}]_+, \ (\hat{Q})^2 = (\hat{\bar{Q}})^2 = 0, \ [\hat{Q}, \hat{H}]_- = [\hat{\bar{Q}}, \hat{H}]_- = 0}. \tag{35}$$

[2]On a super-Lie algebra the bracket satisfies the following rule: $\{f, g\} = -(-1)^{\pi(f)\pi(g)}\{g, f\}$ where $\pi(f)$ is the parity of f (see [20]).

We remark that in our example, contrary to the standard supersymmetric mechanics (Witten), $\hat{Q}$ and $\hat{\bar{Q}}$ are not Hermitially conjugate to each other. Moreover, in general cases our quantum Hamiltonian $\hat{H}$ is no more a non negative nor Hermitian operator in the natural Hilbert space.

Let us assume that the potential V is C^2-smooth.

We see that the system has the two following even conserved charges

$$N = \frac{y^2}{2} - V(x), \tag{36}$$

$$F_c = \psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2. \tag{37}$$

$\hat{F}_c$ is the operator of fermion charge. Its eigenvalues are $0, 1, -1$. $\hat{F}_c$ is diagonal in the following decomposition of the total Hilbert space

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(-1)} \oplus \mathcal{H}^{(0)}, \tag{38}$$

where

$$\mathcal{H}^{(1)} = L^2(\mathbb{R}^2) \otimes \mathbb{C}\theta_1,\ \mathcal{H}^{(-1)} = L^2(\mathbb{R}^2) \otimes \mathbb{C}\theta_2, \qquad \mathcal{H}^{(0)} = L^2(\mathbb{R}^2) \otimes (\mathbb{C} \oplus \mathbb{C}\theta_1\theta_2). \tag{39}$$

Each of these subspace is formally invariant by $\hat{H}$ and $\hat{H}$ is formally Hermitian in $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(-1)}$ **but not in** $\mathcal{H}^{(0)}$ **if** $V'' \neq -1$.

Remark 2.1. The conserved charge N is the Nœther charge corresponding to the infinitesimal symmetry of the Lagrangian: $y \to y + \epsilon\dot{x}$. It easy to check that the Lagrangian (27) is shifted by a total derivative after this transformation.

Moreover we have here an other pair of odd conserved charges,

$$\begin{aligned} T &= \psi_1[p_x - iV'(x)] + \bar{\psi}_2(p_y + iy)\ , \\ \bar{T} &= \bar{\psi}_1(p_y - iy) + \psi_2[p_x + iV'(x)]\ . \end{aligned} \tag{40}$$

Generically for supersymmetric systems with two super charges $\hat{Q}, \hat{T}$ each non vacuum states is 4-fold degenerate, because if Ψ is an eigenstate then we also have the eigenstates $\{\hat{Q}\Psi, \hat{T}\Psi, \hat{Q}\hat{T}\Psi\}$.

Let us consider now the simple case with a quadratic potential:

$$V(X) = -\frac{\omega^2 X^2}{2}. \tag{41}$$

Let us first consider the bosonic component of our Hamiltonian

$$\hat{H}_B = -\partial_x\partial_y + \omega^2 xy.$$

We introduce the annihilation/creation operators

$$a = \frac{y+\partial_y}{\sqrt{2}}, \quad a^* = \frac{y-\partial_y}{\sqrt{2}} \tag{42}$$

$$b = \frac{-\omega^2+\partial_x}{\sqrt{2}\omega}, \quad b^* = \frac{-\omega^2-\partial_x}{\sqrt{2}\omega}. \tag{43}$$

They satisfy

$$[a, a^*] = [b, b^*] = 1, \quad [a, b] = [a, b^*] = [a^*, b] = 0. \tag{44}$$

So we get

$$\hat{H}_B = -\omega(ab - a^*b^*)$$

Let us introduce another pair of annihilation/creation operators, satisfying the commutation relations (44),

$$c = \frac{a+b^*}{\sqrt{2}}, \quad d = \frac{a-b^*}{\sqrt{2}},$$

we get for $\hat{H}_B$ a difference of two independant harmonic oscillators:

$$\hat{H}_B = \omega(dd^* - cc^*).$$

This proves that the spectrum of the Hamiltonian $\hat{H}_B$ is $\omega\mathbb{Z}$.
In $\mathcal{H}^{(0)}$ the Hamiltonian $\hat{H}$ has the matrix form

$$\hat{H}^{(0)} = \begin{pmatrix} \hat{H}_B & 1 \\ \omega^2 & \hat{H}_B \end{pmatrix} \tag{45}$$

This matrix can be put into a diagonal form through a non unitary transformation. If

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\omega} & -\frac{1}{\omega} \\ \omega & \frac{1}{\omega} \end{pmatrix},$$

then we have

$$\hat{H}^{(0)} = M \begin{pmatrix} \hat{H}_B + \omega & 0 \\ 0 & \hat{H}_B - \omega \end{pmatrix} M^{-1}. \tag{46}$$

Finally we get that the spectrum of the full Hamiltonian $\hat{H}$ is real, discrete, infinitely degenerate, involving both positive and negative energies. So, in spite of supersymmetry, the spectrum of $\hat{H}$ has no bottom and hence involves ghosts.

Later on we shall consider the spectral problem when we modify the superpotential (41) by adding quartic interaction term $V(x) = -\frac{\omega^2 X^2}{2} - \frac{\lambda}{4}x^4$ and also for interpolate super-Lagrangians between the Witten example and our model where we shall see connections with the Fokker-Planck equation.

3. Classical dynamics. In a first step, let us disregard the fermion variables and concentrate on the dynamics of the bosonic Hamiltonian

$$H_B \ = \ p_x p_y - yV'(x). \tag{47}$$

This Hamiltonian has two degree of freedom. Its Hamiltonian vector field Ξ_{H_B} is by definition

$$\Xi_{H_B} = \partial_{p_x} H_B \partial_x + \partial_{p_y} H_B \partial_y - \partial_x H_B \partial_{p_x} - \partial_y H_B \partial_{p_y}.$$

So we see that Ξ_{H_B} and Ξ_N are linearly independent outside the closed set defined by $\{V'(x) = 0, y = 0, p_x = p_y = 0\}$ and outside this set the system is integrable. Indeed, we get the following more explicit equations of motion,

$$\ddot{x} - V'(x) = 0; \qquad \ddot{y} - V''(x)y = 0. \tag{48}$$

The first equation is a Newton equation with one degree of freedom for the potential $-V(x)$ and the second equation is a Hill equation (x depends periodically on time with a period depending on initial data for the first equation).

For example, let us choose a quartic potential

$$V(x) \ = \ -\frac{\omega^2 x^2}{2} - \frac{\lambda x^4}{4}. \tag{49}$$

The potential is confining and here the equation of motion has an explicit solution given by an elliptic cosine[3] function with the parameters depending on the integral of motion N,

$$x(t) \ = \ x_0 \,\mathrm{cn}[\Omega t, k]. \tag{50}$$

[3]For definitions and properties of elliptic functions see for example, [22].

We get the full pseudo-classical dynamics for H by adding the equations for the fermionic variables

(51) $$\dot{\psi}_1 = -p_{\psi_2},\ \dot{\psi}_2 = p_{\psi_1},\ \dot{p}_{\psi_1} = -V''(x)\psi_2, \dot{p}_{\psi_2} = V''(x)\psi_1$$

So the fermionic evolution is determined the by same Hill equation.

(52) $$\ddot{\psi}_j - V''(x(t)\psi_j = 0,\ \ j = 1, 2.$$

We can remark that the differential equation (52) has the two following independent solutions

(53) $$y_1(t) = \dot{x}(t),\ \ y_2(t) = \dot{x}(t)\int^t \frac{ds}{\dot{x}^2(s)},$$

the integral in y_2 is well defined outside the turning points $\dot{x}(t) = 0$ nevertheless y_2 can be extended smoothly on all the real axis $\mathbb{R}$. It is an unstable solution of (52) increasing in time as $\mathcal{O}(t)$.

4. Quantum dynamics.

4.1. Bosonic system. Let us now consider the Weyl quantization $\hat{H}_B$ of the bosonic Hamiltonian (47). We shall see that the corresponding evolution for the Schrödinger equation $i\partial_t\psi_t = \hat{H}_B\psi_t$ is unitary and we shall compute this evolution. It is convenient to perform a unitary partial Fourier transform $\mathcal{F}_{y\mapsto p_y}$. Let K_B be the Hamiltonien obtained from H_B by the canonical transformation $(y, p_y) \to (p_y, -y)$. So we get $\hat{K}_B = \mathcal{F}_{y\mapsto p_y}\hat{H}_B\mathcal{F}^{-1}_{y\mapsto p_y}$, where $K_B(x, v; p_x, p_v) = vp_x + V'(x)p_v$. We remark that $i\hat{K}_B$ is the Hamilton vector field for the potential $-V(x)$. So the time dependent Schrödinger equation for $\hat{K}_B$ is solved by integration this vector field Ξ_N where $N(x, v) = \frac{v^2}{2} - V(x)$ (here v is considered as the conjugate momentum of x).

We assume for simplicity that $\lim_{|x|\to+\infty} V(x) = -\infty$ ($-V$ is confining). We denote Γ^t_N the Hamiltonian flow for Ξ_N at time t. Then Γ^t_N is well defined everywhere in $\mathbb{R}^2$ for all times and for every Ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, $\Psi_t(x, v) = \Psi\left(\Gamma^{-t}_N(x, v)\right)$ satisfies the Schrödinger equation for $\hat{K}_B$ and $\Psi_t \in \mathcal{S}(\mathbb{R}^2)$.

From this property it results classically that $\hat{K}_B$ hence $\hat{H}_B$ are essentially self-adjoint operators in $L^2(\mathbb{R}^2)$.

Now we compute a spectral decomposition for $\hat{H}_B$ and a complete family of generalized-eigenstates.

The flow Γ_N^t is integrable, so we can use action-angle variables.

Assume for simplicity that V is concave, with one maximum $V_{\max}$ at 0. Thus, we perform a canonical transformation κ: $(x,v) \mapsto (I,\varphi)$, ($I$ is the action variable, φ is the angle, $I \in]0,+\infty[$, $\varphi \in [0,2\pi[$) such that in this new coordinates system the flow is

$$\Gamma_N^t(\kappa^{-1}(I,\varphi)) = \kappa^{-1}(I,\varphi + t\sigma(I)) \tag{54}$$

where the frequency of the motion is $\sigma(I) = \partial N/\partial I$.

Let us recall that the action I is given by the following integral, for $N_0 > -V_{\max}$,

$$I(N_0) = \frac{1}{2\pi}\oint v dx = \frac{1}{2\pi}\int_{N(x,v)\leq N_0} dxdv,$$

In the action-angle coordinates the state Ψ depends now on I and φ and the solution to the Schrödinger equation takes the form

$$\Psi_t(I,\varphi) \equiv U(t)\Psi_0(I,\varphi) = \Psi_0(I,\varphi - t\sigma(I)).$$

In this representation, $U(t)$ is a unitary evolution in the Hilbert space $L^2(]0,\infty[\times\mathbb{R}/2\pi\mathbb{Z})$. Its generator is the following quantum Hamiltonian:

$$\hat{H}_{aa}\Psi(I,\varphi) = -i\sigma(I)\frac{\partial\Psi}{\partial\varphi}(I,\varphi)\,. \tag{55}$$

The Hamiltonians $\hat{H}_B$ and $\hat{K}_B$ are unitary equivalent to the Hamiltonian H_{aa}, so it is enough to compute a spectral decomposition for $\hat{H}_{aa}$. This is easy to do with a Fourier decomposition in the variable φ and we have an explicit spectral decomposition for $\hat{H}_{aa}$. If $\Psi(I,\varphi) = \sum_{n\in\mathbb{Z}} \Psi_n(I)\mathrm{e}^{in\varphi}$, then

$$\hat{H}_{aa}\Psi(I,\varphi) = \sum_{n\in\mathbb{Z}} n\sigma(I)\Psi_n(I)\mathrm{e}^{in\varphi}. \tag{56}$$

Let us remark that we can interpret the expression $E := E_n = n\sigma(I)$ as a kind of quantization condition (for I fixed). If the frequency $\sigma(I)$ is constant this is really a quantification condition and we recover the spectrum of $\hat{H}_B$ for $V(x) = -\dfrac{\omega^2x^2}{2}$. If the frequency $\sigma(I)$ is non constant then the spectrum of $\hat{H}_B$ is absolutely continuous and consists in a union of bands. This is the case for $V(x) = -\dfrac{\omega^2X^2}{2} - \dfrac{\lambda}{4}x^4$, $\lambda > 0$. The structure of the spectrum of $\hat{H}_B$ is the following:

$$Spec[\hat{H}_B] =]-\infty,-\omega] \cup \{0\} \cup [\omega,+\infty[$$

with two continuous components and an isolated eigenvalue 0.

The spectrum of $\hat{H}_B$ is describe as the set $Spec[\hat{H}_B] = \{n\lambda \quad n \in \mathbb{Z}, \lambda \in]0, +\infty[\}$.

We could get a similar description for more general confining potentials, with a finite set of non degenerate extrema.

The generalized eigenfunctions of the Hamiltonian (55) are labelled by the parameters $I_0 \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\Psi_{I_0,n}(I,\varphi) \;=\; \delta(I - I_0)e^{in\varphi} \;. \tag{57}$$

There are infinitely many states of zero energy. In the action-angle variables, any function $g(I) \to \tilde{g}(N)$ not depending on φ is an eigenfunction of (55) with zero eigenvalue.

Finally, we have obtained an explicit spectral decomposition of our bosonic Hamiltonian $\hat{H}_B$. More precisely, for every complex measurable and bounded function f on $\mathbb{R}$ and and every state $\Psi \in L^2$, we have

$$\langle\Psi|f(H_B)|\Psi\rangle = \sum_{n\in\mathbb{Z}} \int_0^{+\infty} |\Psi_n(I)|^2 f(n\sigma(I))dI. \tag{58}$$

$\Psi_n(I)$ are the Fourier components of the function $\Psi(I,\varphi)$.

4.2. Systems with bosons and fermions. Now we want to consider the full quantum Hamiltonien $\hat{H} = \hat{H}_B + \hat{H}_F$ where $\hat{H}_F = \partial_{\psi_1}\partial_{\psi_2} - V''(x)\psi_2\psi_1$. Note first of all that the time–dependent Schrödinger equation can be easily solved by the same method as in the bosonic case. We introduce $\eta = \bar{\chi}, \bar{\eta} = \chi$ and use the "position variables" (x, v, ψ, η). The Schrödinger equation takes the form

$$i\frac{\partial\Psi}{\partial t} + iv\frac{\partial\Psi}{\partial x} + iV'(x)\frac{\partial\Psi}{\partial v} + \eta\frac{\partial\Psi}{\partial\psi} - \psi V''(x)\frac{\partial\Psi}{\partial\eta} = 0. \tag{59}$$

Instead of $\hat{H}$ we consider the Hamiltonian $\hat{K} = \hat{K}_B + \hat{K}_F$ where $\hat{K}_F = V''(x)\psi\partial_\eta - \eta\partial_\psi$. Equation (59) is a linear first order differential equation and its solution can be written by the characteristics method.

$$\Psi_t(x, v; \psi, \eta) = \;\Psi_0(\Gamma^{-t}(x, v; \psi, \eta)), \tag{60}$$

where Γ^t is the flow of the characteristic system solving equation (59):

$$\Gamma^t(x, v; \psi, \eta) = (\Gamma^t_N(x, v); \psi_t, \eta_t) \tag{61}$$

where

$$\begin{aligned} \dot{\psi}_t &= -i\eta_t\,, \\ \dot{\eta}_t &= iV''(x_t)\psi_t. \end{aligned} \tag{62}$$

Unfortunately, except in the particular case where $V''(x)$ is constant, the L^2-norm of Ψ_t is not preserved by the time evolution. Moreover it is not clear how to define a closed extension for the full Hamiltonian $\hat{H}$ (or equivalently for $\hat{K}$).

Recall that the states are classified by the value of the fermionic charge F_c, which can take values $-1, 0, 1$. In coordinates (x, v, ψ, η), the generalized eigenfunctions functions in the sectors $F_c = 0$ are

$$\begin{aligned} \Psi^{(-1)}(x, v; \psi, \eta) &= \Psi_B(x, v)\,, \\ \Psi^{(1)}(x, v; \psi, \eta) &= \Psi_B(x, v)\psi\eta \end{aligned} \tag{63}$$

with Ψ_B is a generalized eigenfunction for the bosonic part $\hat{K}_B$ computed in section 4 (translated in coordinates (x, v)).

Generalized states in the sector $F_c = 0$ can be obtained from the eigenstates (63) by the action of the supercharges $Q, \bar{Q}, T, \bar{T}$ (which commute with the Hamiltonian). But we do not know how to give a complete spectral description of $\hat{K}$ in this sector.

Let us remark that if $V''(x)$ is uniformy bounded on $\mathbb{R}$ then $\hat{K}$ is a closed operator with domain $Dom(\hat{K}_B) \otimes calG_2$. Hence there are two open questions: is the spectrum of $\hat{K}$ real? What is the spectrum of $\hat{K}$?

5. Another model. Let us consider the following model

$$L = \int d\bar{\theta} d\theta \left[\frac{i}{2}(\bar{\mathcal{D}}X)\frac{d}{dt}(\mathcal{D}X) + \frac{\gamma}{2}\bar{\mathcal{D}}X\mathcal{D}X + V(X)\right], \tag{64}$$

where γ is a coupling between the model considered in section 4 and the Witten model of supersymmetry (section 1.4).

The pseudoclassical expression for the Lagrangian is here

$$L_{psc} = \dot{x}\dot{y} + yV'(x) + V''(x)\chi\psi + \dot{\chi}\dot{\psi} + \gamma\left[\frac{\dot{x}^2 + y^2}{2} + \frac{i}{2}(\dot{\psi}\chi - \psi\dot{\chi})\right]. \tag{65}$$

The corresponding Hamiltonian is

$$\begin{aligned} H &= H_B + H_F, \quad \text{where} \\ H_B &= p_x p_y - yV'(x) - \frac{\gamma}{2}(y^2 + p_y^2) \\ H_F &= \bar{\psi}_1\bar{\psi}_2 + \left(\frac{\gamma^2}{4} - V''(x)\right)\psi_2\psi_1 - \frac{\gamma}{2}F_c \end{aligned} \tag{66}$$

where $F_c = \psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2$.

As in our first model we have two pairs of Nœther supercharges but here the superalgebra is not so simple, so we give here only one pair (see [16] for details),

$$\begin{aligned} Q &= \psi_1(p_x + iV'(x)) + (\bar{\psi}_2 + \tfrac{\gamma}{2}\psi_1)(p_y - iy) \\ \tilde{Q} &= -\psi_2(p_x - iV'(x)) + (\bar{\psi}_1 + \tfrac{\gamma}{2}\psi_1)(p_y + iy). \end{aligned} \tag{67}$$

For this pair we have, as before, the algebra (35).

Let us consider here the sectors $F_c = \pm 1$ where the problem is equivalent to a purely bosonic problem with the Hamiltonian

$$\hat{H}_B \;=\; -\partial_x\partial_y - yV'(x) - \frac{\gamma}{2}(y^2 - \partial_y^2). \tag{68}$$

$\hat{H}_B$ is unitary equivalent, up to a partial Fourier transform, to

$$\hat{K}_B = \frac{1}{i}\left(y\partial_x + V'(x)\partial_y\right) - \frac{\gamma}{2}(y^2 - \partial_y^2). \tag{69}$$

We can see that $\hat{K}_B$ has the same algebraic structure as the Fokker-Planck operator[4] but there is a big difference: the Fokker-Planck operator $\hat{H}_{FP}$ is

$$\hat{H}_{FP}^{V'} = y\partial_x + V'(x)\partial_y - \frac{\gamma}{2}(y^2 - \partial_y^2)$$

so $\hat{H}_{FP}$ is not Hermitean but the Hermitian part of $\hat{H}_{FP}$ has a sign. Here $\hat{K}_B$ is Hermitean and is not bounded above nor below, and we have lost the hypoelliptic character of the Fokker-Planck operator (see [9] for more details).

For harmonic potentials $V(x) = -\dfrac{\omega^2 x^2}{2}$ we can compute the spectrum of $\hat{K}_B$. More precisely let us introduce the annihilation/creations operators

$$a = \tfrac{1}{\sqrt{2}}(y + \partial_y), \quad a^* = \frac{1}{\sqrt{2}}(y - \partial_y) \tag{70}$$

$$b = \tfrac{1}{\sqrt{2}}(x + \partial_x), \quad b^* = \frac{1}{\sqrt{2}}(x - \partial_x). \tag{71}$$

then we have

$$\hat{K}_B = \frac{\omega}{i}(a^*b - ab^*) - \gamma(a^*a + 1).$$

[4]We thank B. Helffer for this remark (oral communication).

Now we are looking for a complex symplectic transformation

$$c_1 = \alpha b + i\delta a, \quad c_2 = i\delta b + \alpha a \tag{72}$$

such that

$$[c_1, c_2] = [c_1^*, c_2] = 0, \quad [c_1, c_1^*] = [c_2, c_2^*] = 1$$

and

$$\hat{K}_B = \omega_1 c_1^* c_1 - \omega_2 c_2^* c_2 - \gamma. \tag{73}$$

We get the following conditions

$$\alpha^2 + \delta^2 = 1 \tag{74}$$

$$\omega(\alpha^2 - \delta^2) = \gamma\alpha\delta. \tag{75}$$

After computations we find, with $\tau = \dfrac{2\omega}{\gamma}$,

$$\omega_1 = \frac{\tau}{\sqrt{1+\tau^2}}\omega - \frac{\gamma}{2}\left(1 - \frac{1}{\sqrt{1+\tau^2}}\right), \tag{76}$$

$$\omega_2 = \frac{\tau}{\sqrt{1+\tau^2}}\omega + \frac{\gamma}{2}\left(1 + \frac{1}{\sqrt{1+\tau^2}}\right). \tag{77}$$

So we get that the spectrum of $\hat{H}_B$ is pure point, with eigenvalues

$$\left(\omega_1 + \frac{1}{2}\right) j - \left(\omega_2 + \frac{1}{2}\right) k - \gamma, \quad j, k \in \mathbb{N}.$$

Remark 5.1. For more general potential V it is not clear that $\hat{H}_B$ is essentially self-adjoint. This can be proved if $V''(x)$ is uniformly bounded on $\mathbb{R}$. But we do not know how is the spectrum of $\hat{H}_B$ for non quadratic potentials. If the coupling γ is imaginary we are in the Fokker-Planck case and if some technical conditions on the potential V are satisfied then it is known that the spectrum of $\hat{K}_B$ is pure point; moreover in this case the resolvent of $\hat{K}_B$ is compact [9].

Moreover it may be interesting to emphasize that the two Hamiltonians $\hat{K}_B$ and $\hat{H}_{FP}^{-V'}$ are conjugated by the complex scaling $y \mapsto iy$, $D_y \mapsto -iD_y$.

A. More on the Pais-Uhlenbeck oscillator. Assumming first $\omega_1 > \omega_2$, let introduce complex valued linear forms a_1, a_2, defined as complex coordinates for the classic flow as follows

$$q(t) = \alpha_1 \mathrm{e}^{-i\omega_1 t} + \alpha_2 \mathrm{e}^{-i\omega_2 t} + h.c, \quad x(t) = -i\omega_1 x(t)\alpha_1 \mathrm{e}^{-i\omega_1 t} - i\omega_2 \alpha_2 \mathrm{e}^{-i\omega_2 t} + h.c$$

$$p_q(t) = i\omega_1\omega_2^2\alpha_1 \mathrm{e}^{-i\omega_1 t} + i\omega_1^2\omega_2\alpha_2\mathrm{e}^{-i\omega_2 t} + h.c,$$
$$p_x(t) = -\omega_1^2\alpha_1\mathrm{e}^{-i\omega_1 t} - \omega_2^2\omega_2\alpha_2\mathrm{e}^{-i\omega_2 t} + h.c \tag{78}$$

We have the following equalities for the Poisson brackets

$$\{\alpha_1, \alpha_2\} = \{\bar\alpha_1, \bar\alpha_2\} = \{\bar\alpha_1, \alpha_2\} = \{\alpha_1, \bar\alpha_2\} = 0$$
$$\{\alpha_1, \bar\alpha_1\} = \frac{1}{2i\omega_1(\omega_1^2 - \omega_2^2)} \ \ \{\alpha_2, \bar\alpha_2\} = \frac{1}{2i\omega_1(\omega_2^2 - \omega_1^2)}. \tag{79}$$

Considering on $\mathbb{C}^2$ the symplectic form $(z, z') \mapsto \Im(z \cdot z')$, we get a linear symplectic map

$$(x, q, p_x, p_q) \mapsto (a_1, a_2)$$

where

$$a_1 = \alpha_1\sqrt{2\omega_1(\omega_1^2 - \omega_2^2)}, \ \ a_2 = \bar\alpha_2\sqrt{2\omega_1(\omega_1^2 - \omega_2^2)}.$$

So we get a metaplectic transformation such that $\hat H$ is unitary equivalent to the oscillator

$$\hat K = \omega_1\hat a_1^*\hat a_1 - \omega_2\hat a_2^*\hat a_2.$$

In [1] the authors claim that for the Pais-Uhlenbeck oscillator ghosts can be easily eliminated. One possibility is to perform a complex scaling like the following. Let us come back to real coordinates:

$$\hat a_j = \frac{x_j - \partial_{x_j}}{\sqrt 2}$$

The dilation operator in the x_2 direction is :$U_\tau f(x_1, x_2) = f(x_1, \mathrm{e}^\tau x_2)$. Its generator is $\hat A := i^{-1}x_2\partial_{x_2} + x_2\partial_{x_2}$, $U_\tau = \mathrm{e}^{-i\tau\hat A}$. Let us consider deformations of $\hat K$,

$$\hat K_\tau = U_\tau\hat K U_{-\tau}$$

For $\tau \in \mathbb{R}$, $\hat H$ and $\hat K_\tau$ are unitary equivalent. But for $\tau = i\frac{\pi}{2}$ we have

$$\hat K_{-\pi/2} = \omega_1\hat a_1^*\hat a_1 + \omega_2\hat a_2^*\hat a_2$$

hence $\hat K_{-\pi/2}$ is a positive operator, without ghosts.

Remark A.1. Another interpretation for $\hat K_{-\pi/2}$ is to see it as a realization of $\hat K$ in the configuration space $\mathbb{R} \times i\mathbb{R}$.

The question of ghosts for the degenerate case is more delicate. It was discussed in [15, 18, 14] and more recently in [2].

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