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SPECTRAL SHIFT FUNCTION FOR THE PERTURBATIONS OF SCHRÖDINGER OPERATORS AT HIGH ENERGY

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Abstract. We give a complete pointwise asymptotic expansion for the Spectral Shift Function for Schrödinger operators that are perturbations of the Laplacian on $\mathbb{R}^n$ with slowly decaying potentials.

1. Introduction. The aim of this paper is to give complete asymptotic expansion of the Spectral Shift Function (SSF) for a large class of perturbations of the Laplace operator on $\mathbb{R}^n$. For a pair of lower-bounded selfadjoint operators (H_1, H_2) in a Hilbert space $\mathcal{H}$ the SSF can be defined as follows: assume that $\|(H_2-E)^{-N}-(H_1-E)^{-N}\|_{\mathrm{tr}} < \infty$ for some $N \geq 1$ and E in the resolvent set of H_j, $j = 1, 2$, the SSF $\xi(\lambda)$ is defined in the sense of distribution by the equation (see [1, 10]),

$$\langle \xi', f \rangle = -\mathrm{tr}(f(H_2) - f(H_1)),$$

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for all $f \in C_0^\infty(\mathbb{R})$. Since the operators H_j, $j = 1, 2$ are lower-bounded, the SSF is normalized by the condition

$$\xi(\lambda) = 0, \quad \lambda < \inf\Big(\sigma(H_1) \cup \sigma(H_2)\Big).$$

Here $\|\cdot\|_{\rm tr}$ denotes the trace-class norm in $\mathcal{H}$.

The SSF is an important object in the spectral theory of perturbations, which covers both discrete and continuous spectrum. The SSF was brought into mathematical use in M. G. Krein's famous paper [9], where the precise statement of the problem was given and the explicit representation of the SSF in terms of the perturbation determinant was obtained. The work of M. G. Krein on the SSF has been described in detail in [2]. The connection between the SSF and the scattering matrix is given in the paper [1]. For more details about the interpretation of the spectral shift function we refer to the survey by D. Robert [13] and chapter 8 of the monograph by D. R. Yafaev [16].

The asymptotic behavior of the SSF at high energy of the Schrödinger operator has been intensively studied in the last twenty years (see [3, 4, 8, 11, 12, 14, 17] and the reference given there).

In [15] D. Robert studied the SSF for a pair of Schrödinger operators $(H_1, H_2) = (-\Delta + V, -\Delta + V + W)$ where V, W are smooth potentials defined on $\mathbb{R}^n$ for which there exist $\delta > 0$ and $\rho > n$ such that

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0 \,\text{s.t}\, |\partial^\alpha V(x)| \leq C_\alpha (1 + |x|)^{-\delta - |\alpha|}, \tag{1}$$

$$\forall \alpha \in \mathbb{N}^n, \exists C'_\alpha > 0 \,\text{s.t}\, |\partial^\alpha W(x)| \leq C'_\alpha (1 + |x|)^{-\rho - |\alpha|}. \tag{2}$$

Under the above assumptions, D. Robert gives a complete pointwise asymptotic expansion of $\xi(\lambda)$ as $\lambda \nearrow +\infty$.

In this paper we give a similar result but for a pair of Schrödinger operators $(-\Delta + V, -\Delta + V + W)$ where V do not satisfy assumption (1). Our result on the pointwise asymptotics for the SSF applies, for example, to perturbations of the Laplacian with logarithmic decreasing potentials. On the other hand, our method is different from the one developed in [15]. In particular, we don't require any parametrix construction as in [12, 13, 15].

2. Weak asymptotics. We consider two Schrödinger operators H_j, $j = 1, 2$ defined as the selfadjoint realizations of $H_1 = -\Delta + V$, $H_2 = -\Delta +$

$V + W$ on $L^2(\mathbb{R}^n)$ where V, W are smooth potentials satisfying the following assumptions: for all $\alpha \in \mathbb{N}^n$ there exists $C_\alpha, c_\alpha > 0$ such that

$$|\partial_x^\alpha V(x)| \leq c_\alpha, \tag{3}$$

$$|\partial_x^\alpha W(x)| \leq C_\alpha (1 + |x|)^{-\delta - |\alpha|}, \tag{4}$$

for some constant $\delta > n$.

The assumptions (3) and (4) enable us to define the SSF, $\xi(\lambda, H_1, H_2) \in \mathcal{D}'(\mathbb{R})$ related to the operators H_1 and H_2 following the general theory (see [1, 10]) by the equality

$$\langle \xi', f \rangle = -\mathrm{tr}(f(H_2) - f(H_1)), \forall f \in C_0^\infty(\mathbb{R}).$$

We normalize $\xi(\lambda) := \xi(\lambda, H_1, H_2)$ by setting $\xi(\lambda) = 0$ for $\lambda < \inf\Big(\sigma(H_1) \cup \sigma(H_2)\Big)$.

Theorem 1 (Weak asymptotics)**.** *Assume that V, W satisfy assumptions* (3) *and* (4)*. For every $f \in C_0^\infty(\mathbb{R})$, the following full asymptotic expansion holds as* $(h \searrow 0)$*:*

$$\mathrm{tr}(f(h^2 H_2) - f(h^2 H_1)) \sim \sum_{j=1}^{\infty} c_j(f)\, h^{2j-n}. \tag{5}$$

In particular,

$$c_1(f) = \frac{n\kappa_0}{(2\pi)^n} \int_0^\infty f'(r^2) r^{n-1} dr \int_{\mathbb{R}^n} W(x) dx, \tag{6}$$

$$c_2(f) = \frac{n\kappa_0}{2(2\pi)^n} \int_0^\infty f''(r^2) r^{n-1} dr \int_{\mathbb{R}^n} (W(x)^2 + 2V(x)W(x)) dx, \tag{7}$$

where $\kappa_0 = \mathrm{vol}(\{x \in \mathbb{R}^n; |x| < 1\})$ *is the measure of the unit ball in* $\mathbb{R}^n$

P r o o f. For $h \in]0, 1]$, set

$$H_2(h) = h^2 H_2, \;\; H_1(h) = h^2 H_1, \quad \text{and } H_0(h) = -h^2 \Delta.$$

Let $\tilde{f} \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. We recall that $\tilde{f} \in C_0^\infty(\mathbb{C})$ satisfies

$$|\overline{\partial} \tilde{f}| \leq C_N |\Im(z)|^N, \forall N \in \mathbb{N} \;\text{ and }\; \tilde{f}_{|\mathbb{R}} = f. \tag{8}$$

Since the spectrum of H_j is bounded from below, we may choose $z_0 \in \mathbb{R}$, so that z_0 is away from $\sigma(H_j)$, $j = 1, 2$. Set

$$g(z) = f(z)(z - z_0)^m.$$

By Helffer-Sjöstrand formula (see [5, 7]) we have

$$g(H_.(h)) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z - z_0)^m \ (z - H_.(h))^{-1} L(dz),$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$ and $H_.$ denotes either H_1 or H_2. Clearly,

$$\begin{aligned} f(H_.(h)) &= (H_.(h) - z_0)^{-m} g(H_.(h)) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z - z_0)^m (H_.(h) - z_0)^{-m} (z - H_.(h))^{-1} L(dz). \end{aligned}$$

From the assumption (4) the operator $\Big[(H_j(h) - z_0)^{-m}(z - H_j(h))^{-1}\Big]_1^2$ is trace class for $m > \frac{n}{2}$. Here we use the notation $[a_j]_1^2 = a_2 - a_1$. Combining this with the above equation we obtain

$$\begin{aligned} (9)\quad & \mathrm{tr}\Big(f(H_2(h)) - f(H_1(h))\Big) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z - z_0)^m \times \mathrm{tr}\Big[(H_j(h) - z_0)^{-m}(z - H_j(h))^{-1}\Big]_1^2 L(dz). \end{aligned}$$

Without any loss of the generality, we may assume that $m = 0$ in the equality (9), (see Remark 1 for the general case).

From the resolvent equation, we obtain

$$(z - H_2(h))^{-1} = \sum_{k=0}^{N} h^{2k}\Big[(z - H_0(h))^{-1}(V + W)\Big]^k (z - H_0(h))^{-1}$$

$$+h^{2(N+1)}(z - H_2(h))^{-1}(V + W)\Big[(z - H_0(h))^{-1}(V + W)\Big]^N (z - H_0(h))^{-1},$$

$$(z - H_1(h))^{-1} = \sum_{k=0}^{N} h^{2k}\Big[(z - H_0(h))^{-1}V\Big]^N (z - H_0(h))^{-1} +$$

$$h^{2(N+1)}(z - H_1(h))^{-1}V\Big[(z - H_0(h))^{-1}V\Big]^N (z - H_0(h))^{-1},$$

so using the two last equations, we find

$$(10)\quad \begin{aligned}&(z-H_2(h))^{-1}-(z-H_1(h))^{-1}\\ &=\sum_{k=1}^{N} h^{2k}\Big(\Big[(z-H_0(h))^{-1}(V+W)\Big]^k-\Big[(z-H_0(h))^{-1}V\Big]^k\Big)(z-H_0(h))^{-1}\\ &\quad + h^{2(N+1)}\Big((z-H_2(h))^{-1}(V+W)\Big[(z-H_0(h))^{-1}(V+W)\Big]^N\\ &\quad - (z-H_1(h))^{-1}V\Big[(z-H_0(h))^{-1}V\Big]^N\Big)(z-H_0(h))^{-1}\\ &=\sum_{k=1}^{N}A_k(z)+\ B(z).\end{aligned}$$

Clearly, $\|B(z)\|_{\mathrm{tr}} = \mathcal{O}(h^{2(N+1)}|\Im z|^{-(N+2)})$, which together with (8) implies

$$\Big\|\int \bar\partial_z \tilde f(z) B(z)\, L(dz)\Big\|_{\mathrm{tr}} = \mathcal{O}(h^{2(N+1)}),\ \ \forall N\in\mathbb{N}.$$

Thus, to prove (5), it suffices to show that for all k

$$(11)\qquad I_k := -\frac{1}{\pi}\int \bar\partial_z \tilde f(z)\,\mathrm{tr}(A_k(z))\, L(dz),$$

has an asymptotic expansion in powers of h^2.

To do this, we notice that $A_k(z)$ can be written as a finite sum of terms of the form

$$A_k(z) = h^{2k}(z-H_0(h))^{-1}G_1(z-H_0(h))^{-1}G_2\ldots(z-H_0(h))^{-1},$$

with $G_i = V+W$, V, W, and there exists at least one i_0 such that $G_{i_0} = W$. Hence, $A_k(z)$ is of trace class.

Next, fix δ in $\left]0,\frac{1}{2}\right[$ and apply (8), we obtain

$$\Big\|\int_{|\Im z|\le h^\delta} \bar\partial_z \tilde f(z) A_k(z)\, L(dz)\Big\|_{\mathrm{tr}} = \mathcal{O}(h^\infty).$$

On the other hand, by the h-pseudodifferential calculus (see for instance Dimassi-Sjöstrand [5, chapters 7,8]) there exists a C^∞ function $(x,\xi)\to\mathcal{G}_k(x,\xi,z,h)$ such that

$$|\partial_x^\alpha\partial_\xi^\beta\mathcal{G}_k(x,\xi,z,h)| \le C_{\alpha,\beta}h^{-(k+1)\delta-\delta(|\alpha|+|\beta|)},$$

uniformly on $z \in \Omega_\delta := \{z \in \mathrm{supp}\widetilde{f};\ |\Im z| > h^\delta\}$, and $A_k(z) = \mathcal{G}_k^w(x, hD_x, z, h)$ for all z in Ω_δ. Moreover

$$\mathcal{G}_k(x,\xi,z,h) \sim \mathcal{G}_{k,0}(x,\xi,z) + h\mathcal{G}_{k,1}(x,\xi,z) + h^2\mathcal{G}_{k,3}(x,\xi,z) + \cdots$$

with $\mathcal{G}_{k,i}(x,\xi,z)$ a finite sum of terms of the form

$$f_i(\xi)(z - |\xi|^2)^{-k-i-1} g_i(x),$$

where $f_i(\xi)$ is a homogeneous polynomial of degree i and g_i are functions depending on V, W and their derivatives. Now, by a classical result on trace class operators, we have

$$I_k \sim \sum_{j=0}^{\infty} -\frac{h^j}{\pi} \int \bar{\partial}_z \tilde{f}(z) \int\int \mathcal{G}_{k,j}(x,\xi,z) \frac{dx d\xi}{(2\pi h)^n}\, L(dz). \tag{12}$$

Since $f_i(-\xi) = -f_i(\xi)$ for i odd, it follows from the above discussion that I_k has an asymptotic expansion in powers of h^2. It remains to compute the terms c_1 and c_2.

Using the cyclicity of the trace as well as the fact that

$$-\frac{1}{\pi} \int \bar{\partial}_z \tilde{f}(z)(z-a)^{-p-1}\, L(dz) = \frac{1}{p!} f^{(p)}(a), \quad \text{for all } a \in \mathbb{R}, \tag{13}$$

we obtain

$$I_1 = -\frac{h^2}{\pi} \mathrm{tr}\left(\int \bar{\partial}_z \tilde{f}(z)\,(z - H_0)^{-2} W\, L(dz)\right) = h^{-n+2} \int_{\mathbb{R}^{2n}} f'(|\xi|^2) W(x) \frac{dx d\xi}{(2\pi)^n},$$

which together with (9) and (10) give (6).

The principal symbol of $A_2(z)$ has the form

$$\mathcal{G}_{2,0}(x,\xi,z) = (z - |\xi|^2)^{-3}((W(x) + V(x))^2 - V(x)^2).$$

Thus

$$\begin{aligned} I_2 = & h^{-n+4} \iint_{\mathbb{R}^{2n}} -\frac{1}{\pi}\left(\int \bar{\partial}_z \tilde{f}(z)\,(z-|\xi|^2)^{-3}((W(x)+V(x))^2 - V(x)^2) L(dz)\right) \frac{dx d\xi}{(2\pi)^n} \\ & + \mathcal{O}(h^{-n+6}) \\ = & h^{-n+4} \int_{\mathbb{R}^{2n}} f''(|\xi|^2)(W(x)^2 + 2V(x)W(x)) \frac{dx d\xi}{(2\pi)^n} + \mathcal{O}(h^{-n+6}). \end{aligned}$$

In the last equality we have used (13). Using (10), (11) and (12) we obtain (7). This completes the proof of Theorem 1. □

Remark 1. For $m \neq 0$, we decompose $\Big[(H_j(h)-z_0)^{-m}(z-H_j(h))^{-1}\Big]_1^2$ as

$$\Big[(H_j(h)-z_0)^{-m}(z-H_j(h))^{-1}\Big]_1^2 = \Big[(H_j(h)-z_0)^{-m}\Big]_1^2 (z-H_2(h))^{-1} + (H_1(h)-z_0)^{-m}\Big((z-H_2(h))^{-1}-(z-H_1(h))^{-1}\Big) = I + II.$$

Since $(H_i(h)-z_0)^{-1}$ is an h-pseudodifferential operator, we can use the same arguments as above and show that the trace of the operators I and II has an asymptotic expansion in powers of h^2 for $z \in \{z \in \mathbb{C}, |\Im z| > h^\delta\}$ with $\delta \in]0, \frac{1}{2}[$. The remainder of the proof is the same.

Corollary 1. *Assume* (3) *and* (4). *For λ large enough, the following asymptotics holds in the sense of distribution:*

$$\xi'(\lambda) = a_1\, \lambda^{\frac{n-4}{2}} + a_2\, \lambda^{\frac{n-6}{2}} + o(\lambda^{\frac{n-6}{2}}),$$

where

$$a_1 = \frac{n(n-2)\kappa_0}{4(2\pi)^n} \int_{\mathbb{R}^n} W(x)dx,$$

$$a_2 = -\frac{n(n-4)\kappa_0}{16(2\pi)^n} \int_{\mathbb{R}^n} (W(x)^2 + 2V(x)W(x))dx.$$

P r o o f. Let f be in $C_0^\infty(]0,+\infty[)$. A simple calculation shows that

$$c_1(f) = -a_1\langle t^{\frac{n-4}{2}}, f(\cdot)\rangle, \quad c_2(f) = a_2\langle t^{\frac{n-6}{2}}, f(\cdot)\rangle. \tag{14}$$

On the other hand, the Lifshits-Krein formula implies that

$$\mathrm{tr}(f(h^2H_2) - f(h^2H_1)) = \int_{-\infty}^{+\infty} \frac{1}{h^2}\xi'(\frac{\lambda}{h^2}, H_2, H_1)\, f(\lambda)\, d\lambda := \langle \frac{1}{h^2}\xi'(\frac{\cdot}{h^2}), f(\cdot)\rangle,$$

which together with (5), (6) and (7) give the corollary.

3. Pointwise asymptotics. Let H_j, $j = 1, 2$ be as above. In this section we replace the assumptions (3) and (4) by the following one: There exist $\tilde{n} > n$ and $C > 0$ such that

$$|V(x)| \leq C, \quad \text{and} \quad |W(x)| \leq C(1+|x|)^{-\tilde{n}}, \text{ for all } x \in \mathbb{R}^n \tag{15}$$

and there exist $\theta_0 > 0$ such that V and W extend to holomorphic functions on a complex domain of the form

$$\Gamma_{\theta_0} = \{z \in \mathbb{C}^n;\ |\Re(z)| \leq c(1 + |\Im(z)|)\}\,,$$

for some $c > 0$ and satisfy the assumption (15) on Γ_{θ_0}.

Remark 2. We note that using the Cauchy inequalities, assumption (15) remains valid for all the derivatives of V and W. Precisely we have

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0 \text{ s.t} \,|\partial^\alpha W(x)| \leq C_\alpha (1 + |x|)^{-\tilde{n}-|\alpha|},$$

and

$$\forall \alpha \in \mathbb{N}^n, \exists C'_\alpha > 0 \text{ s.t} \,|\partial^\alpha V(x)| \leq C'_\alpha (1 + |x|)^{-|\alpha|}.$$

Remark 2 implies that the spectral shift function $\xi(\lambda, H_2, H_1)$ corresponding to H_2 and H_1 is well defined. Our main result is the following:

Theorem 2. *Let V and W be two potentials satisfying* (15)*. Then the following properties hold:*

(*i*) *There exist $\lambda_0 > 0$ such that $\xi(\lambda)$ is C^∞ in $]\lambda_0, +\infty[$.*

(*ii*) *$\xi'(\lambda)$ has a complete asymptotic expansion for $\lambda \nearrow +\infty$ of the form*

$$\xi'(\lambda) \sim \lambda^{\frac{n}{2}} \left(\sum_{j \geq 1} a_j \lambda^{-1-j} \right).$$

The coefficients a_1, a_2 are given in Corollary 1*.*

Remark 3. We can compute explicitly all the coefficients a_j and $c_j(f)$. In fact, the computation of $c_j(f)$ is a simple consequence of the functional calculus of h-pseudodifferential operators (see [5], chapter 8). To compute a_j, we proceed as in the proof of Corollary 1.

The proof of Theorem 2 is based on some lemmas. The next section is concerned with these lemmas and their proofs. In the last section we give the end of the proof of Theorem 2.

4. Some technical lemmas. Let H_j, $j = 1, 2$ be two operators satisfying (15). Consider the functions

(16) $$\sigma_\pm(z) = (z - z_0)^m \mathrm{tr}\Big[(z - H_j)^{-1}(H_j - z_0)^{-m}\Big]_1^2, \ \pm\Im z > 0,$$

where m is an integer and $z_0 \in \rho(H_j)$, for $j = 1, 2$. We recall that $[a_j]_1^2 = a_2 - a_1$. From the assumption (15) and Remark 2 the right hand side of (16) is well defined for $m > \frac{n}{2}$. From now on, m is a fixed integer such that $m > \frac{n}{2}$.

Next we will obtain a representation of the derivative $\xi'(\lambda) := \xi'(\lambda, H_2, H_1)$.

Lemma 1. *Under the assumption (15) we have*

$$\xi'(\lambda) = \frac{1}{\pi}\Im\sigma_+(\lambda + i0).$$

More precisely, for all $f \in C_0^\infty(\mathbb{R})$, we have

$$\langle \xi', f\rangle = \lim_{\epsilon\searrow 0}\frac{1}{\pi}\int f(\lambda)\Im\sigma_+(\lambda + i\epsilon)d\lambda,$$

where the limit is taken in the sense of distributions.

Proof. Let $f \in C_0^\infty(\mathbb{R})$ and let $\tilde{f} \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. According to formulae (9), we have

(17)
$$\operatorname{tr}\Big(f(H_2) - f(H_1)\Big) = -\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)(z-z_0)^m \times \operatorname{tr}\Big[(H_j - z_0)^{-m}(z - H_j)^{-1}\Big]_1^2 L(dz).$$

Since we have $\sigma_\pm(z) = \mathcal{O}(|\Im z|^{-1})$ and $\bar{\partial}_z\tilde{f} = \mathcal{O}(|\Im z|)$, we may write the right hand side of the above equation as

$$\langle \xi', f\rangle = -\operatorname{tr}\Big(f(H_2) - f(H_1)\Big)$$

$$= \lim_{\epsilon\searrow 0}\frac{1}{\pi}\Big(\int_{\Im z>0}\bar{\partial}_z\tilde{f}(z)\sigma_+(z+i\epsilon)L(dz) + \int_{\Im z<0}\bar{\partial}_z\tilde{f}(z)\sigma_-(z-i\epsilon)L(dz)\Big).$$

The function $\sigma_+(z+i\epsilon)$ (resp. $\sigma_-(z-i\epsilon)$) is holomorphic on the complex domain $\{z \in \mathbb{C} : \Im z > 0\}$ (resp. $\{z \in \mathbb{C} : \Im z < 0\}$). Thus applying the Green formula we obtain

$$\langle \xi', f\rangle = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int f(\lambda)\Big(\sigma_+(\lambda + i\epsilon) - \sigma_-(\lambda - i\epsilon)\Big)d\lambda.$$

Using the above equation and the fact that $\sigma_-(\lambda - i\epsilon) = \overline{\sigma_+(\lambda + i\epsilon)}$ we get the lemma.

Lemma 2. *Assume* (15). *There exist $\lambda_0 \gg 1$ such that $\xi(\lambda)$ is C^∞ in $]\lambda_0, +\infty[$ and for every $N \in \mathbb{N}$ there exists C_N such that*

$$|\xi^{(N+1)}(\lambda)| \le C_N\lambda^{m-N-1}, \tag{18}$$

uniformly for $\lambda \in [\lambda_0, +\infty[$.

Proof. In the following, we denote $V_1 = V$ and $V_2 = V + W$. For $\theta \in \mathbb{R}$ set

$$H_{.,\theta} = -e^{-2\theta}\Delta + V_.(e^\theta x).$$

Here $V_.$ denotes either V_1 or V_2. For θ real the operator $(z - H_.)^{-1}(H_. - z_0)^{-m}$ is unitarily equivalent to $(z - H_{.,\theta})^{-1}(H_{.,\theta} - z_0)^{-m}$. Consequently, the cyclicity of the trace yields

$$\sigma_+(z) = (z - z_0)^m \mathrm{tr}\Big[(z - H_{.,\theta})^{-1}(H_{.,\theta} - z_0)^{-m}\Big]_1^2, \tag{19}$$

for all $z \in \mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$ and $\theta \in D(0, \theta_0) \cap \mathbb{R}$.

Now, fix $\delta > 0$ and let $z \in \mathbb{C}_\delta = \{z \in \mathbb{C}; \Im z \geq \delta\}$. Since $H_{.,\theta}$ extends to an analytic type A family of operators on $D(0, \theta_0)$ for sufficiently small θ_0 and $z \in \mathbb{C}_\delta$, the right hand side of (19) extends by analytic continuation in θ to the disc $D(0, \theta_0)$. For $\theta \in D(0, \theta_0)$ with $\Im\theta < 0$, both terms of (19) are analytic on $\mathbb{C}_+$ and consequently (19) remains true for all z in $\mathbb{C}_+$.

From now on, the number θ will be fixed in $D(0, \theta_0)$ with $\theta = -i\eta$, $\eta > 0$. The following estimates holds uniformly on $z \in \{z \in \mathbb{C}; \Re z > 1, \Im z > -a\}$ for some positive constant a:

$$\|(-e^{2\theta}\Delta - z)^{-1}\| \leq \sup_{\xi \in \mathbb{R}^n} |e^{-2\theta}|\xi|^2 - z|^{-1} \leq C\eta^{-1}\Re z^{-1}.$$

Using (15) and the above estimate, we see that

$$H_{.,\theta} - z = (-e^{-2\theta}\Delta - z)\Big(I + (-e^{-2\theta}\Delta - z)^{-1}V_.(e^\theta x)\Big),$$

is invertible for $z \in \mathcal{A}_{a,A} := \{z \in \mathbb{C}; \Re z > A, \Im z > -a\}$ where A is a large positive constant. Moreover,

$$\mathcal{A}_{a,A} \ni z \to (H_{.,\theta} - z)^{-1} \text{is holomorphic , and } \|(H_{.,\theta} - z)^{-1}\| = \mathcal{O}(\Re z^{-1}), \tag{20}$$

uniformly on $z \in \mathcal{A}_{a,A}$. On the other hand, a classical result on trace class operators (see for instance [5]) shows that

$$\|(H_{.,\theta} - z_0)^{-m}W_\theta\|_{\mathrm{tr}} = \mathcal{O}(1). \tag{21}$$

Taking $(k-1)$ derivatives in z in the resolvent identity

$$(z - H_{1,\theta})^{-1} - (z - H_{2,\theta})^{-1} = (z - H_{1,\theta})^{-1}W_\theta(z - H_{2,\theta})^{-1},$$

and setting $z = z_0$, we see that $(z_0 - H_{1,\theta})^{-k} - (z_0 - H_{2,\theta})^{-k}$ is a linear combination of terms of the form

$$(z_0 - H_{1,\theta})^{-j} W_\theta (z_0 - H_{2,\theta})^{-(k+1-j)}$$

with $1 \leq j \leq k$. Combining this with (21) we deduce that for every $k > \frac{n}{2}$,

$$\|(z_0 - H_{1,\theta})^{-k} - (z_0 - H_{2,\theta})^{-k}\|_{\mathrm{tr}} = \mathcal{O}(1). \tag{22}$$

Next, we write $\sigma_+(z) = \sigma_+^1(z) + \sigma_+^2(z)$, where

$$\sigma_+^1(z) = \mathrm{tr}\Big((z - z_0)^m (z - H_{1,\theta})^{-1} \Big[(H_{1,\theta} - z_0)^{-m} - (H_{2,\theta} - z_0)^{-m}\Big]\Big), \tag{23}$$

and

$$\sigma_+^2(z) = \mathrm{tr}\Big((z - z_0)^m \Big[(z - H_{1,\theta})^{-1} - (z - H_{2,\theta})^{-1}\Big](H_{2,\theta} - z_0)^{-m}\Big)$$

$$= \mathrm{tr}\Big[(z - z_0)^m (z - H_{1,\theta})^{-1} W_\theta (H_{2,\theta} - z_0)^{-m} (H_{2,\theta} - z)^{-1}\Big]. \tag{24}$$

From (20), (21) and (22) we deduce that the right hand side of (23) and (24) are holomorphic in $\mathcal{A}_{a,A}$ which implies that $\xi'(\lambda) = \dfrac{1}{\pi}\Im(\sigma_+^1(\lambda + i0) + \sigma_+^1(\lambda + i0))$ is C^∞ in $]\lambda_0, +\infty[$ for some large constant λ_0.

On the other hand, the estimates (20), (21), (22) and the fact that $|\lambda - z_0| = \mathcal{O}(\lambda^m)$ imply that $|\sigma_+^1(\lambda + i\varepsilon)|, |\sigma_+^2(\lambda + i\varepsilon)| = \mathcal{O}(\lambda^{m-1})$, uniformly for $\lambda > \lambda_0 >> 1$ and $\varepsilon \in [0, \varepsilon_0[$ for some ε_0 sufficiently small. Consequently,

$$\xi'(\lambda) = \frac{1}{\pi}\Im\sigma_+(\lambda + i0) = \frac{1}{\pi}\Im(\sigma_+^1(\lambda + i0) + \sigma_+^2(\lambda + i0)) = \mathcal{O}(\lambda^{m-1}).$$

This ends the proof of the lemma for $N = 0$. For $N \geq 1$ we take de derivatives of $\dot{\sigma_+}(z)$ with respect to z and repeat the same arguments as above. □

Lemma 3. *Let $\psi \in C_0^\infty(\mathbb{R})$ and let f_μ be a C^∞ function in $\mathbb{R}$, depending on a parameter $\mu \in [\mu_0, +\infty[$. We suppose that, there exist $m \in \mathbb{R}$ and $\delta \in [0, 1[$ such that for all $k \in \mathbb{N}$,*

$$\left(\frac{\partial}{\partial x}\right)^k f_\mu(x) = \mathcal{O}(\mu^{m-k\delta}), \ \textit{as } \mu \to +\infty \textit{ uniformly for } x \in \mathbb{R}. \tag{25}$$

Then for all $N \in \mathbb{N}$, there exist $\mu_N > 0$ such that :

$$\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = \sum_{k=0}^{N} \frac{(-i)^k}{\mu^k k!}\psi^{(k)}(0)\left(\frac{\partial}{\partial x}\right)^k f_\mu(x) + \mathcal{O}(\mu^{-N(1-\delta)+m}), \tag{26}$$

uniformly for $x \in \mathbb{R}$ and $\mu \in [\mu_N, +\infty[$.
In particular, if $\psi = 1$ near zero, then

$$\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = f_\mu(x) + \mathcal{O}(\mu^{-\infty}). \tag{27}$$

Here

$$\mathcal{F}_\mu^{-1}\psi(t) = \frac{\mu}{2\pi}\int_{\mathbb{R}} e^{itx\mu}\psi(x)dx.$$

Proof. By a change of variable, we have

$$\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = \int_{\mathbb{R}} \mathcal{F}^{-1}\psi(t) f_\mu\left(x - \frac{t}{\mu}\right)dt. \tag{28}$$

Applying Taylor's formula to the function $t \longmapsto f_\mu(x - \frac{t}{\mu})$ at $t = 0$, and using (25), we get

$$f_\mu(x - \frac{t}{\mu}) = \sum_{k=0}^{N} f_\mu^{(k)}(x)\frac{(-t)^k}{\mu^k k!} + \mathcal{O}(\mu^{-N(1-\delta)+m}t^N). \tag{29}$$

Inserting the above equality in (28) and using the fact that
$\int_{\mathbb{R}}(-it)^k\mathcal{F}^{-1}\psi(t)dt = \psi^{(k)}(0)$ we obtain (26). □

5. Proof of Theorem 2. The statement (i) is proved in Lemma 2. To prove (ii), let $g \in C_0^\infty\left(\left]\frac{1}{2}, \frac{3}{2}\right[\right)$ be equal to 1 near one. For $\mu > 1$, we set $f_\mu(x) := g(x)\xi'(\mu^2 x)$. Using Lemma 2, we see that the function f_μ satisfies all the assumptions in Lemma 3 with $\delta = 0$. Let $\psi \in C_0^\infty(\mathbb{R})$ be as in Lemma 3 with $\psi = 1$ near zero. According to Lemma 3, we have

$$\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = f_\mu(x) + \mathcal{O}(\mu^{-\infty}). \tag{30}$$

On the other hand, a simple calculation shows that

$$\mu^2\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = \mu^2\int_{\mathbb{R}} \mathcal{F}_\mu^{-1}\psi(x-t)g(t)\xi'(\mu^2 t)dt$$

$$= \int_{\mathbb{R}} \mathcal{F}_\mu^{-1}\psi\left(x - \frac{t}{\mu^2}\right)g(\mu^{-2}t)\xi'(t)dt = \langle \xi', \mathcal{F}_\mu^{-1}\psi\left(x - \frac{\cdot}{\mu^2}\right)g(\mu^{-2}.)\rangle$$

$$= \operatorname{tr}\left[\mathcal{F}_\mu^{-1}\psi\left(x-\mu^{-2}H_2\right)g(\mu^{-2}H_2) - \mathcal{F}_\mu^{-1}\psi\left(x-\mu^{-2}H_1\right)g(\mu^{-2}H_1)\right].$$

For $\mu \gg 1$, $\mu^{-2}H_{\cdot}$ is an μ^{-1}-pseudodifferential operator. According to [6] (see also [5, chapter 11-12]), the right hand side of the last equality has a complete asymptotic expansion in powers of μ^{-2}. Combining this with (27), we get

$$\mu^2\mathcal{F}_\mu^{-1}\psi * f_\mu(x) = \mu^2 f_\mu(x) + \mathcal{O}(\mu^{-\infty}) = \mu^n\sum_{j=1}^{\infty} a_j(x)\mu^{-2j} + \mathcal{O}(\mu^{-\infty}).$$

Taking $x=1$ and $\lambda=\mu^2$ we obtain

$$\xi'(\lambda) = \lambda^{\frac{n}{2}}\sum_{j=1}^{\infty} a_j(1)\lambda^{-j-1} + \mathcal{O}(\lambda^{-\infty}).$$

We recall that $f_\mu(x) = g(x)\xi'(\mu^2 x)$ and $g(1)=1$. This ends the proof of Theorem 2. The explicit formula of a_j is given by the weak asymptotics of $\operatorname{tr}(f(H_2)-f(H_1))$ (see Theorem 1).

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