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PSEUDODIFFERENTIAL OPERATORS AND WEIGHTED NORMED SYMBOL SPACES

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Abstract. This work is the continuation of two earlier ones by the author and stimulated by many more recent contributions. We develop a very general calculus of pseudodifferential operators with microlocally defined normed symbol spaces. The goal was to attain the natural degree of generality in the case when the underlying metric on the cotangent space is constant. We also give sufficient conditions for our operators to belong to Schatten–von Neumann classes.

1. Introduction. This paper is devoted to pseudodifferential operators with symbols of limited regularity. The author [28] introduced the space of symbols $a(x)$ on the phase space $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$ with the property that

$$|\widehat{\chi_\gamma a}(x^*)| \le F(x^*),\ \forall \gamma \in \Gamma \tag{1.1}$$

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for some L^1 function F on E^*. Here the hat indicates that we take the Fourier transform, $\Gamma \subset E$ is a lattice and $\chi_\gamma(x) = \chi_0(x-\gamma)$ form a partition of unity, $1 = \sum_{\gamma\in\Gamma} \chi_\gamma$, $\chi_0 \in \mathcal{S}(E)$. A. Boulkhemair [4] noticed that this space is identical to a space that he had defined differently in [3].

It was shown among other things that this space of symbols is an algebra for the ordinary multiplication and that this fact persists after quantization, namely the corresponding pseudodifferential operators (say under Weyl quantization) form a non-commutative algebra: If a_1, a_2 belong to the class above with corresponding L^1 functions F_1 and F_2 then $a_1^w \circ a_2^w = a_3^w$ where a_3 belongs to the same class and as a correponding function we may take $F_3 = C_N F_1 * F_2 * \langle\cdot\rangle^{-N}$ for any $N > 2n$. Here $*$ indicates convolution and $a^w : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is the Weyl quantization of the symbol a, given by

$$a^w u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\theta} a(\frac{x+y}{2}, \theta) u(y) dy d\theta. \tag{1.2}$$

The definition (1.1) is independent of the choice of lattice and the corresponding function χ_0. When passing to a different choice, we may have to change the function F to $m(x^*) = F * \langle\cdot\rangle^{-N_0}$ for any fixed $N_0 > 2n$. We then gain the fact that the weight m is an order function in the sense that

$$m(x^*) \le C_0 \langle x^* - y^* \rangle^{N_0} m(y^*),\ \ x^*, y^* \in E^*. \tag{1.3}$$

(See [11] where this notion is used for developing a fairly simple calculus of semi-classical pseudodifferential operators, basically a special case of Hörmander's Weyl calculus [26].)

The space of functions in (1.1) is a special case of the modulation spaces of H.G. Feichtinger (see [12, 14]), and the relations between these spaces and pseudodifferential operators have been developed by many authors; K. Gröchenig [18, 19], Gröchenig, T. Strohmer [22], K. Tachigawa [32], J. Toft [33], A. Holst, J. Toft, P. Wahlberg [25]. Here we could mention that Boulkhemair [5] proved L^2-continuity for Fourier integral operators with symbols and phases in the original spaces of the type (1.1), that T. Strohmer [31] has applied the theory to problems in mobile communications and that Y. Morimoto and N. Lerner [27] have used the original space to prove a version of the Fefferman-Phong inequality for pseudodifferential operators with symbols of low regularity. This result was recently improved by Boulkhemair [8].

Closely related works on pseudodifferential - and Fourier - integral operators with symbols of limited regularity include the works of Boulkhemair [6, 7], and many others also contain a study of when such operators or related Gabor localization operators belong to to Schatten-von Neumann classes: E. Cordero, Gröchenig [9, 10], C. Heil, J. Ramanathan, P. Topiwala [24], Heil [23], J. Toft [34], and M.W. Wong [37].

The present work has been stimulated by these developments and the prospect of using "modulation type weights" to get more flexibility in the calculus of pseudodifferential operators with limited regularity. In the back of our head there were also some very stimulating discussions with J.M. Bony and N. Lerner from the time of the writing of [28, 29] and at that time Bony explained to the author a nice very general point of view of A. Unterberger [36] for a direct microlocal analysis of very general classes of operators. Bony used it in his work [1] and showed how his approach could be applied to recover and generalize the space in [28]. However, the aim of the work [1] was to develop a very general theory of Fourier integral operators related to symplectic metrics of Hörmander's Weyl calculus of pseudodifferential operators, and the relation with [28] was explained very briefly. See [2] for even more general classes of Fourier integral operators.

In the present paper we make a direct generalization of the spaces of [28]. Instead of using order functions only depending on x^* we can now allow arbitrary order functions $m(x,x^*)$. See Definition 2.1 below. In Proposition 2.4 we show that this definition gives back the spaces above when the weight $m(x^*)$ is an order function of x^* only.

In Section 3 we consider the quantization of our symbols and show how to define an associated effective kernel on $E\times E$, $E=T^*\mathbf{R}^n$, which is $\mathcal{O}(1)m(\gamma(x,y))$ where $\gamma(x,y)=\left(\frac{x+y}{2},J^{-1}(y-x)\right)$ and $J:E^*\to E$ is the natural Hamilton map induced by the symplectic structure. We show that if the effective kernel is the kernel of a bounded operator : $L^2(E)\to L^2(E)$ then our pseudodifferential operator is bounded in $L^2(\mathbf{R}^n)$. In particular if $m=m(x^*)$ only depends on x^*, we recover the L^2-boundedness when m is integrable. This result was obtained previously by Bony [1], but our approach is rather different.

In Section 4 we study the composition of pseudodifferential operators in our classes. If a_j are symbols associated to the order functions m_j, $j=1,2$, then the Weyl composition is a well defined symbol associated to the order function $m_3(z,z^*)$ given in (4.11), provided that the integral there converges for at least one value of (z,z^*) (and then automatically for all other values by Proposition 4.1). This statement is equivalent to the corresponding natural one for the effective kernels, namely the composition is well defined if the composition of the majorant kernels $m_1\left(\frac{x+y}{2},J^{-1}(y-x)\right)$ and $m_2\left(\frac{x+y}{2},J^{-1}(y-x)\right)$ is well-defined, see (4.16), (4.17).

In Section 5 we simplify the results further (for those readers who are familiar with Bargmann transforms from the FBI – complex Fourier integral operator point of view).

In Section 6 we use the same point of view to give a simple sufficient condition on the order function m and the index $p \in [1, \infty]$, for the quantization a^w to belong to the Schatten–von Neumann class C_p for every symbol a belonging to the symbol class with weight m. See [34, 35, 25, 20, 21] for related results and ideas.

In Section 7 we finally generalize our results by replacing the underlying space ℓ^∞ on certain lattices by more general translation invariant Banach spaces. We believe that this generalization allows to include modulation spaces, but we have contented ourselves by establishing results allowing to go from properties on the level of lattices to the level of pseudodifferential operators. The results could undoubtedly be even further generalized. In this section and the preceding one, we have been inspired by the use of lattices and amalgan spaces in time frequency analysis, in particular by the work of Gröchenig and Strohmer [22] that uses previous results by Fournier–Stewart [15] and Feichtinger [13].

We have chosen to work with the Weyl quantization, but it is clear that the results carry over with the obvious modifications to other quantizations like the Kohn-Nirenberg one, actually for the general symbol-spaces under consideration the results could also have been formulatated directly for classes of integral operators.

Similar ideas and results have been obtained in many other works, out of which some are cited above and later in the text.

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2. Symbol spaces. Let E be a d-dimensional real vector space. We say that $m : E \to]0, \infty[$ is an order function on E if there exist constants $C_0 > 0$, $N_0 \geq 1$, such that

$$m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu),\ \forall \rho, \mu \in E. \tag{2.1}$$

Here $\langle \rho - \mu \rangle = (1 + |\rho - \mu|^2)^{1/2}$ and $|\,|$ is a norm on E.

Let E be as above, let E^* be the dual space and let Γ be a lattice in $E \times E^*$, so that $\Gamma = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \cdots + \mathbf{Z}e_{2d}$ where $e_1, \ldots, e_{2d}$ is a basis in $E \times E^*$.

Let $\chi \in \mathcal{S}(E \times E^*)$ have the property that

$$\sum_{\gamma \in \Gamma} \tau_\gamma \chi = 1, \quad \tau_\gamma \chi(\rho) = \chi(\rho - \gamma). \tag{2.2}$$

Let m be an order function on $E \times E^*$, $a \in \mathcal{S}'(E)$.

Definition 2.1. *We say that $a \in \widetilde{S}(m)$ if there is a constant $C > 0$ such that*

$$\|\chi_\gamma^w a\| \leq C m(\gamma),\ \gamma \in \Gamma, \tag{2.3}$$

where $\chi_\gamma = \tau_\gamma \chi$ and χ_γ^w denotes the Weyl quantization of χ_γ. The norm will always be the the one in L^2 if nothing else is indicated.

To define the L^2-norm we need to choose a Lebesgue measure on E, but clearly that can only affect the choice of the constant in (2.3).

Proposition 2.2. *$\widetilde{S}(m)$ is a Banach space with $\|a\|_{\widetilde{S}(m)}$ equal to the smallest possible constant in (2.3). Changing Γ, χ and replacing the L^2 norm by the L^p-norm for any $p \in [1, \infty]$ in the above definition, gives rise to the same space with an equivalent norm.*

P r o o f. The Banach space property will follow from the other arguments so we do not treat it explicitly. Let m, Γ, a be as in Definition 2.1.

Let $\widetilde{\Gamma}$ be another lattice and let $\widetilde{\chi}$ be another function with the same properties as χ. We have to show that

$$\|\widetilde{\chi}_{\widetilde{\gamma}}^w a\|_{L^p} \leq \widetilde{C} m(\widetilde{\gamma}), \quad \widetilde{\gamma} \in \widetilde{\Gamma}$$

Lemma 2.3. *$\exists \psi \in \mathcal{S}(E \times E^*)$ such that $\sum_{\gamma \in \Gamma} \psi_\gamma^w \chi_\gamma^w = 1$, where $\psi_\gamma = \tau_\gamma \psi$.*

P r o o f. Let $\widetilde{\chi} \in \mathcal{S}(E \times E^*)$ be equal to 1 near $(0, 0)$, and put $\widetilde{\chi}^\epsilon(x, \xi) = \widetilde{\chi}(\epsilon(x, \xi))$. Then $\sum_{\gamma \in \Gamma}(1 - \widetilde{\chi}_\gamma^\epsilon) \# \chi_\gamma \to 0$ in $S^0(E \times E^*)$, when $\epsilon \to 0$, so for $\epsilon > 0$ small enough,

$$\sum_{\gamma \in \Gamma} (\widetilde{\chi}_\gamma^\epsilon)^w \chi_\gamma^w = 1 - \sum_{\gamma \in \Gamma} (1 - \widetilde{\chi}_\gamma^\epsilon)^w \chi_\gamma^w$$

has a bounded inverse in $\mathcal{L}(L^2, L^2)$. Here S^0 is the space of all $a \in C^\infty(E \times E^*)$ that are bounded with all their derivatives. By a version of the Beals lemma (see for instance [11]), we then know that the inverse is of the form Ψ^w where $\Psi \in S^0$. Also $\tau_\gamma \Psi = \Psi$, $\gamma \in \Gamma$. Put $\psi_\gamma^w = \Psi^w \circ (\widetilde{\chi}_\gamma^\epsilon)^w$ for ϵ small enough and fixed, so that

$\psi_\gamma = \tau_\gamma \psi_0$, $\psi_0 \in \mathcal{S}$ (using for instance the simple pseudodifferential calculus in [11]). Then $\sum_\gamma \psi_\gamma^w \chi_\gamma^w = 1$. □

Now, write

$$\widetilde{\chi}_{\widetilde{\gamma}}^w a = \sum_{\gamma\in\Gamma} \widetilde{\chi}_{\widetilde{\gamma}}^w \psi_\gamma^w \chi_\gamma^w a.$$

Here (using for instance [11])

$$\|\widetilde{\chi}_{\widetilde{\gamma}} \psi_\gamma^w\|_{\mathcal{L}(L^2,L^p)} \le C_{p,N} \langle \widetilde{\gamma}-\gamma\rangle^{-N},\ 1\le p\le\infty,\ N\ge 0.$$

Hence, if N is large enough,

$$\begin{aligned}
\|\widetilde{\chi}_{\widetilde{\gamma}}^w a\|_{L^p} &\le C_{p,N}\sum_{\gamma\in\Gamma}\langle\widetilde{\gamma}-\gamma\rangle^{-N}\|\chi_\gamma^w a\|_{L^2} \\
&\le \widetilde{C}_{p,N,a}\sum_{\gamma\in\Gamma}\langle\widetilde{\gamma}-\gamma\rangle^{-N} m(\gamma) \\
&\le \widehat{C}_{p,N,a,m}\Big(\sum_{\gamma\in\Gamma}\langle\widetilde{\gamma}-\gamma\rangle^{-N+N_0}\Big) m(\widetilde{\gamma}) \\
&\le \check{C} m(\widetilde{\gamma}).
\end{aligned} \tag{2.4}$$

Conversely, if $\|\widetilde{\chi}_{\widetilde{\gamma}}^w a\|_{L^p} \le \mathrm{Const}\, m(\widetilde{\gamma})$, $\widetilde{\gamma}\in\widetilde{\Gamma}$, we see that by the same arguments that $\|\chi_\gamma^w a\|_{L^2} \le \mathcal{O}(1) m(\gamma)$, $\gamma\in\Gamma$. □

Next, we check that this is essentially a generalization of a space introduced by Sjöstrand [28] and independently and in a different way by Boukhemair [3]. It is a special case of more general modulation spaces (see [12, 14]). That follows from the next result if we take an order function $m(x,x^*)$ independent of x.

Proposition 2.4. *Let $m = m(x,x^*)$ be an order function on $E\times E^*$ and let $\chi\in\mathcal{S}(E)$, $\sum_{j\in J}\chi_j = 1$, where $J\subset E$ is a lattice and $\chi_j = \tau_j\chi$. Then*

$$\widetilde{S}(m) = \{a\in\mathcal{S}'(E);\ \exists C>0,\ |\widehat{\chi_j u}(x^*)| \le C m(j,x^*)\}. \tag{2.5}$$

P r o o f. Let $K\subset E^*$ be a lattice and choose $\chi^*\in\mathcal{S}(E^*)$, such that $\sum_{k\in K}\chi_k^* = 1$, where $\chi_k^* = \tau_k\chi^*$. If a belongs to the set in the right hand side of (2.5), then by Parseval's relation,

$$\|\chi_k^*(D)(\chi_j(x)u(x))\|_{L^2} \le \widetilde{C} m(j,k). \tag{2.6}$$

Now $\chi_k^*(D) \circ \chi_j(x) = \chi_{j,k}^w$, where $\chi_{j,k} = \tau_{j,k}\chi_{0,0}$, $\chi_{0,0} \in \mathcal{S}$, $(j,k) \in J \times J^*$, so $a \in \widetilde{S}(m)$. Conversely, if $a \in \widetilde{S}(m)$, we get (2.6). According to Proposition 2.2, we can replace the L^2 norm by any L^p norm, and the proof shows that we can equally well replace the L^2 norm that of $\mathcal{F}L^p$. Taking $\mathcal{F}L^\infty$, we get

$$\|\chi_k^*(x^*)\widehat{\chi_j u}(x^*)\|_{L^\infty} \le \widehat{C}m(j,k),$$

and since m is an order function, we deduce that a belongs to the set in the right hand side of (2.5). □

3. Effective kernels and L^2-boundedness.

A closely related notion for effective kernels in terms of short time Fourier transforms has been introduced by Gröchenig and Heil [20].

We now take $E = \mathbf{R}^{2n} \simeq T^*\mathbf{R}^n$. If $a, b \in \mathcal{S}(E)$, we let

$$a\#b = (e^{\frac{i}{2}\sigma(D_x,D_y)}a(x)b(y))_{y=x} \tag{3.1}$$

denote the Weyl composition so that $(a\#b)^w = a^w \circ b^w$. Here $\sigma(D_{x,\xi}, D_{y,\eta}) = D_\xi \cdot D_y - D_x \cdot D_\eta$ where we write (x,ξ), (y,η) instead of x, y whenever convenient.

We know that the Weyl composition is still well-defined when a, b belong to various symbol spaces like

$$S(m) = \{a \in C^\infty(E);\ |D_x^\alpha a(x)| \le C_\alpha m(x)\}, \tag{3.2}$$

when m is an order function on E. (See Example 4.4 below for a straight forward generalization.)

Let $\ell(x) = x \cdot x^*$ be a linear form on E and let a be a symbol. Then,

$$\begin{aligned} e^{i\ell}\#a &= e^{\frac{i}{2}\sigma(D_x,D_y)}(e^{i\ell(x)}a(y))_{y=x} \\ &= (e^{i\ell(x)}e^{\frac{i}{2}\sigma(\ell'(x),D_y)}a(y))_{y=x} \\ &= e^{i\ell(x)}(e^{\frac{1}{2}H_\ell}a) \end{aligned} \tag{3.3}$$

where $H_\ell = \ell'_\xi \cdot \dfrac{\partial}{\partial x} - \ell'_x \cdot \dfrac{\partial}{\partial \xi}$ (with "$x = (x,\xi)$") is the Hamilton field of ℓ. Similarly,

$$a\#e^{i\ell} = e^{i\ell(x)}(e^{-\frac{1}{2}H_\ell}a). \tag{3.4}$$

From (3.3), (3.4), we get

$$e^{i\ell}\#a\#e^{-i\ell} = e^{H_\ell}a, \tag{3.5}$$

where we notice that $(e^{H_\ell}a)(x) = a(x + H_\ell)$, and

$$e^{i\frac{m}{2}}\#a\#e^{i\frac{m}{2}} = e^{im}a, \tag{3.6}$$

if m is a second linear form on E.

If $a \in \mathcal{S}(E)$ is fixed, we may consider that a is concentrated near $(0,0) \in E \times E^*$. Then we say that $e^{-H_\ell}e^{im}a$ is concentrated near $(H_\ell, m) \in E \times E^*$. Conversely, if b is concentrated near a point $(x_0, x_0^*) \in E \times E^*$, we let $y_0^* \in E^*$ be the unique vector with $x_0 = H_{y_0^*}$ and write

$$b = e^{-H_{y_0^*}}e^{ix_0^*}a = e^{-iy_0^*}\#e^{i\frac{x_0^*}{2}}\#a\#e^{i\frac{x_0^*}{2}}\#e^{iy_0^*}, \tag{3.7}$$

where a is concentrated near $(0,0) \in E \times E^*$.

To make this more precise, let (as in [30])

$$Tu = C\int e^{i\phi(x,y)}u(y)dy, \quad C > 0, \tag{3.8}$$

be a generalized Bargmann transform where $\phi(x,y)$ is a quadratic form on $\mathbf{C}^n \times \mathbf{C}^n$ with $\det \phi''_{xy} \neq 0$, $\operatorname{Im}\phi''_{yy} > 0$, and with $C > 0$ suitably chosen, so that T is unitary $L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n) = \operatorname{Hol}(\mathbf{C}^n) \cap L^2(e^{-2\Phi(x)}L(dx))$, where $L(dx)$ denotes the Lebesgue measure on $\mathbf{C}^n$ and Φ is the strictly plurisubharmonic quadratic form given by

$$\Phi(x) = \sup_{y\in\mathbf{R}^n} -\operatorname{Im}\phi(x,y). \tag{3.9}$$

We know ([30]) that if $\Lambda_\Phi = \left\{\left(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}\right);\, x \in \mathbf{C}^n\right\}$, then

$$\Lambda_\Phi = \kappa_T(E), \tag{3.10}$$

where

$$\kappa_T : \mathbf{C}^{2n} \simeq E^{\mathbf{C}} \ni (y, -\phi'_y(x,y)) \to (x, \phi'_x(x,y)) \in \mathbf{C}^{2n} \tag{3.11}$$

is the linear canonical transformation associated to T. Here $\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} x} + \frac{1}{i}\frac{\partial}{\partial \operatorname{Im} x}\right)$, following standard conventions in complex analysis.

If $a \in S^0(E)$ we have an exact version of Egorov's theorem, saying that

$$Ta^wT^{-1} = \widetilde{a}^w, \tag{3.12}$$

where $\widetilde{a} \in S^0(\Lambda_\Phi)$ is given by $\widetilde{a} \circ \kappa_T = a$. In [30] it is dicussed how to define and estimate the Weyl quantization of symbols on the Bargmann transform side, by means of almost holomorphic extensions and contour deformations. We retain from the proof of Proposition 1.2 in that paper that

$$\widetilde{a}^w u(x) = \int e^{\Phi(x)} K^{\mathrm{eff}}_{\widetilde{a}}(x,y) u(y) e^{-\Phi(y)} L(dy), \ u \in H_\Phi(\mathbf{C}^n), \tag{3.13}$$

where the kernel is non-unique but can be chosen to satisfy

$$K^{\mathrm{eff}}_{\widetilde{a}}(x,y) = \mathcal{O}_N(1)\langle x-y\rangle^{-N}, \tag{3.14}$$

for every $N \geq 0$. (This immediately implies the Calderón-Vaillancourt theorem for the class $\mathrm{Op}\,(S^0(E))$.)

If $a \in \mathcal{S}(E)$, then for every $N \in \mathbf{N}$

$$|K^{\mathrm{eff}}_{Ta^wT^{-1}}(x,y)| \leq C_N(a)\langle x\rangle^{-N}\langle y\rangle^{-N}, \ x,y \in \mathbf{C}^n, \tag{3.15}$$

where $C_N(a)$ are seminorms in $\mathcal{S}$.

Identifying $x \in \mathbf{C}^n$ with $\kappa_T^{-1}\left(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}\right) \in E$, we can view $K^{\mathrm{eff}}_{Ta^wT^{-1}}$ as a function $K^{\mathrm{eff}}_{a^w}(x,y)$ on $E \times E$ and (3.15) becomes

$$|K^{\mathrm{eff}}_{a^w}(x,y)| \leq C_N(a)\langle x\rangle^{-N}\langle y\rangle^{-N}, \ x,y \in E. \tag{3.16}$$

Now, let b in (3.7) be concentrated near $(x_0, x_0^*) = (Jy_0^*, x_0^*) \in E \times E^*$ with $a \in \mathcal{S}(E)$, where we let $J : E^* \to E$ be the map $y^* \mapsto H_{y^*}$ (and we shall prefer to write Jy^* when we do not think of this quantity as a constant coefficient vector field). Then by (3.5)–(3.7), we have

$$b = e^{-iy_0^*} \# e^{ix_0^*/2} \# a \# e^{ix_0^*/2} \# e^{iy_0^*}, \tag{3.17}$$

$$b^w = e^{-i(y_0^*)^w} \circ e^{i(x_0^*)^w/2} \circ a^w \circ e^{i(x_0^*)^w/2} \circ e^{i(y_0^*)^w}. \tag{3.18}$$

Now it is wellknown that if $z^* \in E^*$ then $e^{-i(z^*)^w} = (e^{-iz^*})^w$ is a unitary operator that can be viewed as a quantization of the phase space translation $E \ni x \mapsto x + H_{z^*} \in E$. On the Bargmann transform side these quantizations can be explicitly represented as magnetic translations, i.e. translations made unitary by multiplication by certain weights. In fact, let $\ell(x,\xi) = x_0^* \cdot x + x_0 \cdot \xi$ be a linear form on $\mathbf{C}^{2n}$ which is real on Λ_Φ, so that

$$x_0^* \cdot x + x_0 \cdot \frac{2}{i}\frac{\partial\Phi}{\partial x}(x) \in \mathbf{R}, \ \forall x \in \mathbf{C}^n. \tag{3.19}$$

By essentially the same calculation as in the real setting, we see that

$$(e^{i\ell})^w u(x) = e^{ix_0^*\cdot(x+\frac{1}{2}x_0)}u(x+x_0),\ u\in H_\Phi,$$

and here we recall from the unitary and metaplectic equivalence with $L^2(\mathbf{R}^n)$ (via T) that $(e^{i\ell})^w : H_\Phi \to H_\Phi$ is unitary, or equivalently that

$$-\Phi(x)+\Phi(x+x_0)+\mathrm{Re}\left(ix_0^*\cdot(x+\frac{1}{2}x_0)\right)=0,\ \forall x\in \mathbf{C}^n. \tag{3.20}$$

(A simple calculation shows more directly the equivalence of (3.19) and (3.20).) Notice also that if we identify u with a function $\widetilde{u}(\rho)$ on Λ_Φ via the natural projection $(x,\xi)\mapsto x$, then $u(x+x_0)$ is identified with $\widetilde{u}(\rho+H_\ell)$, where the Hamilton field H_ℓ is viewed as a real constant vector field on Λ_Φ.

It follows that b^w has a kernel satisfying

$$|K_{b^w}^{\mathrm{eff}}(x,y)| = \left|K_{a^w}^{\mathrm{eff}}\left(x+\frac{1}{2}Jx_0^*-x_0, y-\frac{1}{2}Jx_0^*-x_0\right)\right|$$

and from (3.16) we get

$$|K_{b^w}^{\mathrm{eff}}(x,y)| \le C_N(a)\left\langle x-\left(x_0-\frac{1}{2}Jx_0^*\right)\right\rangle^{-N}\left\langle y-\left(x_0+\frac{1}{2}Jx_0^*\right)\right\rangle^{-N}, \tag{3.21}$$

so the kernel of b^w is concentrated near $\left(x_0-\frac{1}{2}Jx_0^*, x_0+\frac{1}{2}Jx_0^*\right)$.

Now, let m be an order function on $E\times E^*$ and let $a\in\widetilde{S}(m)$. Choose a lattice $\Gamma\subset E\times E^*$ and a partition of unity as in (2.2) as well as a function $\psi\in\mathcal{S}(E\times E^*)$ as in Lemma 2.3. Write

$$a=\sum_{\gamma\in\Gamma}a_\gamma,\ a_\gamma=\psi_\gamma^w\widetilde{a}_\gamma,\ \widetilde{a}_\gamma=\chi_\gamma^w a, \tag{3.22}$$

where $\|\widetilde{a}_\gamma\|\le Cm(\gamma)$. Then, using that ψ_0^w is continuous: $L^2(E)\to\mathcal{S}(E)$, we see that a_γ is concentrated near γ in the above sense and more precisely,

$$\begin{aligned}&|K_{a^w}^{\mathrm{eff}}(x,y)|\\ &\le C_N m(\gamma)\left\langle x-\left(\gamma_x-\frac{1}{2}J\gamma_{x^*}\right)\right\rangle^{-N}\left\langle y-\left(\gamma_x+\frac{1}{2}J\gamma_{x^*}\right)\right\rangle^{-N},\quad x,y\in E,\end{aligned} \tag{3.23}$$

where we write $\gamma=(\gamma_x,\gamma_{x^*})\in E\times E^*$.

Let $q(x,y) = \left(\frac{x+y}{2}, J^{-1}(y-x)\right) = (q_x(x,y), q_{x^*}(x,y))$, so that

$$q^{-1}(\gamma) = \left(\gamma_x - \frac{1}{2}J\gamma_x, \gamma_x + \frac{1}{2}J\gamma_x\right),$$

and hence

$$\langle q(x,y)-\gamma\rangle \leq \mathcal{O}(1)\left\langle x-\left(\gamma_x - \frac{1}{2}J\gamma_{x^*}\right)\right\rangle \left\langle y-\left(\gamma_x + \frac{1}{2}J\gamma_{x^*}\right)\right\rangle,$$

so (3.23) implies

$$\text{(3.24)} \qquad \begin{aligned} |K^{\mathrm{eff}}_{a^w_\gamma}(x,y)| &\leq C_N(a)m(\gamma)\langle q(x,y)-\gamma\rangle^{-N} \\ &\leq \widetilde{C}_N(a)m(q(x,y))\langle q(x,y)-\gamma\rangle^{N_0-N}, \end{aligned}$$

where we used that m is an order function in the last inequality. Choose N with $N_0 - N < -4n$, sum over γ and use (3.22) to get

$$\text{(3.25)} \quad |K^{\mathrm{eff}}_{a^w}(x,y)| \leq C(a)m(q(x,y)) = C(a)m\left(\frac{x+y}{2}, J^{-1}(y-x)\right), \ x,y\in E.$$

We get

Theorem 3.1. *Let $a \in \widetilde{S}(m)$, where m is an order function on $E\times E^*$, $E = T^*\mathbf{R}^n$. Then a^w has an effective kernel (rigorously defined after applying a Bargmann transform as above) satisfying* (3.25)*, where $C(a)$ is a $\widetilde{S}(m)$ norm of a. In particular, if $M(x,y) = m\left(\frac{x+y}{2}, J^{-1}(y-x)\right)$ is the kernel of an $L^2(E)$-bounded operator, then a^w is bounded: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$.*

As mentioned in the introduction, the statement on L^2-boundedness here is due to Bony [1], who obtained it in a rather different way. A calculation, similar to the one leading to (3.25), has been given by Gröchenig [18].

Corollary 3.2. *If M is the kernel of a Shur class operator i.e. if*

$$\sup_x \int m(\frac{x+y}{2}, J^{-1}(y-x))dy,\ \sup_y \int m(\frac{x+y}{2}, J^{-1}(y-x))dx \ <\infty,$$

then a^w is bounded: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$.

Corollary 3.3. *Assume $m(x,x^*) = m(x^*)$ is independent of x, for $(x,x^*)\in E\times E^*$ and $m(x^*)\in L^1(E^*)$, then a^w is bounded: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$.*

4. Composition. Let $a, b \in \mathcal{S}(E)$, $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$, (x_0, x_0^*), $(y_0, y_0^*) \in E \times E^*$ and consider the Weyl composition of the two symbols $e^{x\cdot x_0^*}a(x-x_0)$, $e^{x\cdot y_0^*}b(x-y_0)$, concentrated near (x_0, x_0^*) and (y_0, y_0^*) respectively:

$$e^{\frac{i}{2}\sigma(D_x,D_y)}(e^{ix\cdot x_0^*}a(x-x_0)e^{iy\cdot y_0^*}b(y-y_0))(z,z). \tag{4.1}$$

We work in canonical coordinates $x \simeq (x, \xi)$ and identify E and E^*. Then

$$\sigma(x^*, y^*) = Jx^* \cdot y^*,\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ {}^tJ = -J,\ J^2 = -1,$$

and $e^{\frac{i}{2}\sigma(D_x,D_y)}$ is convolution with k, given by

$$k(x,y) = \frac{1}{(2\pi)^{2n}} \iint e^{i(x\cdot x^* + y\cdot y^* + \frac{1}{2}Jx^*\cdot y^*)} dx^* dy^*.$$

The phase $\Phi = x\cdot x^* + y\cdot y^* + \frac{1}{2}Jx^*\cdot y^*$ has a unique nondegenerate critical point $(x^*, y^*) = (2Jy, -2Jx)$ and the corresponding critical value is equal to $-2\sigma(x,y) = -2Jx\cdot y$. Hence $k = Ce^{-2i\sigma(x,y)} = Ce^{-2iJx\cdot y}$ for some (known) constant C.

The composition (4.1) becomes

$$\begin{aligned} &C \iint e^{i(-2J(z-x)\cdot(z-y) + x\cdot x_0^* + y\cdot y_0^*)} a(x-x_0)b(y-y_0)dxdy = \\ &Ce^{iz\cdot(x_0^*+y_0^*)} \iint e^{i(-2Jx\cdot y + x\cdot x_0^* + y\cdot y_0^*)} a(x+z-x_0)b(y+z-y_0)dxdy. \end{aligned} \tag{4.2}$$

The exponent in the last integral can be rewritten as

$$-2Jx\cdot y + x\cdot x_0^* + y\cdot y_0^* = -2J\left(x - \frac{1}{2}J^{-1}y_0^*\right)\cdot\left(y + \frac{1}{2}J^{-1}x_0^*\right) + \frac{1}{2}Jx_0^*\cdot y_0^*,$$

and the composition (4.1) takes the form $e^{iz\cdot(x_0^*+y_0^*)}d(z)$, where

$$\begin{aligned} d(z) = Ce^{\frac{i}{2}\sigma(x_0^*,y_0^*)} \iint e^{-2i\sigma(x,y)} a\left(x + z - \left(x_0 + \frac{1}{2}Jy_0^*\right)\right) \\ \times b\left(y + z - \left(y_0 - \frac{1}{2}Jx_0^*\right)\right) dxdy. \end{aligned}$$

Since $\sigma(x,y)$ is a nondegenerate quadratic form, we have for every $N \geq 0$ by integration by parts,

$$|d(z)| \leq C_N \iint \langle (x,y)\rangle^{-N} \left\langle x+z-\left(x_0+\frac{1}{2}Jy_0^*\right)\right\rangle^{-N} \times \left\langle y+z-\left(y_0-\frac{1}{2}Jx_0^*\right)\right\rangle^{-N} dxdy.$$

Hence for every $N \geq 0$,

$$|d(z)| \leq C_N \left\langle z-\left(x_0+\frac{1}{2}Jy_0^*\right)\right\rangle^{-N} \left\langle z-\left(y_0-\frac{1}{2}Jx_0^*\right)\right\rangle^{-N}.$$

Using the triangle inequality, we get

$$(1+|z-a|)(1+|z-b|) \geq 1+|z-a|+|z-b| \geq 1+\frac{1}{2}|a-b|+\left|z-\frac{a+b}{2}\right|,$$

so

$$(1+|z-a|)(1+|z-b|) \geq \frac{1}{C}(1+|a-b|)^{1/2}\left(1+\left|z-\frac{a+b}{2}\right|\right)^{1/2}$$

and hence for every $N \geq 0$,

$$|d(z)| \leq C_N \left\langle \left(x_0+\frac{1}{2}Jx_0^*\right)-\left(y_0-\frac{1}{2}Jy_0^*\right)\right\rangle^{-N} \times \left\langle z-\frac{1}{2}\left(x_0-\frac{1}{2}Jx_0^*+y_0+\frac{1}{2}Jy_0^*\right)\right\rangle^{-N}. \tag{4.3}$$

Clearly, we have the same estimates for the derivatives of $d(z)$. It follows that the composition (4.1) is equal to $e^{iz\cdot z_0^*}c(z-z_0)$, where

$$z_0^* = x_0^*+y_0^*, \ z_0 = \frac{1}{2}\left(x_0-\frac{1}{2}Jx_0^*+y_0+\frac{1}{2}Jy_0^*\right), \tag{4.4}$$

and where $c \in \mathcal{S}$ and for every seminorm p on $\mathcal{S}$ and every N, there is a seminorm q on $\mathcal{S}$ such that

$$p(c) \leq \left\langle \left(x_0+\frac{1}{2}Jx_0^*\right)-\left(y_0-\frac{1}{2}Jy_0^*\right)\right\rangle^{-N} q(a)q(b). \tag{4.5}$$

It follows that:

$$e^{iz\cdot z_0^*}c(z-z_0)\in \widetilde{S}(\langle \cdot -(z_0,z_0^*)\rangle^{-M})$$

with corresponding norm bounded by

$$q_{N,M}(a)q_{N,M}(b)\left\langle \left(x_0+\frac{1}{2}Jx_0^*\right)-\left(y_0-\frac{1}{2}Jy_0^*\right)\right\rangle^{-N},$$

for all N, $M\geq 0$ where $q_{N,M}$ are suitable seminorms on $\mathcal{S}$.

Let $a_1\in \widetilde{S}(m_1)$, $a_2\in\widetilde{S}(m_2)$ and decompose $a_j=\sum_{\gamma\in\Gamma}a_{j,\gamma}$ as in (3.22), so that $a_{j,\gamma}\in\widetilde{S}(\langle\cdot-\gamma\rangle^{-N})$ for every N with the corresponding estimates on the norms,

$$\|a_{j,\gamma}\|_{\widetilde{S}(\langle\cdot-\gamma\rangle^{-N})}\leq C_N\|a_j\|_{\widetilde{S}(m_j)}m_j(\gamma).$$

Then the above discussion shows that with $\gamma=(\gamma_x,\gamma_{x^*}),\delta=(\delta_x,\delta_{x^*})\in\Gamma$, we have

$$a_{1,\gamma}\# a_{2,\delta}\in\widetilde{S}\left(\left\langle\cdot-\left(\frac{1}{2}(\gamma_x+\delta_x)-\frac{1}{2}J(\gamma_{x^*}-\delta_{x^*}),\gamma_{x^*}+\delta_{x^*}\right)\right\rangle^{-N}\right),$$

and

$$\begin{aligned}&\|a_{1,\gamma}\# a_{2,\delta}\|_{\widetilde{S}(\langle\cdot-(\dots)\rangle^{-N})}\\ &\leq C_M\|a_1\|_{\widetilde{S}(m_1)}\|a_2\|_{\widetilde{S}(m_2)}m_1(\gamma)m_2(\delta)\left\langle\left(\gamma_x+\frac{1}{2}J\gamma_{x^*}\right)-\left(\delta_x-\frac{1}{2}J\delta_{x^*}\right)\right\rangle^{-N}.\end{aligned}$$

Summing over γ, δ we see that $c=a_1\# a_2$ is well-defined and belongs to $\widetilde{S}(m_3^{(N)})$ provided that the sum

$$\begin{aligned}m_3^{(N)}(\epsilon)=\sum_{(\gamma,\delta)\in\Gamma\times\Gamma}\langle\epsilon_{x^*}-(\gamma_{x^*}+\delta_{x^*})\rangle^{-N}\left\langle\epsilon_x-\left(\frac{1}{2}(\gamma_x+\delta_x)-\frac{1}{2}J(\gamma_{x^*}-\delta_{x^*})\right)\right\rangle^{-N}\\ \times\left\langle\left(\gamma_x+\frac{1}{2}J\gamma_{x^*}\right)-\left(\delta_x-\frac{1}{2}J\delta_{x^*}\right)\right\rangle^{-N}m_1(\gamma)m_2(\delta)\end{aligned}$$

converges for all $\epsilon\in\Gamma$. (We will see that this defines an order function if the sum converges for at least one ϵ.) Without changing the convergence or the order of magnitude of $m_3^{(N)}$ we may replace the summations by integrations:

$$(4.6) \quad m_3^{(N)}(z,z^*) = \iiiint \langle z^* - (x^*+y^*)\rangle^{-N} \left\langle z - \frac{1}{2}\left(x - \frac{1}{2}Jx^* + y + \frac{1}{2}Jy^*\right)\right\rangle^{-N} \times \left\langle \left(x+\frac{1}{2}Jx^*\right) - \left(y - \frac{1}{2}Jy^*\right)\right\rangle^{-N} m_1(x,x^*)m_2(y,y^*)dxdydx^*dy^*.$$

In order to understand the integral (4.6), we put $\widetilde{x} = \frac{1}{2}Jx^*$, $\widetilde{y} = \frac{1}{2}Jy^*$, $\widetilde{z} = \frac{1}{2}Jz^*$, and study the set $\Sigma(z,z^*)$ where the arguments inside the three brackets vanish simultaneously:

$$\begin{cases} \widetilde{x} + \widetilde{y} = \widetilde{z}, \\ x + y - \widetilde{x} + \widetilde{y} = 2z, \\ x - y + \widetilde{x} + \widetilde{y} = 0, \end{cases}$$

which can be transformed to

$$(4.7) \quad \Sigma(z,z^*): \begin{cases} \widetilde{x} - x = \widetilde{z} - z, \\ \widetilde{y} + y = \widetilde{z} + z, \\ \widetilde{x} + \widetilde{y} = \widetilde{z}. \end{cases}$$

Now it is clear that for every $M > 0$ there is an $N > 0$ such that

$$(4.8) \quad m_3^{(N)}(z,z^*) \le \mathcal{O}(1) \iiiint \mathrm{dist}\,(x,x^*,y,y^*;\Sigma(z,z^*))^{-M} m_1(x,x^*)m_2(y,y^*)dxdydx^*dy^*.$$

Since m_1, m_2 are order functions, we have

$$\begin{aligned} m_1(x,x^*) &\le \mathcal{O}(1)\mathrm{dist}\,(x,x^*,y,y^*;\Sigma(z,z^*))^{N_0} m_1(\Pi_\Sigma^{(1)}(x,x^*,y,y^*)) \\ m_2(y,y^*) &\le \mathcal{O}(1)\mathrm{dist}\,(x,x^*,y,y^*;\Sigma(z,z^*))^{N_0} m_2(\Pi_\Sigma^{(2)}(x,x^*,y,y^*)), \end{aligned}$$

where $\Pi_\Sigma : (E\times E^*)^2 \to \Sigma(z,z^*)$ is the affine orthogonal projection and we write $\Pi_\Sigma(x,x^*;y,y^*) = (\Pi_\Sigma^{(1)}(x,x^*;y,y^*), \Pi_\Sigma^{(2)}(x,x^*;y,y^*))$. We conclude that for N large enough,

$$(4.9) \quad m_3^{(N)}(z,z^*) \le \mathcal{O}(1)m_3(z,z^*),$$

where

$$m_3(z,z^*)=\int_{\Sigma(z,z^*)} m_1(x,x^*)m_2(y,y^*)d\Sigma \tag{4.10}$$

or more explicitly,

$$m_3(z,z^*)=\int_{\substack{\frac{1}{2}Jx^*-x=\frac{1}{2}Jz^*-z\\ \frac{1}{2}Jy^*+y=\frac{1}{2}Jz^*+z\\ x^*+y^*=z^*}} m_1(x,x^*)m_2(y,y^*))dx. \tag{4.11}$$

Reversing the above estimates, we see that $m_3(z,z^*)\le \mathcal{O}(1)m_3^{(N)}(z,z^*)$, if $N>0$ is large enough.

Proposition 4.1. *If the integral in* (4.10) *converges for one value of* (z,z^*)*, then it converges for all values and defines an order function* m_3*.*

Proof. Suppose the integral converges for the value (z,z^*) and consider any other value $(z+t,z^*+t^*)$. We have the measure preserving map

$$\Sigma(z,z^*)\ni(x,x^*,y,y^*)\mapsto\left(x+t,x^*+t^*,y+\frac{1}{2}Jt^*+t,y^*\right)\in\Sigma(z+t,z^*+t^*),$$

so

$$\begin{aligned} m_3(z+t,z^*+t^*) &= \int_{\Sigma(z,z^*)} m_1(x+t,x^*+t^*)m_2\left(y+\frac{1}{2}Jt^*+t,y^*\right)dx \\ &\le C\langle(t,t^*)\rangle^{N_0}\left\langle t+\frac{J}{2}t^*\right\rangle^{N_0} m_3(z,z^*) \\ &\le \widetilde{C}\langle(t,t^*)\rangle^{2N_0}m_3(z,z^*). \end{aligned}$$

The proposition follows. □

From the above discussion, we get

Theorem 4.2. *Let* m_1*,* m_2 *be order functions on* $E\times E^*$ *and define* m_3 *by* (4.11)*. Assume that* $m_3(z,z^*)$ *is finite for at least one* (z,z^*) *so that* m_3 *is a well-defined order function by Proposition* 4.1*. Then the composition map*

$$\mathcal{S}(E)\times\mathcal{S}(E)\ni(a_1,a_2)\mapsto a_1\#a_2\in\mathcal{S}(E) \tag{4.12}$$

has a bilinear extension

$$\widetilde{S}(m_1)\times\widetilde{S}(m_2)\ni(a_1,a_2)\mapsto a_1\#a_2\in\widetilde{S}(m_3), \tag{4.13}$$

Moreover,

$$\|a_1\#a_2\|_{\widetilde{S}(m_3)} \le \mathcal{O}(1)\|a_1\|_{\widetilde{S}(m_1)}\|a_2\|_{\widetilde{S}(m_2)}. \tag{4.14}$$

Remark 4.3. In the remainder of the paper we will further develop the characterization of symbols and operators by means of generalized Bargmann transforms. Theorem 4.2 will be generalized in Theorem 7.8. We shall also see that under the more general assumptions of that theorem, the composed symbol $a_1\#a_2$ is characterized by the relation (5.17) for the effective kernels (defined in Section 5) and that the integral, defining the composition of the effective kernels of a_1^w and a_2^w, is absolutely convergent. Theorem 7.11 below can be used to define spaces $\mathcal{H}_j$ sandwiched between $\mathcal{S}$ and $\mathcal{S}'$ such that $a_1^w : \mathcal{H}_1 \to \mathcal{H}_2$, $a_2^w : \mathcal{H}_2 \to \mathcal{H}_3$ and the composition $a_2^w \circ a_1^w : \mathcal{H}_1 \to \mathcal{H}_3$ will have the Weyl symbol $a_2\#a_1$.

Alternatively, one could (most likely) show that $\mathcal{S}(E)$ is dense in $\widetilde{S}(m)$ for sequences in $\mathcal{S}(E)$ that are bounded in $\widetilde{S}(m)$ and converge in $\mathcal{S}'(E)$ and deduce the uniqueness of the extension (4.13) by showing that it is sequentially continuous in the same way. A simple proof of that would probably follow from using the associated effective kernels as in Section 5 together with the Lebesgue dominated convergence theorem.

We end this section by establishing a first connection with the effective kernels of Section 4. Let a_j be as in the theorem with $a_3 = a_1\#a_2$. According to Theorem 3.1, we then know that a_j^w has an effective kernel $K_j = K^{\mathrm{eff}}_{a_j^w}(x,y)$ satisfying

$$K_j(x,y) = \mathcal{O}(1)m_j(q(x,y)), \text{ where } q(x,y) = \left(\frac{x+y}{2}, J^{-1}(y-x)\right). \tag{4.15}$$

Since the composition of the effective kernels of a_1^w and a_2^w is an effective kernel for $a_3^w = a_1^w \circ a_2^w$ we expect that

$$m_3(q(\widetilde{x},\widetilde{y})) = C\int m_1(q(\widetilde{x},\widetilde{z}))m_2(q(\widetilde{z},\widetilde{y}))d\widetilde{z}, \tag{4.16}$$

or more explicitly,

$$\begin{aligned} m_3\left(\frac{\widetilde{x}+\widetilde{y}}{2}, J^{-1}(\widetilde{y}-\widetilde{x})\right) \\ = C\int m_1\left(\frac{\widetilde{x}+\widetilde{z}}{2}, J^{-1}(\widetilde{z}-\widetilde{x})\right) m_2\left(\frac{\widetilde{z}+\widetilde{y}}{2}, J^{-1}(\widetilde{y}-\widetilde{z})\right) d\widetilde{z}, \end{aligned} \tag{4.17}$$

Writing

$$\begin{aligned} z &= \frac{\widetilde{x}+\widetilde{y}}{2}, \\ z^* &= J^{-1}(\widetilde{y}-\widetilde{x}), \\ x &= \frac{\widetilde{x}+\widetilde{z}}{2}, \\ x^* &= J^{-1}(\widetilde{z}-\widetilde{x}), \\ y &= \frac{\widetilde{z}+\widetilde{y}}{2}, \\ y^* &= J^{-1}(\widetilde{y}-\widetilde{z}), \end{aligned}$$

we check that the integral in (4.17) coincides with the one in (4.11) up to a constant Jacobian factor, so the results of this section fit with the ones of Section 3.

Example 4.4. Let $a_j \in \widetilde{S}(m_j)$, $j = 1, 2$, where m_j are order functions on $E \times E^*$ of the form

$$\begin{aligned} m_j(x, x^*) &= \widetilde{m}_j(x)\langle x^* \rangle^{-N_j}, \ N_j \in \mathbf{R}, \\ \widetilde{m}_j(x) &\leq C\langle x-y\rangle^{M_j}\widetilde{m}_j(y), \ x, y \in E, \ M_j \geq 0. \end{aligned}$$

Then, the effective kernels K_1, K_2 of a_1^w, a_2^w satisfy

$$K_j(x,y) = \mathcal{O}(1)m_j\left(\frac{x+y}{2}, J^{-1}(y-x)\right) = \mathcal{O}(1)\widetilde{m}_j\left(\frac{x+y}{2}\right)\langle x-y\rangle^{-N_j}.$$

Then $a_1 \# a_2$ is well-defined and belongs to $\widetilde{S}(m_3)$, where

$$m_3\left(\frac{x+y}{2}, J^{-1}(y-x)\right) = \int \widetilde{m}_1\left(\frac{x+z}{2}\right)\langle x-z\rangle^{-N_1}\langle z-y\rangle^{-N_2}\widetilde{m}_2\left(\frac{z+y}{2}\right)dz,$$

provided that the last integral converges for at least one (and then all) value(s) of $((x+y)/2, J^{-1}(y-x))$. If we use that

$$\begin{aligned} \widetilde{m}_1\left(\frac{x+z}{2}\right) &\leq \mathcal{O}(1)\widetilde{m}_1\left(\frac{x+y}{2}\right)\langle z-y\rangle^{M_1} \\ \widetilde{m}_2\left(\frac{z+y}{2}\right) &\leq \mathcal{O}(1)\widetilde{m}_2\left(\frac{x+y}{2}\right)\langle x-z\rangle^{M_2}, \end{aligned}$$

we get

$$m_3\left(\frac{x+y}{2}, J^{-1}(y-x)\right) \le \mathcal{O}(1)\widetilde{m}_1\left(\frac{x+y}{2}\right)\widetilde{m}_2\left(\frac{x+y}{2}\right)\int \langle x-z\rangle^{-N_1+M_2}\langle z-y\rangle^{-N_2+M_1}dz. \tag{4.18}$$

Thus m_3 and $a_1\# a_2 \in \widetilde{S}(m_3)$ are well-defined if

$$-(N_1+N_2)+M_1+M_2 < -2n. \tag{4.19}$$

The integral I in (4.18) is $\mathcal{O}(1)$ in any region where $x-y=\mathcal{O}(1)$. For $|x-y|\ge 2$, we write $I \le I_1+I_2+I_3$, where

- I_1 is the integral over $|x-z| \le \frac{2}{3}|x-y|$. Here $\langle z-y\rangle \backsim \langle x-y\rangle$.
- I_2 is the integral over $|z-y| \le \frac{2}{3}|x-y|$. Here $\langle x-z\rangle \backsim \langle x-y\rangle$.
- I_3 is the integral over $|x-z|, |z-y| \ge \frac{2}{3}|x-y|$. Here $\langle x-z\rangle \backsim \langle y-z\rangle \ge \frac{1}{C}\langle x-y\rangle$.

We get

$$I_1 \backsim \langle x-y\rangle^{-N_2+M_1}\int_0^{\langle x-y\rangle}\langle r\rangle^{-N_1+M_2+2n-1}dr \backsim \langle x-y\rangle^{-N_2+M_1+(-N_1+M_2+2n)_+},$$

with the convention that we tacitly add a factor $\ln\langle x-y\rangle$ when the expression inside $(\dots)_+$ is equal to 0. Similarly (with the same convention),

$$I_2 \backsim \langle x-y\rangle^{-N_1+M_2+(-N_2+M_1+2n)_+}.$$

In view of (4.19), we have

$$I_3 \backsim \int_{\langle x-y\rangle}^{\infty} r^{-(N_1+N_2)+M_1+M_2+2n-1}dr \backsim \langle x-y\rangle^{-(N_1+N_2)+M_1+M_2+2n}.$$

it follows that

$$I \backsim \langle x-y\rangle^{\max(-N_2+M_1+(-N_1+M_2+2n)_+,-N_1+M_2+(-N_2+M_1+2n)_+)}, \tag{4.20}$$

so with the same convention, we have

$$\begin{aligned}(4.21)\quad & m_3(x,x^*)\\ & \le \mathcal{O}(1)\widetilde{m}_1(x)\widetilde{m}_2(x)\langle x^*\rangle^{\max(-N_2+M_1+(-N_1+M_2+2n)_+,-N_1+M_2+(-N_2+M_1+2n)_+)}.\end{aligned}$$

This simplifies to

$$m_3(x,x^*)\le \mathcal{O}(1)\widetilde{m}_1(x)\widetilde{m}_2(x)\langle x^*\rangle^{\max(-N_2+M_1,-N_1+M_2)} \tag{4.22}$$

if we strengthen the assumption (4.19) to:

$$-N_1+M_2,\ -N_2+M_1<-2n. \tag{4.23}$$

5. More direct approach using Bargmann transforms. By using Bargmann transforms more systematically (from the point of view of Fourier integral operators with complex phase) the results of Section 3, 4 can be obtained more directly. The price to pay however, is the loss of some aspects that might be helpful in other situations like the ones with variable metrics.

Let F be a real d-dimensional space as in Section 2 and define $T : L^2(F)\to H_\Phi(F^{\mathbf{C}})$ as in (3.8)–(3.11). Then we have

Proposition 5.1. *If m is an order function on $F\times F^*$, then*

$$\widetilde{S}(m)=\left\{u\in\mathcal{S}'(F);\ e^{-\Phi(x)}|Tu(x)|\le Cm\left(\kappa_T^{-1}\left(x,\frac{2}{i}\frac{\partial\Phi}{\partial x}(x)\right)\right)\right\}, \tag{5.1}$$

where the best constant $C=C(m)$ is a norm on $\widetilde{S}(m)$.

Proof. Assume first that u belongs to $\widetilde{S}(m)$ and write $u=\sum_{\gamma\in\Gamma}\psi_\gamma^w\chi_\gamma^w u$ as in Lemma 2.3. The effective kernel of ψ_γ^w satisfies

$$|K^{\mathrm{eff}}_{\psi_\gamma^w}(x,y)|\le C_N\langle x-\gamma\rangle^{-N}\langle y-\gamma\rangle^{-N}, \tag{5.2}$$

for every $N>0$, where throughout the proof we identify $F^{\mathbf{C}}$ with $F\times F^*$ by means of $\pi\circ\kappa_T$ and work on the latter space. Here $\pi:\Lambda_\Phi\to F^{\mathbf{C}}$ is the natural projection. Then we see that

$$|e^{-\Phi}Tu(x)|\le C_N(u)\sum_{\gamma\in\Gamma}m(\gamma)\langle x-\gamma\rangle^{-N}=\mathcal{O}(m(x)).$$

Conversely, if $e^{-\Phi}Tu = \mathcal{O}(m(x))$, then since the effective kernel of χ_γ^w also satisfies (5.2), we see that $e^{-\Phi}T\chi_\gamma^w u = \mathcal{O}_N(\langle x-\gamma\rangle^{-N}m(\gamma))$, implying $\|e^{-\Phi}T\chi_\gamma^w u\|_{L^2} = \mathcal{O}(m(\gamma))$, and hence $\|\chi_\gamma^w u\| = \mathcal{O}(m(\gamma))$. □

With this in mind, we now take $a \in \widetilde{S}(\mathbf{R}^n \times (\mathbf{R}^n)^*; m)$ and look for an explicit choice of effective kernel for a^w. Let $T : L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n)$ be a Bargmann transform as above. Consider first the map $a \mapsto K_{a^w}(x,y) \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ from a to the distribution kernel of a^w, given by

$$
\begin{aligned}
K_{a^w}(x,y) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\tau} a\left(\frac{x+y}{2},\tau\right)d\tau \\
&= \frac{1}{(2\pi)^{2n}}\iiint e^{i(x-y)\cdot\tau + i(\frac{x+y}{2}-t)\cdot s} a(t,\tau)\,dt\,ds\,d\tau.
\end{aligned}
\tag{5.3}
$$

We view this as a Fourier integral operator $B : a \mapsto K_{a^w}(x,y)$ with quadratic phase. The associated linear canonical transformation is given by:

$$
\kappa_B : (t,\tau;t^*,\tau^*) = \left(\frac{x+y}{2},\tau;s,y-x\right) \mapsto \left(x,\tau+\frac{s}{2};y,-\tau+\frac{s}{2}\right) = (x,x^*;y,y^*),
$$

which we can write as

$$
\kappa_B : (t,\tau;t^*,\tau^*) \mapsto \left(t-\frac{\tau^*}{2},\tau+\frac{t^*}{2};t+\frac{\tau^*}{2},-\tau+\frac{t^*}{2}\right). \tag{5.4}
$$

From the unitarity of T, we know that $T^*T = 1$, where

$$
T^*v(y) = C\int e^{-i\overline{\phi(x,y)}}v(x)e^{-2\Phi(x)}L(dx). \tag{5.5}
$$

We can therefore define the effective kernel of a^w to be

$$
K^{\mathrm{eff}}(x,y) = e^{-\Phi(x)}K(x,\overline{y})e^{-\Phi(y)}, \tag{5.6}
$$

where

$$
\begin{aligned}
Ta^wT^*v(x) &= \int K(x,\overline{y})v(y)e^{-2\Phi(y)}L(dy),\ v \in H_\Phi(\mathbf{C}^n), \\
K(x,\overline{y}) &= C^2\iint e^{i(\phi(x,t)-\overline{\phi(y,s)})}K_{a^w}(t,s)\,dt\,ds.
\end{aligned}
\tag{5.7}
$$

We write this as

$$
K(x,y) = C^2\iint e^{i(\phi(x,t)-\phi^*(y,s))}K_{a^w}(t,s)\,dt\,ds,
$$

with $\phi^*(y,s) = \overline{\phi(\overline{y},\overline{s})}$, so

$$K(x,y) = (T \otimes \widetilde{T})(K_{a^w})(x,y), \tag{5.8}$$

where

$$(\widetilde{T}u)(y) = C \int e^{-i\phi^*(y,s)} u(s) ds = \overline{(T\overline{u})(\overline{y})}. \tag{5.9}$$

We see that $\widetilde{T} : L^2(\mathbf{R}^n) \to H_{\Phi^*}(\mathbf{C}^n)$ is a unitary Bargmann transform, where

$$\Phi^*(y) = \sup_{s\in\mathbf{R}^n} \operatorname{Im} \phi^*(y,s) = \sup_{s\in\mathbf{R}^n} \operatorname{Im} \overline{\phi(\overline{y},s)} = \Phi(\overline{y}). \tag{5.10}$$

The canonical transformation associated to $\widetilde{T}$ is

$$\kappa_{\widetilde{T}} : \left(s, \frac{\partial \phi^*}{\partial s}(y,s)\right) \mapsto \left(y, -\frac{\partial \phi^*}{\partial y}(y,s)\right). \tag{5.11}$$

If

$$\iota(s,\sigma) = (\overline{s}, -\overline{\sigma}), \tag{5.12}$$

we check that

$$\kappa_{\widetilde{T}} = \iota \kappa_T \iota, \quad \iota : \left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right) \mapsto \left(\overline{x}, \frac{2}{i}\frac{\partial \Phi^*}{\partial y}(\overline{x})\right). \tag{5.13}$$

Clearly $T \otimes \widetilde{T}$ is a Bargmann transform with associated canonical transformation $\kappa_T \times (\iota\kappa_T\iota)$, so in view of (5.4) the map $a \mapsto K$ is also a Bargmann transform with associated canonical transformation

$$\begin{aligned} &(E \times E^*)^{\mathbf{C}} \ni (t,\tau;t^*,\tau^*) \\ &\quad \mapsto \left(\kappa_T\left((t,\tau) - \frac{1}{2}J(t^*,\tau^*)\right), \iota\kappa_T\left(\overline{(t,\tau)} + \frac{1}{2}\overline{J(t^*,\tau^*)}\right)\right), \end{aligned} \tag{5.14}$$

where $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$. The restriction to the real phase space is

$$\begin{aligned} &E \times E^* \ni (t,\tau;t^*,\tau^*) \mapsto \\ &\left(\kappa_T\left((t,\tau) - \frac{1}{2}J(t^*,\tau^*)\right), \iota\kappa_T\left(t,\tau) + \frac{1}{2}(t^*,\tau^*)\right)\right) \in \Lambda_\Phi \times \iota\Lambda_\Phi = \Lambda_\Phi \times \Lambda_{\Phi^*}, \end{aligned} \tag{5.15}$$

and this restriction determines our complex linear canonical transformation uniquely.

As in Section 3 we may view the effective kernel $K^{\mathrm{eff}}(x,y)$ in (5.6) as a function on $E\times E$, by identifying $x,y\in\mathbf{C}^n$ with $\kappa_T^{-1}\left(x,\frac{2}{i}\frac{\partial\Phi}{\partial x}(x)\right)$, $\kappa_T^{-1}\left(y,\frac{2}{i}\frac{\partial\Phi}{\partial x}(y)\right)\in E$ respectively. With this identification and using also the general characterization in (5.1) (with T replaced by $T\otimes\widetilde{T}$)), we see that if $a\in\mathcal{S}'(E)$, then $a\in\widetilde{S}(m)$ iff

$$K^{\mathrm{eff}}(t-\frac{1}{2}Jt^*,t+\frac{1}{2}Jt^*)=\mathcal{O}(1)m(t,t^*),\ (t,t^*)\in E\times E^*, \tag{5.16}$$

where we shortened the notation by writing t instead of (t,τ) and t^* instead of (t^*,τ^*).

Theorem 3.1 now follows from (5.16), (5.6), (5.7).

Theorem 4.2 also follows from (5.16), (5.6), (5.7) together with the remark that the kernel $K(x,y)=K_a(x,y)$ is the unique kernel which is holomorphic on $\mathbf{C}^n\times\mathbf{C}^n$, such that the corresponding $K^{\mathrm{eff}}_{a^w}$ given in (5.6) is of temperate growth at infinity and (5.7) is fulfilled. Indeed, then it is clear that

$$K^{\mathrm{eff}}_{(a_1\#a_2)^w}(x,y)=\int K^{\mathrm{eff}}_{a_1^w}(x,z)K^{\mathrm{eff}}_{a_2^w}(z,y)L(dz) \tag{5.17}$$

and the bound (5.16) for $a_1\#a_2$ with $m=m_3$ follows directly from the corresponding bounds for a_j with $m=m_j$.

6. C_p classes. In this section we give a simple condition on an order function m on $E\times E^*$ ($E=T^*\mathbf{R}^n$) and a number $p\in[1,\infty]$ that implies the property:

$$\exists C>0 \text{ such that: } a\in\widetilde{S}(m)\Rightarrow a^w\in C_p(L^2,L^2) \text{ and } \|a^w\|_{C_p}\le C\|a\|_{\widetilde{S}(m)}. \tag{6.1}$$

Here $C_p(L^2,L^2)$ is the Schatten–von Neumann class of operators: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$, see for instance [16].

Let m be an order function on $E\times E^*$ and let $p\in[1,+\infty]$. Consider the following property, where q is given in (4.15) and $\Gamma\subset E$ is a lattice,

$$\begin{aligned}&\exists C>0 \text{ such that if } |a_{\alpha,\beta}|\le m(q(\alpha,\beta)),\ \alpha,\beta\in\Gamma,\\ &\text{then } (a_{\alpha,\beta})_{\alpha,\beta\in\Gamma}\in C_p(\ell^2(\Gamma),\ell^2(\Gamma)) \text{ and } \|(a_{\alpha,\beta})\|_{C_p}\le C.\end{aligned} \tag{6.2}$$

Notice that if (6.2) holds and if we fix some number $N_0 \in \mathbf{N}^*$, then if $(A_{\alpha,\beta})_{\alpha,\beta\in\Gamma}$ is a block matrix where every $A_{\alpha,\beta}$ is an $N_0 \times N_0$ matrix then

(6.3) same as (6.2) with $a_{\alpha,\beta}$ replaced by $A_{\alpha,\beta}$ and $|\cdot|$ by $\|\cdot\|_{\mathcal{L}(\mathbf{C}^{N_0},\mathbf{C}^{N_0})}$.

Proposition 6.1. *The property* (6.2) *only depends on* m, p *but not on the choice of* Γ.

Proof. Let m, p, Γ satisfy (6.2) and let $\widetilde{\Gamma}$ be a second lattice in E. Let $(a_{\widetilde{\alpha},\widetilde{\beta}})$ be a $\widetilde{\Gamma}\times\widetilde{\Gamma}$ matrix satisfying $|a_{\widetilde{\alpha},\widetilde{\beta}}| \le m(q(\widetilde{\alpha},\widetilde{\beta}))$. Let $\pi(\widetilde{\alpha}) \in \Gamma$ be a point that realizes the distance from $\widetilde{\alpha}$ to Γ, so that $|\pi(\widetilde{\alpha}) - \widetilde{\alpha}| \le C_0$ for some constant $C_0 > 0$. Let $N_0 = \max \#\pi^{-1}(\alpha)$ and choose an enumeration $\pi^{-1}(\alpha) = \{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{N(\alpha)}\}$, $N(\alpha) \le N_0$, for every $\alpha \in \Gamma$. Then we can identify $(a_{\widetilde{\alpha},\widetilde{\beta}})_{\widetilde{\Gamma}\times\widetilde{\Gamma}}$ with the matrix $(A_{\alpha,\beta})_{\alpha,\beta\in\Gamma\times\Gamma}$ where $A_{\alpha,\beta}$ is the $N_0\times N_0$ matrix with the entries

$$(A_{\alpha,\beta})_{j,k} = \begin{cases} a_{\widetilde{\alpha}_j,\widetilde{\beta}_k}, & \text{if } 1 \le j \le N(\alpha),\ 1 \le k \le N(\beta), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|A_{\alpha,\beta}\| \le Cm(q(\alpha,\beta))$ and we can apply (6.3) to conclude. □

Theorem 6.2. *Let* m *be an order function and* $p \in [1,\infty]$. *If* (6.2) *holds, then we have* (6.1).

Proof. Assume that (6.2) holds and let $a \in \widetilde{S}(m)$. Define $K(x,\overline{y})$ as in (5.7). It suffices to estimate the C_p norm of the operator $A : L^2(e^{-2\Phi}L(dx)) \to L^2(e^{-2\Phi}L(dx))$, given by

$$Au(x) = \int K(x,\overline{y})u(y)e^{-2\Phi(y)}L(dy),$$

or equivalently the one of $A_{\rm eff} : L^2(\mathbf{C}^n) \to L^2(\mathbf{C}^n)$, given by

$$A_{\rm eff}u(x) = \int K^{\rm eff}(x,y)u(y)L(dy), \tag{6.4}$$

with $K^{\rm eff}$ given in (5.6). Recall that $K^{\rm eff}(x,y) = \mathcal{O}(1)m(q(x,y))$ (identifying $\mathbf{C}^n$ with $T^*\mathbf{R}^n$ via $\pi_x \circ \kappa_T$), so $K(x,\overline{y}) = \mathcal{O}(1)m(q(x,y))e^{\Phi(x)+\Phi(y)}$.

For $\alpha, \beta \in \Gamma$ we have (identifying Γ with a lattice in $\mathbf{C}^n$)

$$K(x,\overline{y}) = e^{F_\alpha(x-\alpha)}\widetilde{K}_{\alpha,\beta}(x,\overline{y})e^{\overline{F_\beta(y-\beta)}}, \tag{6.5}$$

where

$$F_\alpha(x-\alpha) = \Phi(\alpha) + 2\frac{\partial \Phi}{\partial x}(\alpha)\cdot(x-\alpha) \tag{6.6}$$

is holomorphic with

$$\operatorname{Re} F_\alpha(x-\alpha) = \Phi(x) + R_\alpha(x-\alpha),\ \ R_\alpha(x-\alpha) = \mathcal{O}(|x-\alpha|^2), \tag{6.7}$$

and

$$|\nabla_x^k \nabla_y^\ell \widetilde{K}_{\alpha,\beta}(x,\overline{y})| \le \widetilde{C}_{k,\ell}\, m(q(\alpha,\beta)),\ \ |x-\alpha|, |y-\beta| \le C_0. \tag{6.8}$$

Here we identify $\alpha, \beta \in E$ with their images $\pi_x\kappa_T(\alpha), \pi_x\kappa_T(\beta) \in \mathbf{C}^n$ respectively. In fact, the case $k = \ell = 0$ is clear and we get the extension to arbitrary k, ℓ from the Cauchy inequalities, since $\widetilde{K}_{\alpha,\beta}$ is holomorphic.

We can also write

$$K^{\mathrm{eff}}(x,y) = e^{iG_\alpha(x-\alpha)} K_{\alpha,\beta}(x,y) e^{-iG_\beta(y-\beta)}, \tag{6.9}$$

where

$$G_\alpha(x-\alpha) = \operatorname{Im} F_\alpha(x-\alpha), \quad K_{\alpha,\beta} = e^{R_\alpha(x-\alpha)}\widetilde{K}_{\alpha,\beta}(x,\overline{y}) e^{R_\beta(y-\beta)},$$

so

$$|\nabla_x^k \nabla_y^\ell K_{\alpha,\beta}(x,y)| \le C_{k,\ell} m(q(\alpha,\beta)),\ \ |x-\alpha|, |y-\beta| \le C_0. \tag{6.10}$$

Consider a partition of unity

$$1 = \sum_{\alpha\in\Gamma} \chi_\alpha(x), \quad \chi_\alpha(x) = \chi_0(x-\alpha),\ \ \chi_0 \in C_0^\infty(\Omega_0; \mathbf{R}), \tag{6.11}$$

where Ω_0 is open with smooth boundary. Let $\Omega_\alpha = \Omega_0 + \alpha$, so that (6.10) holds for $(x,y) \in \Omega_\alpha \times \Omega_\beta$.

Let $W : L^2(\mathbf{C}^n) \to \bigoplus_{\beta\in\Gamma} L^2(\Omega_\beta)$ be defined by

$$Wu = \Big((e^{-iG_\beta(x-\beta)} u(x))_{|\Omega_\beta} \Big)_{\beta\in\Gamma},$$

so that the adjoint of W is given by

$$W^* v = \sum_{\alpha\in\Gamma} e^{iG_\alpha(x-\alpha)} v_\alpha(x) 1_{\Omega_\alpha}(x), \quad v = (v_\alpha)_{\alpha\in\Gamma} \in \bigoplus_{\alpha\in\Gamma} L^2(\Omega_\alpha).$$

Then W and its adjoint are bounded operators and

$$A_{\mathrm{eff}} = W^* \mathcal{A} W, \tag{6.12}$$

where $\mathcal{A} = (A_{\alpha,\beta})_{\alpha,\beta\in\Gamma}$ and $A_{\mathrm{eff}} : L^2(\mathbf{C}^n) \to L^2(\mathbf{C}^n)$, $A_{\alpha,\beta} : L^2(\Omega_\beta) \to L^2(\Omega_\alpha)$ are given by the kernels $K^{\mathrm{eff}}(x,y)$ and $\chi_\alpha(x) K_{\alpha,\beta}(x,y)\chi_\beta(y)$ respectively. It now suffices to show that

$$\mathcal{A} : \bigoplus_{\beta\in\Gamma} L^2(\Omega_\beta) \to \bigoplus_{\beta\in\Gamma} L^2(\Omega_\beta)$$

belongs to C_p with a norm that is bounded by a constant times the $\widetilde{S}(m)$-norm of a.

Let $e_0, e_1, \dots \in L^2(\Omega_0)$ be an orthonormal basis of eigenfunctions of minus the Dirichlet Laplacian in Ω_0, arranged so that the corresponding eigenvalues form an increasing sequence. Then $e_{\alpha,j} := \tau_\alpha e_j$, $j = 0, 1, \dots$ form an orthonormal basis of eigenfunctions of the corresponding operator in $L^2(\Omega_\alpha)$. From (6.10) it follows that the matrix elements $K_{\alpha,j;\beta,k}$ of $A_{\alpha,\beta}$ with respect to the bases $(e_{\alpha,\cdot})$ and $(e_{\beta,\cdot})$ satisfy

$$|K_{\alpha,j;\beta,k}| \le C_N m(q(\alpha,\beta)) \langle j\rangle^{-N} \langle k\rangle^{-N}, \tag{6.13}$$

for every $N \in \mathbf{N}$. We notice that $(K_{\alpha,j;\beta,k})_{(\alpha,j),(\beta,k)\in\Gamma\times\mathbf{N}}$ is the matrix of $\mathcal{A}$ with respect to the orthonormal basis $(e_{\alpha,j})_{(\alpha,j)\in\Gamma\times\mathbf{N}}$. We can represent this matrix as a block matrix $(K^{j,k})_{j,k\in\mathbf{N}}$, where $K^{j,k} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ has the matrix $(K_{\alpha,j;\beta,k})_{\alpha,\beta\in\Gamma}$. Since (6.2) holds and $a \in \widetilde{S}(m)$, we deduce from (6.13) that

$$\|K^{j,k}\|_{C_p} \le \widetilde{C}_N \langle j\rangle^{-N} \langle k\rangle^{-N}. \tag{6.14}$$

Choosing $N > 2n$, we get

$$\|\mathcal{A}\|_{C_p} \le \sum_{j,k} \|K^{j,k}\|_{C_p} < \infty. \tag{6.15}$$

Hence $a^w \in C_p$ and the uniform bound $\|a^w\|_{C_p} \le \|a\|_{\widetilde{S}(m)}$ also follows from the proof. □

Example 6.3. Assume that

$$\int_{E^*} \|m(\cdot, x^*)\|_{L^p(E)} dx^* < \infty. \tag{6.16}$$

Then

$$(m(q(\alpha,\beta)))_{\alpha,\beta\in\Gamma} = \left(m\left(\frac{\alpha+\beta}{2}, J^{-1}(\beta-\alpha)\right)\right)_{\alpha,\beta\in\Gamma} \tag{6.17}$$

is a matrix where each translated diagonal $\{(\alpha,\beta)\in\Gamma\times\Gamma;\ \alpha-\beta=\delta\}$ has an ℓ^p norm which is summable with respect to $\delta\in\Gamma$. Now a matrix with non-vanishing elements in only one translated diagonal has a C_p norm equal to the ℓ^p norm of that diagonal, so we conclude that the C_p norm of the matrix in (6.17) is bounded by

$$\sum_{\delta\in\Gamma} \|m(\frac{\cdot}{2},\delta)\|_{\ell^p} < \infty.$$

We clearly have the same conclusion for every matrix $(a_{\alpha,\beta})_{\alpha,\beta\in\Gamma}$ satisfying $|a_{\alpha,\beta}|\le m(q(\alpha,\beta))$, so (6.2) holds and hence by Theorem 6.2 we have the property (6.1).

7. Further generalizations. Let E be a d-dimensional real vector space and let $\Gamma\subset E$ be a lattice. We shall extend the preceding results by replacing the $\ell^\infty(\Gamma)$-norm in the definition of the symbol spaces by a more general Banach space norm. Let B be a Banach space of functions $u:\Gamma\to\mathbf{C}$ with the following properties:

$$\text{If } u\in B,\ \gamma\in\Gamma,\ \text{then } \tau_\gamma u\in B,\ \text{and } \|\tau_\gamma u\|_B = \|u\|_B. \tag{7.1}$$

$$\delta_\gamma\in B,\ \forall\gamma\in\Gamma, \tag{7.2}$$

where $\tau_\gamma u(\alpha)=u(\alpha-\gamma)$, $\delta_\gamma(\alpha)=\delta_{\gamma,\alpha}$, $\alpha\in\Gamma$. (The last assumption will soon be replaced by a stronger one.)

If $u=\sum_{\gamma\in\Gamma} u(\gamma)\delta_\gamma\in B$, we get

$$\|u\|_B \le \sum |u(\gamma)|\|\delta_\gamma\|_B = C\|u\|_{\ell^1},$$

where $C=\|\delta_\gamma\|_B$ (is independent of γ). Thus

$$\ell^1(\Gamma)\subset B. \tag{7.3}$$

We need to strengthen (7.2) to the following assumption:

$$\begin{aligned}&\text{If } u\in B \text{ and } v:\Gamma\to\mathbf{C} \text{ satisfies } |v(\gamma)|\le|u(\gamma)|,\ \forall\gamma\in\Gamma,\\ &\text{then } v\in B \text{ and } \|v\|_B\le C\|u\|_B,\ \text{where } C \text{ is independent of } u,v.\end{aligned} \tag{7.4}$$

It follows that $\|u(\gamma)\delta_\gamma\|_B \le C\|u\|_B$, for all $u\in B$, $\gamma\in\Gamma$, or equivalently that

$$|u(\gamma)| \le \frac{C}{\|\delta_\gamma\|_B}\|u\|_B = \widetilde{C}\|u\|_B,$$

so

$$B\subset \ell^\infty(\Gamma),\ \text{and } \|u\|_{\ell^\infty}\le \widetilde{C}\|u\|_B,\ \forall u\in B. \tag{7.5}$$

If $f\in\ell^1(\Gamma)$ then using only the translation invariance (7.1), we get

$$u\in B \Rightarrow \begin{cases} f*u\in B,\\ \|f*u\|_B\le \|f\|_{\ell^1}\|u\|_B. \end{cases} \tag{7.6}$$

Using also (7.4) we get the following partial strengthening: Let $k:\Gamma\times\Gamma\to\Gamma$ satisfy $|k(\alpha,\beta)|\le f(\alpha-\beta)$ where $f\in\ell^1(\Gamma)$. Then

$$u\in B\Rightarrow v(\alpha):=\sum_{\beta\in\Gamma}k(\alpha,\beta)u(\beta)\in B \text{ and } \|v\|_B\le C\|f\|_{\ell^1}\|u\|_B, \tag{7.7}$$

where C is independent of k,u. In fact,

$$u\in B\Rightarrow |u|\in B\Rightarrow f*|u|\in B,$$

and v in (7.7) satisfies $|v|\le f*|u|$ pointwise.

Let $\widetilde{\Gamma}\subset E$ be a second lattice and let $\widetilde{B}\subset\ell^\infty(\widetilde{\Gamma})$ satisfy (7.1), (7.4). We say that $B\prec\widetilde{B}$ if the following property holds for some $N>d$:

(7.8) If $u\in B$ and $\widetilde{u}:\widetilde{\Gamma}\to\mathbf{C}$ satisfies $|\widetilde{u}(\widetilde{\gamma})|\le \sum_{\gamma\in\Gamma}\langle\widetilde{\gamma}-\gamma\rangle^{-N}|u(\gamma)|,\ \widetilde{\gamma}\in\widetilde{\Gamma}$,

then $\widetilde{u}\in\widetilde{B}$ and $\|\widetilde{u}\|_{\widetilde{B}}\le C\|u\|_B$, where C is independent of $u,\widetilde{u}$.

If (7.8) holds for one $N>d$ and $M>d$ then it also holds with N replaced by M. This is obvious when $M\ge N$ and if $d<M<N$, it follows from the observation that

$$\langle\widetilde{\gamma}-\gamma\rangle^{-M}\le C_{N,M}\sum_{\widetilde{\beta}\in\widetilde{\Gamma}}\langle\widetilde{\gamma}-\widetilde{\beta}\rangle^{-M}\langle\widetilde{\beta}-\gamma\rangle^{-N}$$

(cf. (4.20), where I is the integral in (4.18), $2n$ is replaced by d, and we take $M_1=M_2=0$), which allows us to write

$$\sum_{\gamma\in\Gamma}\langle\widetilde{\gamma}-\gamma\rangle^{-M}|u(\gamma)|\le C_{N,M}\langle\cdot\rangle^{-M}*v,$$

where $v(\widetilde{\beta}) := \sum_\gamma \langle \widetilde{\beta} - \gamma \rangle^{-N} |u(\gamma)|$ and v belongs to $\widetilde{B}$ since (7.8) holds.

Definition 7.1. *Let $\Gamma, \widetilde{\Gamma}$ be two lattices in E and let $B, \widetilde{B}$ be Banach spaces of functions on Γ and $\widetilde{\Gamma}$ respectively, satisfying* (7.1), (7.4). *Then we say that $B \equiv \widetilde{B}$, if $B \prec \widetilde{B}$ and $\widetilde{B} \prec B$. Notice that this is an equivalence relation.*

We can now introduce our generalized symbol spaces. With $E \simeq \mathbf{R}^d$ as above, let $\Gamma \subset E \times E^*$ be a lattice and $B \subset \ell^\infty$ a Banach space satisfying (7.1), (7.4). Let $a \in \mathcal{S}'(E)$.

Definition 7.2. *We say that $a \in \widetilde{S}(m, B)$ if the function*

$$\Gamma \ni \gamma \mapsto \frac{1}{m(\gamma)} \|\chi_\gamma^w a\|$$

belongs to B. Here χ_γ is the partiction of unity (2.2).

Proposition 2.2 extends to

Proposition 7.3. *$\widetilde{S}(m, B)$ is a Banach space with the natural norm. If we replace Γ, χ, B by $\widetilde{\Gamma}, \widetilde{\chi}, \widetilde{B}$, having the same properties, and with $\widetilde{B} \subset \ell^\infty(\widetilde{\Gamma})$ equivalent to B, and if we further replace the L^2 norm by the L^p norm for any $p \in [1, \infty]$, we get the same space, equipped with an equivalent norm.*

P r o o f. It suffices to follow the proof of Proposition 2.2: From the estimate (2.4) we get for any $N \geq 0$,

$$\frac{1}{m(\widetilde{\gamma})} \|\chi_{\widetilde{\gamma}}^w a\|_{L^p} \leq C_{p,N} \sum_{\gamma \in \Gamma} \langle \widetilde{\gamma} - \gamma \rangle^{-n} \frac{1}{m(\gamma)} \|\chi_\gamma^w a\|_{L^2},$$

where we also used that m is an order function. Hence, since B, $\widetilde{B}$ are equivalent,

$$\left\| \frac{1}{m(\cdot)} \right\| \widetilde{\chi}_a^w \cdot \|_{L^p} \|_{\widetilde{B}} \leq \left\| \frac{1}{m(\cdot)} \right\| \chi_\cdot^w a \|_{L^2} \|_B.$$

The reverse estimate is obtained the same way. □

As a preparation for the use of Bargmann transforms, we next develop a "continuous" version of B-spaces; a kind of amalgam spaces in the sense of [22, 13, 15]. Let Γ be a lattice in a d-dimensional real vector space E and let $B \subset \ell^\infty(\Gamma)$ satisfy (7.1), (7.4). Let $0 \leq \chi \in C_0^\infty(E)$ satisfy $\sum_{\gamma \in \Gamma} \tau_\gamma \chi > 0$.

Definition 7.4. *We say that the locally bounded measurable function $u : E \to \mathbf{C}$ is of class $[B]$, if there exists $v \in B$ such that*

$$|u(x)| \leq \sum_{\gamma \in \Gamma} v(\gamma) \tau_\gamma \chi(x). \tag{7.9}$$

The space of such functions is a Banach space that we shall denote by $[B]$, equipped with the norm

$$\|u\|_{[B]} = \inf\{\|v\|_B;\ (7.9) \text{ holds }\}. \tag{7.10}$$

This space does not depend on the choice of χ and we may actually characterize it as the space of all locally bounded measurable functions u on E such that

$$|u(x)| \le \sum_{\gamma\in\Gamma} w(\gamma)\langle x-\gamma\rangle^{-N},\ \text{for some } w\in B, \tag{7.11}$$

where $N > d$ is any fixed number. Clearly (7.8) implies (7.11). Conversely, if u satisfies (7.11) and χ is as in Definition 7.4, then

$$\langle x\rangle^{-N} \le C\sum_{\alpha\in\Gamma}\langle\alpha\rangle^{-N}\tau_\alpha\chi(x),$$

so if (7.11) holds, we have,

$$\begin{aligned} |u(x)| &\le C\sum_\gamma w(\gamma)\sum_\alpha\langle\alpha\rangle^{-N}\chi(x-(\gamma+\alpha)) \\ &= C\sum_\beta(\langle\cdot\rangle^{-N} * w)(\beta)\chi(x-\beta), \end{aligned}$$

and $\langle\cdot\rangle^{-N} * w \in B$.

Similarly, the definition does not change if we replace $B\subset\ell^\infty(\Gamma)$ by an equivalent space $\widetilde{B}\subset\ell^\infty(\widetilde{\Gamma})$.

Let m_1, m_2, m_3 be order functions on $E_1\times E_2$, $E_2\times E_3$, $E_1\times E_3$ respectively, where E_j is a real vectorspace of dimension d_j. Let $\Gamma_j\subset E_j$ be lattices and let

$$B_1\subset\ell^\infty(\Gamma_1\times\Gamma_2),\ B_2\subset\ell^\infty(\Gamma_2\times\Gamma_3),\ B_3\subset\ell^\infty(\Gamma_1\times\Gamma_3)$$

be Banach spaces satisfying (7.1), (7.4). Introduce the

Assumption 7.5. *If $k_j\in m_jB_j$, $j=1,2$, then*

$$k_3(\alpha,\beta) := \sum_{\gamma\in\Gamma_2} k_1(\alpha,\gamma)k_2(\gamma,\beta)$$

converges absolutely for every $(\alpha,\beta)\in\Gamma_1\times\Gamma_3$. Moreover, $k_3\in m_3B_3$ and

$$\|k_3/m_3\|_{B_3} \le C\|k_1/m_1\|_{B_1}\|k_2/m_2\|_{B_2}$$

where C is independent of k_1, k_2.

Again, it is an easy exercise to check that the assumption is invariant under changes of the lattices Γ_j and the passage to corresponding equivalent B-spaces.

Proposition 7.6. *We make the Assumption* 7.5*, where B_j satisfy* (7.1)*,* (7.4)*. Let $K_j \in m_j[B_j]$ for $j = 1, 2$ in the sense that $K_j/m_j \in [B_j]$. Then the integral*

$$K_3(x,y) := \int_{E_2} K_1(x,z)K_2(z,y)dz, \ \ (x,y) \in E_1 \times E_3,$$

converges absolutely and defines a function $K_3 \in m_3[B_3]$. Moreover,

$$\|K_3/m_3\|_{[B_3]} \leq C\|K_1/m_1\|_{[B_1]}\|K_2/m_2\|_{[B_2]},$$

where C is independent of K_1, K_2.

Proof. Write

$$\begin{aligned} |K_1(x,z)| &\leq \sum_{\Gamma_1\times\Gamma_2} k_1(\alpha,\gamma)\chi^{(1)}(x-\alpha, z-\gamma) \\ |K_2(z,y)| &\leq \sum_{\Gamma_2\times\Gamma_3} k_2(\gamma,\beta)\chi^{(2)}(z-\gamma, y-\beta), \end{aligned}$$

with $\chi^{(1)} \in C_0^\infty(E_1 \times E_2)$, $\chi^{(2)} \in C_0^\infty(E_2 \times E_3)$ as in Definition 7.4 and with $k_j \in m_j B_j$. Then

$$\begin{aligned} |K_3(x,y) &\leq \int_{E_2} |K_1(x,z)||K_2(z,y)|dz \\ &\leq \sum_{\substack{(\alpha,\beta)\in\Gamma_1\times\Gamma_3 \\ \gamma,\gamma'\in\Gamma_2}} k_1(\alpha,\gamma)k_2(\gamma',\beta)F(x-\alpha, y-\beta; \gamma-\gamma'), \end{aligned}$$

where

$$\begin{aligned} F(x,y;\gamma-\gamma') &= \int \chi^{(1)}(x, z-\gamma)\chi^{(2)}(z-\gamma', y)dz \\ &= \int \chi^{(1)}(x, z-(\gamma-\gamma'))\chi^{(2)}(z,y)dz. \end{aligned}$$

We notice that $0 \leq F(x,y;\gamma) \in C_0^\infty(E_1 \times E_3)$ and that $F(x,y;\gamma) \not\equiv 0$ only for finitely many $\gamma \in \Gamma$. Hence for some $R_0 > 0$,

$$|K_3(x,y)| \leq \sum_{|\gamma|\leq R_0} \sum_{(\alpha,\beta)\in\Gamma_1\times\Gamma_3} \left(\sum_{\gamma'} k_1(\alpha, \gamma'+\gamma)k_2(\gamma',\beta) \right) F(x-\alpha, y-\beta; \gamma).$$

Since

$$\frac{1}{m_1(\cdot,\cdots)}k_1(\cdot,\cdots+\gamma)\in B_1,$$

for every fixed γ, and $k_2/m_2 \in B_2$, the Assumption 7.5 implies that

$$k_3(\alpha,\beta;\gamma):=\sum_{\gamma'}k_1(\alpha,\gamma'+\gamma)k_2(\gamma',\beta)\in m_3B_3,$$

for every $\gamma\in\Gamma$. The proposition follows. □

We next generalize (5.1). Let $F=\mathbf{R}^d$ and define $T: L^2(F)\to H_\Phi(F^{\mathbf{C}})$ as in (3.8)–(3.11). Let m be an order function on $F\times F^*$, let $\Gamma\subset F\times F^*$ be a lattice and let $B\subset\ell^\infty(\Gamma)$ satisfy (7.1), (7.4). Then we get

Proposition 7.7. *We have*

$$\widetilde{S}(m,B)=\{u\in\mathcal{S}'(F);\ \frac{1}{m}\Big((e^{-\Phi}Tu)\circ\pi\circ\kappa_T\Big)\in[B]\},\tag{7.12}$$

where $\pi:\Lambda_\Phi\ni(x,\xi)\mapsto x\in F^{\mathbf{C}}$ *is the natural projection.*

Proof. This will be a simple extension of the proof of (5.1). As there, we identify $F^{\mathbf{C}}$ with $F\times F^*$ by means of $\pi\circ\kappa_T$ and work on the latter space. Assume first that $u\in\widetilde{S}(m,B)$ and write $u=\sum_{\gamma\in\Gamma}\psi_\gamma^w\chi_\gamma^w u$ as in Lemma 2.3, so that $(\|\chi_\gamma^w u\|)_{\gamma\in\Gamma}\in mB$. Using (5.2), we see that

$$|e^{-\Phi}Tu(x)|\le C_N\sum_{\gamma\in\Gamma}\|\chi_\gamma^w u\|\langle x-\gamma\rangle^{-N},$$

and hence $e^{-\Phi}Tu\in m[B]$, i.e. u belongs to the right hand side of (7.12) (with the identification $\pi\circ\kappa_T$).

Conversely, if $e^{-\Phi}Tu\in m[B]$, then since the effective kernel of χ_γ^w satisfies (5.2), we see that

$$|e^{-\Phi}T\chi_\gamma^w u(x)|\le C_N\int\langle x-\gamma\rangle^{-N}\langle y-\gamma\rangle^{-N}\sum_{\alpha\in\Gamma}\langle y-\alpha\rangle^{-N}a_\alpha dy,$$

where $(a_\alpha)\in mB$. It follows that

$$|e^{-\Phi}T\chi_\gamma^w u(x)|\le\widetilde{C}_N\langle x-\gamma\rangle^{-N}\sum_{\alpha\in\Gamma}\langle\gamma-\alpha\rangle^{-N}a_\alpha=\widetilde{C}_N\langle x-\gamma\rangle^{-N}b_\gamma,$$

where $(b_\gamma)_{\gamma\in\Gamma}\in mB$, and hence $\|\chi_\gamma^w u\|\le\widehat{C}_N b_\gamma$, so $u\in\widetilde{S}(m,B)$. □

From this, we deduce as in (5.16) that if $a \in \mathcal{S}'(E)$, $E = F \times F^*$, then $a \in \widetilde{S}(m, B)$ iff

$$K^{\mathrm{eff}}_{a^w}\left(t - \frac{1}{2}Jt^*, t + \frac{1}{2}Jt^*\right) \in m[B], \tag{7.13}$$

where $K^{\mathrm{eff}}_{a^w}$ is the effective kernel of a^w in (5.6), (5.7) after identification of $\mathbf{C}^d = F^{\mathbf{C}}$ with E via the map $\pi \circ \kappa_T = E \to F^{\mathbf{C}}$. We recall the identity (5.17) for the composition of two symbols.

(7.13) can also be written

$$K^{\mathrm{eff}}_{a^w}(x, y) \in \widetilde{m}[\widetilde{B}], \text{ where } \widetilde{m} = m \circ q,\ [\widetilde{B}] = [B] \circ q, \tag{7.14}$$

where q is given in (4.15).

The following generalization of Theorem 4.2 now follows from Proposition 7.6.

Theorem 7.8. *For $j = 1, 2, 3$, let m_j be an order function $E \times E^*$, where $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$, let $\Gamma_j \subset E \times E^*$ be a lattice and let $B_j \subset \ell^\infty(\Gamma_j)$ satisfy* (7.1), (7.4). *Let $\widetilde{m}_j = m_j \circ q$, $\widetilde{\Gamma}_j = q^{-1}(\Gamma_j)$, $\ell^\infty(\widetilde{\Gamma}_j) \supset \widetilde{B}_j = B_j \circ q$. Assuming (as we may without loss of generality) that $\widetilde{\Gamma}_j = \Gamma \times \Gamma$ where $\Gamma \subset E$ is a lattice, we make the Assumption* 7.5 *for $\widetilde{m}_j \widetilde{B}_j$.*

Then if $a_j \in \widetilde{S}(m_j, B_j)$, $j = 1, 2$, the composition $a_3 = a_1 \# a_2$ is well defined and belongs to $\widetilde{S}(m_3, B_3)$, in the sense that the corresponding composition of effective kernels in (5.17) *is given by an absolutely convergent integral and $K^{\mathrm{eff}}_{a_3^w} \in \widetilde{m}_3[\widetilde{B}_3]$.*

We next consider the action of pseudodifferential operators on generalized symbol spaces. Our result will be essentially a special case of the preceding theorem. We start by "contracting" Assumption 7.5 to the case when $E_3 = 0$.

Let m_1, m_2, m_3 be order functions on $E_1 \times E_2$, E_2, E_1 respectively. Let $\Gamma_j \subset E_j$, $j = 1, 2$ be lattices and let

$$B_1 \subset \ell^\infty(\Gamma_1 \times \Gamma_2),\ B_2 \subset \ell^\infty(\Gamma_2),\ B_3 \subset \ell^\infty(\Gamma_1)$$

be Banach spaces satisfying (7.1), (7.4). Assumption 7.5 becomes

Assumption 7.9. *If $k_j \in m_j B_j$, $j = 1, 2$, then*

$$k_3(\alpha) = \sum_{\beta \in \Gamma_2} k_1(\alpha, \beta) k_2(\beta)$$

converges absolutely for every $\alpha \in \Gamma_1$, *and we have* $k_3 \in m_3 B_3$. *Moreover,*

$$\|k_3/m_3\|_{B_3} \le C \|k_1/m_1\|_{B_1} \|k_2/m_2\|_{B_2}$$

where C *is independent of* k_1, k_2.

The corresponding "contraction" of Proposition 7.6 becomes

Proposition 7.10. *Let Assumption* 7.9 *hold, where* B_j *satisfy* (7.1), (7.4). *Let* $K_j \in m_j[B_j]$ *for* $j = 1, 2$. *Then the integral*

$$K_3(x) := \int_{E_2} K_1(x, z) K_2(z) dz, \;\; x \in E_1,$$

converges absolutely and defines a function $K_3 \in m_3[B_3]$. *Moreover,*

$$\|K_3/m_3\|_{[B_3]} \le C \|K_1/m_1\|_{[B_1]} \|K_2/m_2\|_{[B_2]},$$

where C *is independent of* K_1, K_2.

We get the following result for the action of pseudodifferential operators on generalized symbol spaces.

Theorem 7.11. *Let* m_2, m_3 *be order functions on* $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$ *and let* m_1 *be an order function on* $E \times E^*$. *Let* $\widehat{\Gamma} \subset E \times E^*$ *be a lattice such that* $\widetilde{\Gamma} := q^{-1}(\widehat{\Gamma}) = \Gamma \times \Gamma$ *where* $\Gamma \subset E$ *is a lattice. Let* $\widehat{B}_1 \subset \ell^\infty(\widehat{\Gamma})$, $B_2, B_3 \subset \ell^\infty(\Gamma)$ *satisfy* (7.1), (7.4). *We make the Assumption* 7.9 *with* $\Gamma_1, \Gamma_2 = \Gamma$ *and with* m_1, B_1 *replaced with* $\widetilde{m}_1 = m_1 \circ q$, $\widetilde{B}_1 = B_1 \circ q$, *where* q *is given in* (4.15).

Then, if $a_1 \in \widetilde{S}(m_1, B_1)$, $u \in \widetilde{S}(m_2, B_2)$, *the distribution* $v = a_1^w(u)$ *is well-defined in* $\widetilde{S}(m_3, B_3)$ *in the sense that*

$$e^{-\Phi(x)} T v(x) = \int K^{\mathrm{eff}}_{a_1^w}(x, y) e^{-\Phi(y)} T u(y) L(dy),$$

with $K^{\mathrm{eff}}_{a_1^w}(x, y)$ *as in* (5.6), *converges absolutely for every* $x \in \mathbf{C}^n$ *and*

$$\frac{1}{m_3}((e^{-\Phi} T v) \circ \pi \circ \kappa_T) \in [B_3],$$

as in (7.12).

We shall finally generalize Theorem 6.2.

Theorem 7.12. *Let* $p \in [1, \infty]$ *and let* m *be an order function on* $E \times E^*$ *where* $E = \mathbf{R}^n \times (\mathbf{R}^n)^*$. *Let* $\Gamma \subset E$ *be a lattice and* $B \subset \ell^\infty(q(\Gamma \times \Gamma))$ *a Banach space satisfying* (7.1), (7.4). *Assume that*

$$\text{(7.15)} \qquad \text{if } (a_{\alpha,\beta})_{\alpha,\beta \in \Gamma} \in (m \circ q) B \circ q, \text{ then } (a_{\alpha,\beta}) \in C_p(\ell^2(\Gamma), \ell^2(\Gamma)) \text{ and } \|(a_{\alpha,\beta})\|_{C_p} \le C \|(a_{\alpha,\beta})\|_{(m \circ q) B \circ q},$$

where q is given in (4.15) *and* $C > 0$ *is independent of* $(a_{\alpha,\beta})$*. Then there is a (new) constant* $C > 0$ *such that*

$$\text{If } a \in \widetilde{S}(m,B), \text{ then } a^w \in C_p(L^2,L^2) \text{ and } \|a^w\|_{C_p} \le C\|a\|_{\widetilde{S}(m,B)}. \tag{7.16}$$

The proof of Proposition 6.1 shows that the property (7.15) is invariant under changes $(\Gamma,B) \mapsto (\widetilde{\Gamma},\widetilde{B})$ with $\widetilde{B} \subset \ell^\infty(q(\widetilde{\Gamma}\times\widetilde{\Gamma}))$ equivalent to B.

Proof. We follow the proof of Theorem 6.2. Assume that (7.15) holds and let $a \in \widetilde{S}(m,B)$ be of norm ≤ 1. It suffices to show that $A_{\mathrm{eff}} : L^2(\mathbf{C}^n) \to L^2(\mathbf{C}^n)$ is in C_p with norm $\le C$, where A_{eff} is given in (6.4) and K^{eff} there belongs to $m \circ q[B \circ q]$, provided that we identify $\mathbf{C}^n$ with E via $\pi \circ \kappa_T$.

We see that we still have (6.9) where (6.10) should be replaced by

$$\begin{aligned} &|\nabla_x^k \nabla_y^\ell K_{\alpha,\beta}(x,y)| \le C_{k,\ell} a_{\alpha,\beta}, \ |x-\alpha|, |y-\beta| \le C_0, \\ &(a_{\alpha,\beta})_{\alpha,\beta\in\Gamma} \in (m\circ q) B \circ q, \ \alpha,\beta\in\Gamma. \end{aligned} \tag{7.17}$$

Write $A_{\mathrm{eff}} = W^*\mathcal{A}W$ as in (6.12),

$$\mathcal{A} : \bigoplus_{\beta\in\Gamma} L^2(\Omega_\beta) \to \bigoplus_{\beta\in\Gamma} L^2(\Omega_\beta), \quad \mathcal{A} = (A_{\alpha,\beta}).$$

The matrix elements $K_{\alpha,j;\beta;k}$ of $A_{\alpha,\beta}$ now obey the estimate (cf. (6.13)):

$$|K_{\alpha,j;\beta,k}| \le C_N \langle j\rangle^{-N} \langle k\rangle^{-N} a_{\alpha,\beta} \tag{7.18}$$

with $a_{\alpha,\beta}$ as in (7.18). Using (7.15), this leads to (6.14) and from that point on the proof is identical to that of Theorem 7.12. □

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