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ON THE CAUCHY PROBLEM FOR NON EFFECTIVELY HYPERBOLIC OPERATORS, THE IVRII-PETKOV-HÖRMANDER CONDITION AND THE GEVREY WELL POSEDNESS

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Abstract. In this paper we prove that for non effectively hyperbolic operators with smooth double characteristics with the Hamilton map exhibiting a Jordan block of size 4 on the double characteristic manifold the Cauchy problem is well posed in the Gevrey 6 class if the strict Ivrii-Petkov-Hörmander condition is satisfied.

1. Introduction.

Let

$$P(x, D) = D_0^2 + \sum_{|\alpha|\leq 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0$$

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be a second order differential operator, defined in an open neighborhood of the origin of $\mathbb{R}^{n+1}$, hyperbolic with respect to the x_0 direction and with principal symbol $p(x,\xi)$ where $x = (x_0, x_1, \ldots, x_n)$, $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$. Let $\bar\rho \in T^*\mathbb{R}^{n+1} \setminus \{0\}$ be a double characteristic of p, that is $p(\bar\rho) = 0$, $dp(\bar\rho) = 0$. Since $\bar\rho$ is a singular point of the Hamilton vector field H_p of p then we consider the linearization of H_p at $\bar\rho$ which is called the Hamilton map $F_p(\bar\rho)$ of p at $\bar\rho$ defined as (see e.g. [6], [5])

$$p_{\bar\rho}(X,Y) = \sigma(X, F_p(\bar\rho)Y), \quad X, Y \in T^*\mathbb{R}^{n+1}$$

where $p_{\bar\rho}(X,Y)$ is the polar form of the quadratic form $p_{\bar\rho}$

$$p_{\bar\rho}(x,\xi) = \sum_{|\alpha+\beta|=2} \frac{\partial^2 p}{\partial x^\alpha \partial \xi^\beta}(\bar\rho) x^\alpha \xi^\beta$$

and $\sigma = \sum_{j=0}^n d\xi_j \wedge dx_j$ is the canonical symplectic two form on $T^*\mathbb{R}^{n+1}$. It is well known that all eigenvalues of $F_p(\bar\rho)$ are on the imaginary axis, with a possible exception of a pair of non zero real eigenvalues ([6], [5]). When all the eigenvalues of $F_p(\bar\rho)$ are on the imaginary axis then p is called non effectively hyperbolic at $\bar\rho$. We denote by $P_{sub}(x,\xi)$ the subprincipal symbol of P and the positive trace $\mathrm{Tr}^+ F_p(\bar\rho)$ of $F_p(\bar\rho)$ is defined by

$$\mathrm{Tr}^+ F_p(\bar\rho) = \sum \mu_j$$

where $i\mu_j$ are the eigenvalues of $F_p(\bar\rho)$ on the positive imaginary axis repeated according to their multiplicities.

Theorem 1.1 ([6], [5]). *Assume that* p *is non effectively hyperbolic at* $\bar\rho$ *then in order that the Cauchy problem is* C^∞ *well posed it is necessary that the following Ivrii-Petkov-Hörmander condition is verified;*

$$-\mathrm{Tr}^+ F_p(\bar\rho) \le P_{sub}(\bar\rho).$$

Our aim in this paper is to study the Cauchy problem around non effectively double characteristics under the strict Ivrii-Petkov-Hörmander condition

$$-\mathrm{Tr}^+ F_p(\bar\rho) < P_{sub}(\bar\rho).$$

We now state more precisely our assumptions. We shall assume in the following that p vanishes exactly of order 2 on a C^∞ submanifold Σ on which σ has constant rank and p is non effectively hyperbolic; that is we assume that $\Sigma = \{(x,\xi) \mid p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and

$$\text{Sp}(F_p(\rho)) \subset i\mathbb{R}, \quad \rho \in \Sigma, \tag{1.1}$$

$$\text{dim}T_\rho\Sigma = \text{dimKer}F_p(\rho), \quad \rho \in \Sigma, \tag{1.2}$$

$$\text{rank}\sigma = \text{constant}, \quad \text{on } \Sigma \tag{1.3}$$

where $\text{Sp}(F_p(\rho))$ denotes the spectrum of $F_p(\rho)$. In this paper we always assume (1.1), (1.2) and (1.3). According to the spectral structure of $F_p(\rho)$ two different possible cases may arise:

$$\text{Ker}F_p^2(\rho) \cap \text{Im}F_p^2(\rho) = \{0\}, \quad \text{Ker}F_p^2(\rho) \cap \text{Im}F_p^2(\rho) \neq \{0\}.$$

In the first case, assuming the strict Ivrii-Petkov-Hörmander condition, the Cauchy problem is C^∞ well posed ([5], [7]). Note that the Ivrii-Petkov-Hörmander condition is not enough in general to assure the C^∞ well posedness (see [10]). On the other hand, in the second case, the linear algebraic properties of $F_p(\rho)$ are not enough by themselves to determine completely the behavior of the null bicharacteristics of the principal symbol. It can be readily verified that perturbing the quadratic part of the principal symbol with a suitable term vanishing of order three on the double manifold Σ may cause the Hamilton system to exhibit null bicharacteristics landing on Σ. (See model (1.5) below and [11]).

Here let us recall that we say that $f(x) \in C^\infty(\mathbb{R}^n)$ belongs to $\gamma^{(s)}(\mathbb{R}^n)$, the Gevrey space of order s, where $s \geq 1$, if for any compact set $K \subset \mathbb{R}^n$ there exist $C > 0$, $h > 0$ such that

$$\left|\partial_x^\alpha f(x)\right| \leq Ch^{|\alpha|}|\alpha|!^s, \quad x \in K \tag{1.4}$$

for every $\alpha \in \mathbb{N}^n$. In this paper we prove

Theorem 1.2. *Assume* $\text{Ker}F_p^2(\rho) \cap \text{Im}F_p^2(\rho) \neq \{0\}$, $\rho \in \Sigma$ *and that the strict Ivrii-Petkov-Hörmander condition is verified everywhere on* Σ. *Then the Cauchy problem for* P *is well posed in the Gevrey* 6 *class.*

We remark that it is proved in [3] that the Cauchy problem is well posed in the Gevrey 5 class if $\text{Ker}F_p^2(\rho) \cap \text{Im}F_p^2(\rho) \neq \{0\}$, $\rho \in \Sigma$ and the Levi condition is satisfied, that is $P_{sub} = 0$ everywhere on Σ. We do not know whether the Gevrey index 6 in Theorem 1.2 is optimal or not. We only show an example suggesting effects of null bicharacteristics tangent to the doubly characteristic manifold on the Gevrey well posedness. Consider

$$P = -D_0^2 + 2x_1D_0D_n + D_1^2 + x_1^3D_n^2 \tag{1.5}$$

where the double characteristic manifold is given by $\Sigma = \{\xi_0 = \xi_1 = 0, x_1 = 0\}$. This is a model (canonical) operator such that $\mathrm{Im}F_p^2(\rho) \cap \mathrm{Ker}F_p^2(\rho) \neq \{0\}$, $\rho \in \Sigma$ and admitting a null bicharacteristic with a limit point in the doubly characteristic set.

We say that the Cauchy problem for P is locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (u_0, u_1) \in (\gamma^{(s)}(\mathbb{R}^n))^2$, there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 \quad \text{in } U_\Phi \\ D_0^j u(0, x') = u_j(x'), \quad j = 0, 1, \quad x \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a solution $u(x) \in C^\infty(U_\Phi)$ (see for example [8]). Then we have

Proposition 1.1 ([3]). *The Cauchy problem for P is not locally solvable at the origin in any Gevrey class of order $s > 5$.*

2. Non effectively hyperbolic characteristics. Without restrictions, we may assume that the principal symbol $p(x, \xi)$ of P has the form

$$p(x, \xi) = -\xi_0^2 + q(x, \xi') \tag{2.1}$$

where $q \geq 0$. As stated in Introduction, we always assume (1.1), (1.2) and (1.3). We recall

Proposition 2.1 ([3]). *Suppose that p vanishes exactly of order 2 on a C^∞ submanifold Σ of $T^*\mathbb{R}^{n+1} \setminus \{0\}$ on which the canonical 2 form has constant rank and such that $\mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}$, $\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) \neq 0$, $\forall \rho \in \Sigma$. Then one can write, near a reference point $\bar{\rho}$;*

$$p(x, \xi) = -\xi_0^2 + \sum_{j=1}^{r} \phi_j(x, \xi')^2$$

where Σ is given by $\{\xi_0 = 0, \phi_1 = \cdots = \phi_r = 0\}$ and

$$\{\xi_0 + \phi_1, \phi_j\} = 0, \ j = 1, \ldots, r, \quad \text{on } \Sigma. \tag{2.2}$$

In this proposition it can be concluded $\{\phi_1, \phi_2\} \neq 0$ on Σ which was essential in [3], but in this paper we do not need this fact. Let us set

$$Q = \sum_{j=2}^{r} \phi_j(x, \xi')^2$$

then we have

Lemma 2.1. *Let $\rho \in \Sigma$ then we have*

$$\mathrm{Tr}^+ F_p(\rho) = \mathrm{Tr}^+ F_Q(\rho).$$

P r o o f. One can write $p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + Q(x, \xi')$ which gives

$$p_\rho = -(\xi_0 + d\phi_1(\rho))(\xi_0 - d\phi_1(\rho)) + Q_\rho.$$

By a linear symplectic change of coordinates one may assume that

$$p_\rho = -\xi_0(\xi_0 - 2\xi_1) + Q_\rho(x, \xi').$$

Since $\{\xi_0, Q_\rho\} = 0$ one concludes that Q_ρ is independent of x_0 and hence $Q_\rho = Q_\rho(x', \xi')$. Now it is easy to see that

$$\left|\lambda - F_{p_\rho}\right| = \lambda^2 \left|\lambda - F_{Q_\rho}\right|$$

which proves that non zero eigenvalues of F_{p_ρ} coincides with those of F_{Q_ρ} including the multiplicities. $\square$

We are working near $\bar\rho \in \Sigma$. Let us set

$$\psi_i(x, \xi') = \sum_{j=2}^{r} O_{ij}\phi_j(x, \xi'), \quad i = 2, \dots, r$$

where $O = (O_{ij})$ is an orthogonal matrix. Since $(\{\phi_i, \phi_j\}(\bar\rho))$ is a real anti-symmetric then taking into account $(\{\psi_i, \psi_j\}(\bar\rho)) = O(\{\phi_i, \phi_j\}(\bar\rho))O^{-1}$ we may assume that, with a suitable O

$$\begin{cases} \{\psi_{2i-1}, \psi_j\}(\bar\rho) = \delta_{2i,j}\mu_i, \quad 1 \le j \le 2k + \ell,\ 1 \le i \le k \\ \{\psi_{2i}, \psi_j\}(\bar\rho) = -\delta_{2i-1,j}\mu_i, \quad 1 \le j \le 2k + \ell,\ 1 \le i \le k \\ \{\psi_{2k+j}, \psi_i\}(\bar\rho) = 0, \quad 1 \le j \le \ell,\ 1 \le i \le 2k + \ell \end{cases}$$

where $r = 2k + \ell$ and $i\mu_j$ are the eigenvalues of $(\{\phi_i, \phi_j\}(\bar\rho))$ on the positive imaginary axis counting the multiplicities. Here we remark that

$$\sum_{i=1}^{k} \mu_i = \mathrm{Tr}^+ F_Q(\bar\rho). \tag{2.3}$$

Indeed put

$$\begin{aligned}\Xi_i &= \frac{1}{\sqrt{\mu_i}} d\psi_{2i-1}(x',\xi'),\ X_i = \frac{1}{\sqrt{\mu_i}} d\psi_{2i}(x',\xi'), \quad 1\le i\le k,\\ X_{k+j} &= d\psi_{2k+j}(x',\xi'),\ 1\le j\le \ell.\end{aligned}$$

Since Ξ_i, $1 \le i \le k$, X_i, $1 \le i \le k+\ell$ verifies the canonical commutation relations and hence these extend to a full symplectic coordinates $\{X_i, \Xi_i\}_{i=1}^n$. It is clear that

$$Q_{\bar\rho} = \sum_{i=1}^{k} \mu_i (X_i^2 + \Xi_i^2) + \sum_{i=k+1}^{k+\ell} X_i^2.$$

Since $\mathrm{Tr}^+ Q_{\bar\rho}$ is symplectically invariant and then we have (2.3). Therefore by Lemma 2.1 we have

$$\mathrm{Tr}^+ F_p(\bar\rho) = \sum_{i=1}^{k} \mu_i.$$

We assume that P satisfies the strict Ivrii-Petkov-Hörmander condition near $\bar\rho$;

$$-\mathrm{Tr}^+ F_p(\rho) < P_{sub}(\rho), \quad \rho\in\Sigma, \quad \rho \text{ near } \bar\rho.$$

In particular the condition implies that $\mathsf{Im}\, P_{sub}(\rho) = 0$ for $\rho \in \Sigma$, ρ near $\bar\rho$. Denoting by P_1^s a real valued extension of $P_{sub}(x,\xi)$ outside Σ, we see that $P_{sub}(x,\xi) - P_1^s(x,\xi)$ vanishes on Σ near $\bar\rho$. Hence we conclude that we may assume

$$P_{sub} = P_1^s + R, \quad P_1^s(\rho) \ge -\mathrm{Tr}^+ F_p(\rho) + \epsilon \tag{2.4}$$

near $\bar\rho$ with some $\epsilon > 0$ where $R = \sum_{j=0}^r C_j(x,\xi')\phi_j(x,\xi')$ with $\phi_0 = \xi_0$ near $\bar\rho$.

3. A lemma for Weyl calculus in the Gevrey class. In this section we introduce a class of symbols of pseudodifferential operators which will be used in section 4 to derive Gevrey a priori estimate for P. For $a(x,D)$ with a such symbol we prove a composition formula $e^{\pm\phi(D)} a(x,D) e^{\mp\phi(D)}$ where $\phi(D) \sim (1+|D|)^{1/6}$.

Let

$$\bar g = \langle \mu\xi\rangle^{\delta}\big\{|dx|^2 + \langle\xi\rangle_\mu^{-2}|d\xi|^2\big\}, \ \ \langle\xi\rangle_\mu = (\mu^{-2} + |\xi|^2)^{1/2}, \ \ (x,\xi)\in\mathbb{R}^{2n},$$

be a metric where $0 < \delta < 1$. δ will eventually take the value $2/3$. We say $b(x,\xi,\mu) \in \gamma^{(s)}S(m(\xi,\mu),\bar{g})$ if $b(x,\xi,\mu)$ verifies the following estimates;

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi,\mu)| \le C_\beta m(\xi,\mu)(\langle\mu\xi\rangle^{-\delta/2}\langle\xi\rangle_\mu)^{-|\beta|} \\ \times A^{|\alpha|}|\alpha|!^{s/2}(|\alpha|^{s/2} + \langle\mu\xi\rangle^{\delta/2})^{|\alpha|}$$

for every α, $\beta \in \mathbb{N}^n$. We assume that $b(x,\xi,\mu)$ is independent of x for $|x| \ge M$ with a large M.

Lemma 3.1 ([3]). *Let $s \ge 4$. Assume that*

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi,\mu)| \le C_\beta m(\xi,\mu)(\langle\mu\xi\rangle^{-\delta/2}\langle\xi\rangle_\mu)^{-|\beta|}A^{|\alpha|}|\alpha|!^s$$

for every α, $\beta \in \mathbb{N}^n$. Then we have

$$w(x,\xi,\mu) = \sqrt{f(x,\xi,\mu)^2 + \langle\mu\xi\rangle^{-\delta}} \in \gamma^{(s)}S(m(\xi,\mu),\bar{g}).$$

Lemma 3.2. *Let $a_i(x,\xi,\mu) \in \gamma^{(s)}S(m_i(\xi,\mu),\bar{g})$, $i = 1,2$. Then we have*

$$a_1(x,\xi,\mu)a_2(x,\xi,\mu) \in \gamma^{(s)}S(m_1(\xi,\mu)m_2(\xi,\mu),\bar{g}).$$

Note that if

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi,\mu)| \le C_\beta m(\xi,\mu)\langle\xi\rangle_\mu^{-|\beta|}A^{|\alpha|}|\alpha|!^s, \quad \forall\alpha,\beta$$

then it is obvious that $b(x,\xi,\mu) \in \gamma^{(s)}S(m(\xi,\mu),\bar{g})$.

We now consider

$$e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}.$$

Let us denote $\kappa = 1/s$. As for $\phi(\xi,\mu)$ we assume that

$$\text{(3.1)} \qquad \begin{cases} \phi(\xi,\mu) \in S(\langle\mu\xi\rangle^\kappa, |dx|^2 + \langle\xi\rangle_\mu^{-2}|d\xi|^2), \\ \phi(\xi+\eta,\mu) - \phi(\xi-\eta,\mu) \le C\langle\mu\eta\rangle^\kappa. \end{cases}$$

Then the following proposition holds.

Proposition 3.1 ([3]). *Let $\delta+\kappa\leq 1$ and $b(x,\xi,\mu)\in\gamma^{(1/\kappa)}S(m(x,\xi'),\bar{g})$, $1/\kappa\geq 4$. Assume (3.1). Let $e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}=c^w(x,D,\mu)$ then one can write*

$$c(x,\xi,\mu)=\sum_{j=0}^{N-1}c_j(x,\xi,\mu)+R_N(x,\xi,\mu)$$

where

$$c_j=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0}b_{(\alpha)}(x,\xi,\mu)$$

$$\in\mu^jS(m(\xi,\mu)\langle\mu\xi\rangle^{-j(1-\kappa-\delta/2)},\bar{g}),$$

$$R_N(x,\xi,\mu)\in\mu^NS(m(\xi,\mu)\langle\mu\xi\rangle^{-N(1-\kappa-\delta/2)+n\delta/2},\bar{g}).$$

4. A priori estimate in the Gevrey class.

In this section we derive a priori estimates in the Gevrey class assuming that (2.2) and (2.4) are globally satisfied and then prove Theorem 1.2. Since the routine arguments of partition of unity allow us to reduce our estimate to a global one, we will skip this standard step.

4.1. Preparatory lemmas. We start with

$$p=-(\xi_0+\phi_1)(\xi_0-\phi_1)+\sum_{j=2}^{r}\phi_j^2$$

where we assume that our assumptions are satisfied globally;

$$\{\xi_0+\phi_1,\phi_j\}=\sum_{k=1}^{r}C_{jk}\phi_k,\quad j=1,\ldots,r,\tag{4.1}$$

$$\{\phi_{2i-1},\phi_{2i}\}(x,\xi')\geq(\mu_i-\epsilon_1)|\xi'|,\quad i=1,\ldots,k,\tag{4.2}$$

$$P_{sub}(x,\xi)=P_1^s(x,\xi')+R(x,\xi'),\tag{4.3}$$

$$P_1^s(x,\xi')\geq(-A+\epsilon_2)|\xi'|,\;\; R(x,\xi')=\sum_{j=0}^{r}C_j(x,\xi')\phi_j(x,\xi)\tag{4.4}$$

where $C_{jk}(x,\xi')$, $C_j(x,\xi')$ are homogeneous of degree 0 and $A=\mathrm{Tr}^+F_p(\bar{\rho})=\sum_{i=1}^{k}\mu_i$ and $\epsilon_2>0$ is a fixed positive constant while one can take $\epsilon_1>0$ as small as we please.

We now make a dilation of the variable: $x_0 \to \mu x_0$ so that we have

$$\begin{aligned} P(x,\xi,\mu) &= \mu^2 P(\mu x_0, x', \mu^{-1}\xi_0, \xi') \\ &= p(\mu x_0, x', \xi_0, \mu\xi') + \mu P_1(\mu x_0, x', \xi_0, \mu\xi') + \mu^2 P_0(\mu x_0, x') \\ &= p(x,\xi,\mu) + P_1(x,\xi,\mu) + P_0(x,\xi,\mu) \end{aligned}$$

where

$$\begin{aligned} p(x,\xi,\mu) &= -(\xi_0 - \phi_1(\mu x_0, x', \mu\xi'))(\xi_0 + \phi_1(\mu x_0, x', \mu\xi')) \\ &\quad + \sum_{j=2}^{r} \phi_j(\mu x_0, x', \mu\xi')^2 \\ &= -(\xi_0 - \phi_1(x,\xi',\mu))(\xi_0 + \phi_1(x,\xi',\mu)) + \sum_{j=2}^{r} \phi_j(x,\xi',\mu)^2. \end{aligned}$$

In what follows sometimes we simply write $p(x,\xi,\mu)$, $P_j(x,\xi',\mu)$ as $p(x,\xi)$, $P_j(x,\xi')$. Then (4.1) and (4.2) become

$$\begin{aligned} \{\xi_0 + \phi_1, \phi_j\} &= \mu \sum_{k=1}^{r} C_{jk}\phi_k, \quad j = 1, \ldots, r \\ \{\phi_{2i-1}, \phi_{2i}\} &\geq (\mu_i - \epsilon_1)\mu|\mu\xi'|, \quad i = 1, \ldots, k \end{aligned} \tag{4.5}$$

while (4.3) and (4.4) become

$$\begin{aligned} P_{sub}(x,\xi,\mu) &= P_1^s(x,\xi',\mu) + R(x,\xi,\mu), \quad R = \mu \sum_{j=0}^{r} C_j\phi_j \\ P_1^s(x,\xi,\mu) &\geq (-A + \epsilon_2)\mu|\mu\xi'|. \end{aligned} \tag{4.6}$$

Let g_0 be the standard $S_{1,0}$ metric with small parameter $\mu > 0$;

$$g_0 = \mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2, \quad \langle\xi'\rangle_\mu = (\mu^{-2} + |\xi'|^2)^{1/2}.$$

Let $\chi(s)$ be $\chi(s) = 0$ near 0 and identically 1 outside $|s| \geq 2$ and cut off a reference symbol $\phi(x,\xi',\mu)$ by $\chi(\mu|\xi'|)$ so that we consider $\phi(x,\xi',\mu)\chi(\mu|\xi'|)$. Remark that

Lemma 4.1. *Let $\phi(x,\xi')$ be homogeneous of degree k in ξ'. Then we have*

$$\phi(\mu x_0, x', \mu\xi')\chi(\mu|\xi'|) \in S(\langle\mu\xi'\rangle^k, g_0).$$

Proof. Note that $\phi(x,\xi')\chi(|\xi'|) \in S(\langle\xi'\rangle^k, |dx|^2 + \langle\xi'\rangle^{-2}|d\xi'|^2)$ and $a(\mu x_0, x', \mu\xi') \in S(\langle\mu\xi'\rangle^k, g_0)$ if $a(x,\xi') \in S(\langle\xi'\rangle^k, |dx|^2 + \langle\xi'\rangle^{-2}|d\xi'|^2)$. □

Let us set

$$w = \sqrt{\langle\mu\xi'\rangle^{-2}\phi_1^2 + \langle\mu\xi'\rangle^{-2/3}}, \quad \Phi = \sqrt{1-aw} \tag{4.7}$$

with some constant $a > 0$ so that $1 - aw \geq c > 0$. We introduce a metric

$$g = w^{-2}g_0 = w^{-2}\{\mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2\}$$

which will be used throughout this paper. We also put $\bar{g} = \langle\mu\xi'\rangle^\delta g_0$ as before. Note that

$$w \in S(w,g), \quad \phi_1 \in S(\langle\mu\xi'\rangle w, g).$$

We can rewrite p as

$$p = -(\xi_0 + \phi_1\Phi)(\xi_0 - \phi_1\Phi) + \sum_{j=2}^{r}\phi_j^2 + aw\phi_1^2 \tag{4.8}$$

because $1 - \Phi^2 = aw$. Remark also that

$$\xi_0 + \phi_1\Phi = \xi_0 + \phi_1 + \phi_1(\Phi - 1) = \xi_0 + \phi_1 - \phi_1\psi$$

where

$$\psi = 1 - \Phi = \frac{aw}{1 + \sqrt{1-aw}} \in S(w,g).$$

Lemma 4.2. *We have*

$$\begin{aligned} \phi_1\psi &\in S(w^2\langle\mu\xi'\rangle, g), \quad \phi_1(1+\psi) \in S(w\langle\mu\xi'\rangle, g), \\ \partial_x^\alpha\partial_\xi^\beta(\phi_1\Phi) &\in \mu^{|\beta|}S(\langle\mu\xi'\rangle^{1-|\beta|}, g), \quad |\alpha+\beta| = 2. \end{aligned}$$

Proof. The first two assertions are clear. To check the third one it suffices to note that $\partial_x^\alpha\partial_\xi^\beta\Phi \in \mu^{|\beta|}S(\langle\mu\xi'\rangle^{-|\beta|}, g)$ for $|\alpha+\beta| = 1$. □

In what follows κ and δ are fixed as

$$\kappa = 1/6, \quad \delta = 2/3$$

and assume that $a = 1$ without restrictions. Since $w \geq \langle\mu\xi'\rangle^{-\delta/2}$ and hence $w^{-1/2} \leq \langle\mu\xi'\rangle^{\delta/4} = \langle\mu\xi'\rangle^{1/6} = \langle\mu\xi'\rangle^\kappa$ so that $w^{-1/2} \in S(\langle\mu\xi'\rangle^\kappa, g)$.

Lemma 4.3. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_0)$. Then* (see e.g. Theorem 18.5.5 in [4]) *we have*

(i) $a\#a - a^2 \in \mu^2 S(m_1^2 w^{-4}\langle\mu\xi'\rangle^{-2}, g)$

(ii) $a\#b - b\#a - \frac{1}{i}\{a,b\} \in \mu^3 S(m_1 m_2 w^{-3}\langle\mu\xi'\rangle^{-3}, g)$

(iii) $a\#b + b\#a - 2ab \in \mu^2 S(m_1 m_2 w^{-2}\langle\mu\xi'\rangle^{-2}, g)$.

Corollary 4.1. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_0)$ be real. Then we have*

$$([ab]^w u, u) = \mathsf{Re}\,(b^w u, a^w u) + (T^w u, u)$$

with $T \in \mu^2 S(m_1 m_2 w^{-2}\langle\mu\xi'\rangle^{-2}, g)$.

Lemma 4.4 ([3]). *Let $a \in \mu S(1, g)$. Then we have*

$$\begin{aligned}\mathsf{Re}\,([a\phi_1^2 w]^w u, u) &\leq C\mu\mathsf{Re}\,([\phi_1^2 w]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{\kappa} u\|^2,\\ \mathsf{Re}\,([a\phi_j^2]^w u, u) &\leq C\mu\mathsf{Re}\,([\phi_j^2]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{2\kappa} u\|^2, \quad j \geq 2.\end{aligned}$$

Let $a \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$. Then we have

$$\begin{aligned}\mathsf{Re}\,([a\phi_1^2 w]^w u, u) &\leq C\mu\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2} u\|^2,\\ \mathsf{Re}\,([a\phi_j^2]^w u, u) &\leq C\mu\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{5\kappa/2} u\|^2, \quad j \geq 2.\end{aligned}$$

Let $a \in \mu S(1, g)$ then we have

$$\begin{aligned}\|[a\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 &\leq C\mu^2\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + C\mu^4\|\langle\mu D'\rangle^{5\kappa/2} u\|^2,\\ \|[a\langle\mu\xi'\rangle^{\kappa/2}\sqrt{w}\phi_1]^w u\|^2 &\leq C\mu^2\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa} w\phi_1^2]^w u, u) + C\mu^4\|\langle\mu D'\rangle^{3\kappa/2} u\|^2, \quad j \geq 2.\end{aligned}$$

Lemma 4.5 ([3]). *Let $j \neq 1$ and $a \in \mu S(1, g)$. Then*

$$\begin{aligned}\mathsf{Re}\,([a\phi_1\phi_j]^w u, u) \leq\; & C\mu\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u)\\ &+ C\mu\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2} u\|^2.\end{aligned}$$

Lemma 4.6. *Let $a \in \mu S(\langle\mu\xi'\rangle w, g)$. Then for $j \neq 1$ we have*

$$\begin{aligned}\mathsf{Re}\,([a\phi_j]^w u, u) \leq\; & C\mu^{1/2}\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^{1/2}\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u)\\ &+ C\mu^{3/2}\|\langle\mu D'\rangle^{7\kappa/2} u\|^2 + C\mu^{7/2}\|\langle\mu D'\rangle^{3\kappa/2} u\|^2.\end{aligned}$$

P r o o f. Let us write

$$a\phi_j = \mathsf{Re}\,(\mu^{1/4}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{-1/4}\langle\mu\xi'\rangle^{-\kappa/2}a) + \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g)$$

and hence

$$\begin{aligned}\mathsf{Re}\,([a\phi_j]^w u, u) &\le \mu^{1/2}\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2\\ &\quad +\mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^w u\|^2 + C\mu^3\|\langle\mu D'\rangle^{\kappa}u\|^2.\end{aligned}$$

Note that $\langle\mu\xi'\rangle^{-\kappa/2}a\#\langle\mu\xi'\rangle^{-\kappa/2}a = \langle\mu\xi'\rangle^{-\kappa}a^2 + \mu^4 S(\langle\mu\xi'\rangle^{3\kappa}, g)$ and write

$$\begin{aligned}\langle\mu\xi'\rangle^{-\kappa}a^2 &= (w^{-2}a^2\langle\mu\xi'\rangle^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa}\\ &= b(\langle\mu\xi'\rangle^{-2}\phi_1^2 + \langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}\\ &= b\langle\mu\xi'\rangle^{-2\kappa}w^{-1}(\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}) + b\langle\mu\xi'\rangle^{2-\delta-\kappa}\end{aligned}$$

where $b = w^{-2}a^2\langle\mu\xi'\rangle^{-2} \in \mu^2 S(1, g)$. Since $2-\delta-\kappa = 7\kappa$ thanks to Lemma 4.4 we get

$$\begin{aligned}\mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^w u\|^2 &\le C\mu^{3/2}\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u)\\ &\quad +C\mu^{7/2}\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + C\mu^{3/2}\|\langle\mu D'\rangle^{7\kappa/2}u\|^2.\end{aligned}$$

Then we get the assertion. □

We now estimate $\{\xi_0 + \phi_1\Phi, \phi_j^2\}$, $\{\xi_0 + \phi_1\Phi, w\phi_1^2\}$. Recall that

$$\xi_0 + \phi_1\Phi = \xi_0 + \phi_1 - \phi_1\psi.$$

We first estimate $\{\xi_0 + \phi_1, \phi_j^2\}$ and $\{\xi_0 + \phi_1, w\phi_1^2\}$. From the assumption we have

$$\{\xi_0 + \phi_1, \phi_j^2\} = 2\{\xi_0 + \phi_1, \phi_j\}\phi_j = \sum_{k=1}^{r} C_{jk}\phi_k\phi_j$$

where $C_{jk} \in \mu S(1, g_0)$. Note that for $j, k \ge 2$

$$\begin{aligned}\mathsf{Re}\,([C_{jk}\phi_k\phi_j]^w u, u) &\le C\mu\mathsf{Re}\,([\phi_k^2]^w u, u)\\ &\quad +C\mu\mathsf{Re}\,([\phi_j^2]^w u, u) + C\mu^3\|u\|^2.\end{aligned}$$

For $C_{j1}\phi_1\phi_j$ we apply Lemma 4.5 to get

$$\begin{aligned}\mathsf{Re}\,([C_{j1}\phi_1\phi_j]^w u, u) &\le C\mu\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u, u)\\ &\quad +C\mu\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{5\kappa/2}u\|^2.\end{aligned}$$

We turn to consider

$$\{\xi_0+\phi_1, w\phi_1^2\} = 2\{\xi_0+\phi_1,\phi_1\}\phi_1 w + \{\xi_0+\phi_1, w\}\phi_1^2.$$

For the first term of the right-hand side we remark that

$$\{\xi_0+\phi_1,\phi_1\} = \sum_{k=1}^{r} C_{1k}\phi_k$$

and apply Lemma 4.5 and Lemma 4.4. Recalling $w=\sqrt{\phi_1^2\langle\mu\xi'\rangle^{-2}+\langle\mu\xi'\rangle^{-\delta}}$ we have

$$\begin{aligned}
\{\xi_0+\phi_1, w\} &= \frac{1}{2}w^{-1}\{\xi_0+\phi_1, \langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta}\}\\
&= \frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^2 + w^{-1}\{\xi_0+\phi_1,\phi_1\}\phi_1\langle\mu\xi'\rangle^{-2}\\
&\quad+\frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}.
\end{aligned}$$

Note that $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^2$, $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}\in \mu S(w,g)$ and apply Lemma 4.4 to $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^4$ and $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}\phi_1^2$. Note that

$$w^{-1}\{\xi_0+\phi_1,\phi_1\}\phi_1^3\langle\mu\xi'\rangle^{-2} = \sum_{k=1}^{r} T_k\phi_k\phi_1$$

with $T_k\in\mu S(w,g)$ and apply Lemma 4.5 and Lemma 4.4.

We next estimate $\{\phi_1\psi,\phi_j^2\}$ and $\{\phi_1\psi, w\phi_1^2\}$. Let us consider for $j\geq 2$

$$\{\phi_1\psi,\phi_j^2\} = 2\{\phi_1,\phi_j\}\phi_j\psi + 2\{\psi,\phi_j\}\phi_j\phi_1.$$

Write

$$\{\phi_1,\phi_j\}\phi_j\psi = \mathsf{Re}\,(\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\{\phi_1,\phi_j\}\psi\langle\mu\xi'\rangle^{-\kappa/2}) + \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g)$$

and note that

$$\begin{aligned}
\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-\kappa} &= (\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa}\\
&= Tw^2\langle\mu\xi'\rangle^{2-\kappa} = T(\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}\\
&= (Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa})w\langle\mu\xi'\rangle^{\kappa}\phi_1^2 + T\langle\mu\xi'\rangle^{2-\delta-\kappa}
\end{aligned}$$

with $T=\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2}\in\mu^2S(1,g)$ and hence we have $Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa}\in\mu^2S(1,g)$, $T\langle\mu\xi'\rangle^{2-\delta-\kappa}\in\mu^2S(\langle\mu\xi'\rangle^{7\kappa},g)$. We now apply Lemma 4.5. We turn to consider

$$\{\phi_1\psi,\phi_1^2w\}=\{\phi_1,w\}\phi_1^2\psi+\{\psi,w\}\phi_1^3+2\{\psi,\phi_1\}\phi_1^2w.$$

Note that $\{\phi_1,w\}$, $\{\psi,\phi_1\}\in\mu S(1,g)$, $\{\psi,w\}\phi_1\in\mu S(w,g)$ and apply Lemma 4.4. We summarize

Proposition 4.1. *We have*

$$|\mathsf{Re}\,(\{\xi_0+\phi_1\Phi,\phi_j^2\}^wu,u)|,\ |\mathsf{Re}\,(\{\xi_0+\phi_1\Phi,w\phi_1^2\}^wu,u)|$$
$$\le C\mu\left\{\sum_{j=2}^r\mathsf{Re}\,([\langle\mu\xi'\rangle^\kappa\phi_j^2]^wu,u)+\mathsf{Re}\,([w\langle\mu\xi'\rangle^\kappa\phi_1^2]^wu,u)\right\}$$
$$+C\mu^2\|\langle\mu D'\rangle^{7\kappa/2}u\|^2.$$

4.2. Energy inequality (proof of Theorem 1.2.). Let us consider

$$P=-M\Lambda+B\Lambda+Q$$

where

$$\Lambda=D_0-i\langle\mu D'\rangle^\kappa-\lambda.$$

Proposition 4.2 ([3]). *We have*

$$\begin{aligned}2\mathsf{Im}\,(Pv,\Lambda v)&=\frac{d}{dx_0}(||\Lambda v||^2+(\mathsf{Re}\,Qv,v))\\&+2||\langle\mu D'\rangle^{\kappa/2}\Lambda v||+2\mathsf{Re}\,(\langle\mu D'\rangle^\kappa\mathsf{Re}\,Qv,v)\\&+2(\mathsf{Im}\,B\Lambda v,\Lambda v)+2(\mathsf{Im}\,m\Lambda v,\Lambda v)+2\mathsf{Re}\,(\Lambda v,\mathsf{Im}\,Qv)\\&+\mathsf{Im}\,([D_0-\mathsf{Re}\,\lambda,\mathsf{Re}\,Q]v,v)+2\mathsf{Re}\,(\mathsf{Re}\,Qv,\mathsf{Im}\,\lambda v).\end{aligned}$$

Let us denote $\Lambda=(\xi_0+\phi_1\Phi)^w=D_0-\lambda^w$, $M=(\xi_0-\phi_1\Phi)^w=D_0-m^w$ then thanks to Lemma 4.2 we see that

$$(\xi_0-\phi_1\Phi)\#(\xi_0+\phi_1\Phi)=\xi_0^2-\phi_1^2\Phi^2+\frac{1}{2i}\{\xi_0-\phi_1\Phi,\xi_0+\phi_1\Phi\}+\mu^2S(1,g)$$

and from (4.5) it follows that $\{\xi_0, \phi_1\} = \mu \sum_{j=1}^{r} C_j \phi_j$ and hence

$$\{\xi_0 - \phi_1 \Phi, \xi_0 + \phi_1 \Phi\} = \mu \sum_{j=1}^{r} C_j \phi_j, \quad C_j \in S(1, g).$$

Noting that $P = (p + P_{sub})^w + \mu^2 S(1, g_0)$ and (4.6) one can write

$$P = -M\Lambda + \left[\sum_{j=2}^{r} \phi_j^2 + w\phi_1^2 + P_1^s + R\right]^w + \mu^2 S(1, g)$$

where $R = \mu \sum_{j=0}^{r} C_j \phi_j$, $C_j \in S(1, g)$. We rewrite $C_0 \phi_0 = C_0 \xi_0$ as

$$C_0 \xi_0 = C_0 \# (\xi_0 + \phi_1 \Phi) - C_0 \Phi \phi_1 + \mu^2 S(\langle \mu \xi' \rangle^{2\kappa}, \bar{g})$$

which gives

$$\begin{aligned} P &= -M\Lambda + B\Lambda + Q, \\ Q &= \left[\sum_{j=2}^{r} \phi_j^2 + w\phi_1^2 + P_1^s + R\right]^w + \mu^2 S(\langle \mu \xi' \rangle^{2\kappa}, \bar{g}) \end{aligned} \tag{4.9}$$

where $B \in \mu S(1, g)$ and

$$R = \sum_{j=1}^{r} c_j \phi_j, \quad c_j \in \mu S(1, g). \tag{4.10}$$

Here we note that by Lemma 3.1 with $f(x, \xi', \mu) = \phi_1(x, \xi', \mu) \langle \mu \xi' \rangle^{-1}$ we have $w \in \gamma^{(1/\kappa)} S(1, \bar{g})$. We now conjugate $e^{-x_0 \langle \mu D' \rangle^\kappa} = e^\phi$ to P;

$$e^\phi P e^{-\phi} = -e^\phi M e^{-\phi} e^\phi \Lambda e^{-\phi} + e^\phi B e^{-\phi} e^\phi \Lambda e^{-\phi} + e^\phi Q e^{-\phi}.$$

Let us denote $e^\phi M e^{-\phi}$, $e^\phi \Lambda e^{-\phi}$, $e^\phi B e^{-\phi}$, $e^\phi Q e^{-\phi}$ by M, Λ, B, Q again. We first study

$$M = e^\phi (D_0 - m^w) e^{-\phi} = D_0 - i \langle \mu D' \rangle^\kappa - e^\phi m^w e^{-\phi}.$$

Since $m \in S(w \langle \mu \xi' \rangle, g)$ we apply Proposition 3.1 for Weyl calculus with $\delta = 2/3 = 4\kappa$. Then we have

$$e^\phi m^w e^{-\phi} = -[m_0 + m_1 + m_2]^w, \quad m_0 = -\phi_1 \Phi \tag{4.11}$$

where $m_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $m_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$ by Proposition 3.1 where we recall $\bar{g} = \langle \mu\xi' \rangle^{4\kappa} g_0$. Let us consider

$$\Lambda = e^\phi (D_0 - \lambda^w) e^{-\phi} = D_0 - i\langle \mu D' \rangle^\kappa - e^\phi \lambda^w e^{-\phi}.$$

Since $\lambda \in S(w\langle \mu\xi' \rangle, g)$ repeating the same arguments we have

$$e^\phi \lambda^w e^{-\phi} = -[\lambda_0 + \lambda_1 + \lambda_2]^w, \quad \lambda_0 = \phi_1 \Phi \tag{4.12}$$

where $\lambda_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $\lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. An immediate consequence of Proposition 3.1 is $B \in \mu S(1, \bar{g})$.

We now consider $e^\phi Q e^{-\phi}$. Note that

$$e^\phi [\phi_j^2]^w e^{-\phi} = [\phi_j^2 + a_j \phi_j + r_j]^w$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $r_j \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. We next consider

$$e^\phi [w\phi_1^2]^w e^{-\phi} = [w\phi_1^2 + a_1 w\phi_1 + r_1]^w$$

where $a_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ is pure imaginary and $r_1 \in \mu^2 S(w\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Since $P_1^s \in \mu S(\langle \mu\xi' \rangle, g_0)$ we have $e^\phi [P_1^s]^w e^{-\phi} = [P_1^s + r_2]^w$ where $r_2 \in \mu^2 S(\langle \mu\xi' \rangle^\kappa, \bar{g})$ and

$$e^\phi R e^{-\phi} = \left[\sum_{j=1}^r c_j \phi_j + \tilde{r} \right]^w$$

where $c_j \in \mu S(1, \bar{g})$ and $\tilde{r} \in \mu^2 S(\langle \mu\xi' \rangle^{3\kappa}, \bar{g})$. One can write

$$e^\phi Q e^{-\phi} = \left[\sum_{j=2}^r \phi_j^2 + w\phi_1^2 + P_1^s + \sum_{j=2}^r a_j \phi_j + a_1 w\phi_1 + \sum_{j=1}^r c_j \phi_j + r \right]^w \tag{4.13}$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ are pure imaginary and $r \in \mu^2 S(\langle \mu\xi' \rangle^{3\kappa}, \bar{g})$. Let us put

$$q = \sum_{j=2}^r \phi_j^2 + w\phi_1^2 + P_1^s = q_0 + P_1^s, \;\; q_1 = \sum_{j=2}^r a_j \phi_j + a_1 w\phi_1, \;\; q_2 = \sum_{j=1}^r c_j \phi_j.$$

We summarize

Proposition 4.3. *We can write*

$$e^\phi P e^{-\phi} = -M\Lambda + B\Lambda + Q$$

with $B \in \mu S(1, \bar{g})$ *and*

$$\begin{aligned} M &= D_0 - i\langle \mu D' \rangle^\kappa + [m_0 + m_1 + m_2]^w = D_0 - i\langle \mu D' \rangle^\kappa - m^w, \\ \Lambda &= D_0 - i\langle \mu D' \rangle^\kappa + [\lambda_0 + \lambda_1 + \lambda_2]^w = D_0 - i\langle \mu D' \rangle^\kappa - \lambda^w \end{aligned}$$

where $m_1, \lambda_1 \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ *are pure imaginary and* $m_2, \lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. *As for* Q *we have*

$$\begin{aligned} Q &= [q + q_1 + q_2 + r]^w, \quad q = \sum_{j=2}^r \phi_j^2 + w\phi_1^2 + P_1^s = q_0 + P_1^s, \\ q_1 &= \sum_{j=2}^r a_j\phi_j + a_1 w\phi_1, \quad q_2 = \sum_{j=1}^r c_j\phi_j \end{aligned}$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ *are pure imaginary,* $c_j \in \mu S(1, \bar{g})$, $r \in \mu^2 S(\langle \mu\xi' \rangle^{3\kappa}, \bar{g})$.

Here we note

$$-2\mathsf{Im}\,(\Lambda w, w) = \frac{d}{dx_0}\|w\|^2 + 2\|\langle \mu D' \rangle^{\kappa/2} w\|^2 + 2(\mathsf{Im}\,\lambda w, w)$$

and from this it follows that

$$\begin{aligned} \alpha^{-1}\|\langle \mu D' \rangle^{-\kappa/2}\Lambda\langle \mu D' \rangle^\kappa w\|^2 \geq \frac{d}{dx_0}\|\langle \mu D' \rangle^\kappa w\|^2 \\ +(2-\alpha)\|\langle \mu D' \rangle^{3\kappa/2} w\|^2 + 2(\mathsf{Im}\,\lambda\langle \mu D' \rangle^\kappa w, \langle \mu D' \rangle^\kappa w) \end{aligned} \tag{4.14}$$

with a small $0 < \alpha$ (< 2). Since $\mathsf{Im}\,\lambda \in \mu S(\langle \mu\xi' \rangle^\kappa, g)$ one sees that

$$\alpha^{-1}\|\langle \mu D' \rangle^{-\kappa/2}\Lambda\langle \mu D' \rangle^\kappa u\|^2 \geq \frac{d}{dx_0}\|\langle \mu D' \rangle^\kappa u\|^2 + (2 - \alpha - C\mu)\|\langle \mu D' \rangle^{3\kappa/2} u\|^2.$$

Since

$$\langle \mu D' \rangle^{-\kappa/2}\Lambda\langle \mu D' \rangle^\kappa = \langle \mu D' \rangle^{\kappa/2}\Lambda + \langle \mu D' \rangle^{-\kappa/2}[\Lambda, \langle \mu D' \rangle^\kappa]$$

and noting $\lambda_0 \in S(w\langle \mu\xi' \rangle, g)$ and hence $[\Lambda, \langle \mu D' \rangle^\kappa] \in \mu S(\langle \mu\xi' \rangle^\kappa, \bar{g})$ we have

$$\|\langle \mu D' \rangle^{-\kappa/2}[\Lambda, \langle \mu D' \rangle^\kappa]u\|^2 \leq C\mu\|\langle \mu D' \rangle^{\kappa/2} u\|^2.$$

Then we have

Lemma 4.7. *We have*

$$\|\langle \mu D' \rangle^{\kappa/2}\Lambda u\|^2 \geq \frac{d}{dx_0}\|\langle \mu D' \rangle^\kappa u\|^2 + (1 - C\mu)\|\langle \mu D' \rangle^{3\kappa/2} u\|^2. \tag{4.15}$$

Since $\mathsf{Im}\, m \in \mu S(\langle\mu\xi'\rangle^{\kappa}, \bar{g})$ it follows that

$$|2(\mathsf{Im}\, m\Lambda u, \Lambda u)| \le C\mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2. \tag{4.16}$$

Let us study $2\mathsf{Re}\,(\Lambda u, \mathsf{Im}\, Qu)$. Note that $|2\mathsf{Re}\,(\Lambda u, \mathsf{Im}\, Qu)| \le \mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + \mu^{-1}\|\langle\mu D'\rangle^{-\kappa/2}\mathsf{Im}\, Qu\|^2$. Recall that $\mathsf{Im}\, Q = [q_1 + \mathsf{Im}\, q_2 + r_1]^w$ with $r_1 \in \mu^2 S(\langle\mu\xi'\rangle^{3\kappa}, \bar{g})$. Remark that with $q_2' = \mathsf{Im}\, q_2$

$$\langle\mu\xi'\rangle^{-\kappa/2}\#(q_1 + q_2' + r_1) = \langle\mu\xi'\rangle^{-\kappa/2}(q_1 + q_2') + \mu^2 S(\langle\mu\xi'\rangle^{5\kappa/2}, \bar{g})$$

because $q_1 \in \mu S(\langle\mu\xi'\rangle^{1+\kappa}, g)$. Here we remark that

$$\langle\mu\xi'\rangle^{-\kappa/2}c_1\phi_1 = \big(c_1\langle\mu\xi'\rangle^{-\kappa}w^{-1/2}\big)\big(\langle\mu\xi'\rangle^{\kappa/2}w^{1/2}\phi_1\big)$$

where $c_1\langle\mu\xi'\rangle^{-\kappa}w^{-1/2} \in \mu S(1, g)$. Applying Lemma 4.4 to $\|[\langle\mu\xi'\rangle^{-\kappa/2}(q_1 + q_2')]^w u\|^2$ to get

Lemma 4.8. *We have*

$$\begin{aligned}|2\mathsf{Re}\,(\Lambda u, \mathsf{Im}\, Qu)| \;\le\; & \mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + C\mu\Big\{\sum_{j=2}^{r}\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) \\ & + \mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u, u)\Big\} + C\mu^3\|\langle\mu D'\rangle^{5\kappa/2}u\|^2.\end{aligned}$$

Let us consider $\mathsf{Re}\,(\mathsf{Re}\, Qu, \mathsf{Im}\,\lambda u)$. From Proposition 4.3 we have

$$\mathsf{Re}\, Q = \sum_{j=2}^{r}\phi_j^2 + w\phi_1^2 + \mu S(\langle\mu\xi'\rangle, \bar{g}) = q_0 + \mu S(\langle\mu\xi'\rangle, \bar{g})$$

and $\mathsf{Im}\,\lambda \in \mu S(\langle\mu\xi'\rangle^{\kappa}, \bar{g})$ hence it is clear that it suffices to study $\mathsf{Re}\,([q_0]^w u, \mathsf{Im}\,\lambda u)$ modulo $\mu^2\|\langle\mu D'\rangle^{7\kappa/2}u\|^2$ because $1 + \kappa = 7\kappa$. Since one can write

$$\mathsf{Im}\,\lambda = \lambda_1 + \mu^N S(\langle\mu\xi'\rangle^{1-3N\kappa+2n\kappa}, \bar{g}), \quad \lambda_1 \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$$

for any N we may assume $\mathsf{Im}\,\lambda = \lambda_1 \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$ modulo $\mu^2\|\langle\mu D'\rangle^{7\kappa/2}u\|^2$. Note that

$$\mathsf{Re}\,(\lambda_1\# q_0) = \lambda_1 q_0 + \mu^3 S(\langle\mu\xi'\rangle^{5\kappa}, g).$$

Applying Lemma 4.4 we get

Lemma 4.9. *We have*

$$|2\mathsf{Re}\,(\mathsf{Re}\,Qu, \mathsf{Im}\,\lambda u)| \le C\mu\Big\{\sum_{j=2}^{r} \mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u)$$
$$+\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa} w\phi_1^2]^w u, u)\Big\} + C\mu^2\|\langle\mu D'\rangle^{7\kappa/2}u\|^2.$$

We now estimate $\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u, u)$. Note that one can write

$$\begin{aligned}\mathsf{Re}\,Q &= q + q_2'' + r + \mu^N S(\langle\mu\xi'\rangle^{2-3N\kappa+2n\kappa}, \bar g), \quad r \in \mu^2 S(\langle\mu\xi'\rangle^{3\kappa}, g),\\ \mathsf{Re}\,\lambda &= -\lambda_0 - \lambda_2 + \mu^N S(\langle\mu\xi'\rangle^{1-3N\kappa+2n\kappa}, \bar g), \quad \lambda_2 \in \mu^2 S(\langle\mu\xi'\rangle^{-\kappa}, g)\end{aligned}$$

where $q_2'' = \mathsf{Re}\,q_2$. This proves that $|\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u, u)| = |\mathsf{Re}\,(\{\xi_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q\}^w u, u)|$ modulo $\mu^3\|\langle\mu D'\rangle^{2\kappa}u\|^2$. Note that

$$\{\xi_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q\} = \{\xi_0 - \lambda_0, q + q_2''\} - \{\lambda_2, q\} + \mu^3 S(\langle\mu\xi'\rangle^{5\kappa}, \bar g).$$

Since one can write $\{\lambda_2, q\} = \sum_{j=2}^{r} a_j\langle\mu\xi'\rangle^{\kappa}\phi_j + a_1\langle\mu\xi'\rangle^{\kappa}w\phi_1 + \mu^3 S(\langle\mu\xi'\rangle^{\kappa}, \bar g)$ with $a_j \in \mu^3 S(1, g)$ and $\{\xi_0 - \lambda_0, q_2''\} = \sum_{j=1}^{r} c_j\phi_j + \mu^2 S(\langle\mu\xi'\rangle, \bar g)$ with $c_j \in \mu^2 S(w^{-1}, g) \subset \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, \bar g)$ and $\{\xi_0 - \lambda_0, P_1^s\} \in \mu^2 S(\langle\mu\xi'\rangle, \bar g)$ we have, recalling $q = q_0 + P_1^s$ and $3\kappa = 1/2$

$$|\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u, u)| \le |\mathsf{Re}\,(\{\xi_0 - \lambda_0, q_0\}^w u, u)|$$
$$+C\mu^2\{\sum_{j=2}^{r}\|[\phi_j]^w u\|^2 + \|[\sqrt{w}\phi_1]^w u\|^2 + \|\langle\mu D'\rangle^{1/2}u\|^2\}$$
$$\le |\mathsf{Re}\,(\{\xi_0 - \lambda_0, q_0\}^w u, u)|$$
$$+C\mu^2\{\sum_{j=2}^{r}\mathsf{Re}\,([\phi_j^2]^w u, u) + \mathsf{Re}\,([w\phi_1^2]^w u, u) + \|\langle\mu D'\rangle^{1/2}u\|^2\}.$$

Thanks to Proposition 4.1 we conclude that (note that $1/2 < 7\kappa/2$)

Lemma 4.10. *We have*

$$|\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u, u)| \le C\mu\Big\{\sum_{j=2}^{r}\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u)$$
$$+\mathsf{Re}\,([w\langle\mu\xi'\rangle^{\kappa}\phi_1^2]^w u, u)\Big\} + C\mu^2\|\langle\mu D'\rangle^{7\kappa/2}u\|^2.$$

It remains to estimate $\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}\mathsf{Re}\,Qu,u)$. We first note that

$$|\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}[q_2'']^w u,u)|\leq C\mu\{\sum_{j=2}^{r}\|[\phi_j]^w u\|^2$$
$$+\|[\sqrt{w}\langle\mu\xi'\rangle^{\kappa/2}\phi_1]^w u\|^2+\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2\}$$

and hence it is enough to estimate $\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}q^w u,u)$ modulo $\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2$. Note that

$$\mathsf{Re}\,(\langle\mu\xi'\rangle^{\kappa}\#\phi_j^2)=\langle\mu\xi'\rangle^{\kappa}\phi_j^2+\mu^2 S(\langle\mu\xi'\rangle^{\kappa},g)$$
$$=\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j+\mu^2 S(\langle\mu\xi'\rangle^{\kappa},g),$$
$$\mathsf{Re}\,(\langle\mu\xi'\rangle^{\kappa}\#w\phi_1^2)=\langle\mu\xi'\rangle^{\kappa}w\phi_1^2+\mu^2 S(w\langle\mu\xi'\rangle^{\kappa},g)$$

and hence

$$\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}q_0^w u,u)\geq\sum_{j=2}^{r}\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2$$
$$+\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u,u)-C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.$$

With $\psi_i=\langle\mu\xi'\rangle^{\kappa/2}\phi_i$ note that $2\mathsf{Im}\,(\psi_{2i-1}^w u,\psi_{2i}^w u)=-i([\psi_{2i}^w,\psi_{2i-1}^w]u,u)$ and

$$-i[\psi_{2i}^w,\psi_{2i-1}^w]=\{\psi_{2i-1},\psi_{2i}\}^w+\mu^3 S(\langle\mu\xi'\rangle^{\kappa-1},g_0).$$

Recall that $\{\psi_{2i-1},\psi_{2i}\}=\{\phi_{2i-1},\phi_{2i}\}\langle\mu\xi'\rangle^{\kappa}+c_1\phi_{2i-1}+c_2\phi_{2i}$ where $c_j\in\mu S(\langle\mu\xi'\rangle^{\kappa},g_0)$, $i=1,\ldots,k$ and $\{\phi_{2i-1},\phi_{2i}\}\geq(\mu_i-\epsilon_1)\mu\langle\mu\xi'\rangle$. Thus we have

$$\mathsf{Re}\,([\{\phi_{2i-1},\phi_{2i}\}\langle\mu\xi'\rangle^{\kappa}]^w u,u)\leq\|\psi_{2i-1}^w u\|^2+\|\psi_{2i}^w u\|^2$$
$$+C\mu\{\|\psi_{2i}^w u\|^2+\|\psi_{2i-1}^w u\|^2+\|\langle\mu D'\rangle^{\kappa/2}u\|^2\}.$$

Let us denote $a=\{\phi_{2i-1},\phi_{2i}\}\langle\mu\xi'\rangle^{\kappa}-(\mu_i-\epsilon_1)\mu\langle\mu\xi'\rangle^{7\kappa}$. Since $0\leq a\in\mu S(\langle\mu\xi'\rangle^{7\kappa},g_0)$, noting that $\langle\xi'\rangle_\mu=\mu^{-1}\langle\mu\xi'\rangle$, $7\kappa-1=\kappa$ we see

$$\begin{aligned}0&\leq b=\mu^{-2}\langle\mu\xi'\rangle^{1-7\kappa}a\in S(\langle\xi'\rangle_\mu,g_0),\\ a&=\mu\langle\mu\xi'\rangle^{\kappa/2}\#b\#\mu\langle\mu\xi'\rangle^{\kappa/2}+\mu^3 S(\langle\mu\xi'\rangle^{-5\kappa},g_0).\end{aligned}$$

Thus from the sharp Gårding inequality (e.g. Theorem 18.6.7 in [4]) it follows that

$$\begin{aligned}\mathsf{Re}\,(a^w u,u) \geq \mathsf{Re}\,(b^w \mu\langle\mu D'\rangle^{\kappa/2}u, \mu\langle\mu D'\rangle^{\kappa/2}u)\\ -C\mu^3\|\langle\mu D'\rangle^{-5\kappa/2}u\|^2 \geq -C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2\end{aligned}$$

and this shows

$$\begin{aligned}\mathsf{Re}\,([\{\phi_{2i-1},\phi_{2i}\}\langle\mu\xi'\rangle^{\kappa}]^w u,u) \geq \mu(\mu_i-\epsilon_1)\|\langle\mu D'\rangle^{7\kappa/2}u\|^2\\ -C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.\end{aligned} \tag{4.17}$$

Summing over i and taking another ϵ_1 we obtain

$$\begin{aligned}\big(\sum_{i=1}^{k}\mu_i-\epsilon_1\big)\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2 \leq (1+C\mu)\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}q_0^w u,u)\\ +C\mu\|\langle\mu D'\rangle^{\kappa/2}u\|^2.\end{aligned}$$

We turn to $\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}[P_1^s]^w u,u)$. Note that $\mathsf{Re}\,\langle\mu\xi'\rangle^{\kappa}\#P_1^s = \langle\mu\xi'\rangle^{\kappa}P_1^s + \mu^3 S(\langle\mu\xi'\rangle^{-5\kappa},g_0)$ and $\langle\mu\xi'\rangle^{\kappa}P_1^s \geq -(\sum_{i=1}^{k}\mu_i-\epsilon_2)\mu\langle\mu\xi'\rangle^{7\kappa}$. Repeating the same arguments deriving (4.17) we have

$$\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}[P_1^s]^w u,u) \geq -(\sum_{i=1}^{k}\mu_i-\epsilon_2)\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2 - C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.$$

Then we have

$$\begin{aligned}\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}q^w u,u) \geq (1-\delta)\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}q_0^w u,u) + \mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}[P_1^s]^w u,u)\\ +\delta\mathsf{Re}\,(\langle\mu D'\rangle q_0^w u,u) \geq (1-\delta)(1+C\mu)^{-1}(\sum_{i=1}^{k}\mu_i-\epsilon_1)\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2\\ -(\sum_{i=1}^{k}\mu_i-\epsilon_2)\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2 + \delta\Big\{\sum_{j=2}^{r}\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u,u)\\ +\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u,u)\Big\} - C\mu\|\langle\mu D'\rangle^{\kappa/2}u\|^2.\end{aligned}$$

We take ϵ_1, δ so that $(1-\delta)(1+C\mu)^{-1}(\sum_{i=1}^{k}\mu_i-\epsilon_1) > (\sum_{i=1}^{k}\mu_i-\epsilon_2)$ for small $0<\mu\leq\mu_0$ and hence

Lemma 4.11. *We have*

$$\begin{aligned}\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}\mathsf{Re}\,Qu,u) \;\ge\;& c_1\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2\\&+c_2\Big\{\sum_{j=2}^{r}\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u,u)+\mathsf{Re}\,([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u,u)\Big\}\\&-C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2\end{aligned}$$

with some positive $c_i > 0$ for $0 < \mu \le \mu_0$.

From Lemmas 4.7, 4.8, 4.9, 4.10, 4.11 and (4.16) we have

Proposition 4.4. *There exist $\mu_0 > 0$, $C > 0$, $c > 0$ such that we have*

$$C\|\langle\mu D'\rangle^{-\kappa/2}Pu\|^2 \ge \frac{d}{dx_0}\{\|\Lambda u\|^2+(\mathsf{Re}Qu,u)+\|\langle\mu D'\rangle^{\kappa}u\|^2\}$$

$$+c\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2+c\{\sum_{j=2}^{r}\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)+\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)\}$$

$$+c\|\langle\mu D'\rangle^{3\kappa/2}u\|^2+c\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2$$

for $0 < \mu < \mu_0$.

Taking into account that $\phi_j^2\langle\mu\xi'\rangle^{\kappa} = \phi_j\langle\mu\xi'\rangle^{\kappa/2}\#\phi_j\langle\mu\xi'\rangle^{\kappa/2}+\mu^2 S(\langle\mu\xi'\rangle^{\kappa}, g_0)$, $\phi_1^2 w\langle\mu\xi'\rangle^{\kappa} = \sqrt{w}\phi_1\langle\mu\xi'\rangle^{\kappa/2}\#\sqrt{w}\phi_1\langle\mu\xi'\rangle^{\kappa/2}+\mu^2 S(\langle\mu\xi'\rangle^{3\kappa}, g)$ it is easy to see

$$\begin{aligned}\sum_{j=2}^{r}\mathsf{Re}\,([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u,u)+\mathsf{Re}\,([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u)&\\ \ge -C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.&\end{aligned}$$

The same argument shows that

$$(\mathsf{Re}\,Qu,u) \ge -C\mu^2\|\langle\mu D'\rangle^{\kappa}u\|^2.$$

Integrating the inequality in Proposition 4.4 from $-\infty$ to t with respect to x_0 we get

$$C\int_{-\infty}^{t}\|\langle\mu D'\rangle^{-\kappa/2}Pu\|^2dx_0 \ge \{\|\Lambda u(t,\cdot)\|^2+c\|\langle\mu D'\rangle^{\kappa}u(t,\cdot)\|^2\}$$

$$+c\int_{-\infty}^{t}\{\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2+\|\langle\mu D'\rangle^{3\kappa/2}u\|^2+\mu\|\langle\mu D'\rangle^{7\kappa/2}u\|^2\}dx_0$$

for $0 < \mu < \mu_0$. Returning to the original $P = -M\Lambda + B\Lambda + Q$ and replacing u by $e^{-x_0\langle\mu D'\rangle^\kappa}u$ we obtain

Proposition 4.5. *We have*

$$
\begin{aligned}
&C\int_{-\infty}^{t} \|\langle\mu D'\rangle^{-\kappa/2} e^{-x_0\langle\mu D'\rangle^\kappa} Pu\|^2 dx_0 \\
&\geq \left\{\|e^{-t\langle\mu D'\rangle^\kappa}\Lambda u(t,\cdot)\|^2 + c\|\langle\mu D'\rangle^\kappa e^{-t\langle\mu D'\rangle^\kappa} u(t,\cdot)\|^2\right\} \\
&+c\int_{-\infty}^{t} \left\{\|\langle\mu D'\rangle^{\kappa/2} e^{-x_0\langle\mu D'\rangle^\kappa}\Lambda u\|^2 + \|\langle\mu D'\rangle^{3\kappa/2} e^{-x_0\langle\mu D'\rangle^\kappa} u\|^2\right. \\
&\left. +\mu\|\langle\mu D'\rangle^{7\kappa/2} e^{-x_0\langle\mu D'\rangle^\kappa} u\|^2\right\} dx_0
\end{aligned}
$$

for $0 < \mu < \mu_0$.

Since we have the same a priori estimate for P^*, applying the standard duality arguments we can prove Theorem 1.2.

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