Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **33** (2007), 475–494

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

EQUIVALENCE BETWEEN *K*-FUNCTIONALS BASED ON CONTINUOUS LINEAR TRANSFORMS

Borislav R. Draganov*, Kamen G. Ivanov

Communicated by D. Leviatan

Dedicated to Academician Blagovest Sendov on occasion of his 75th birthday

Abstract. The paper presents a method of relating two K-functionals by means of a continuous linear transform of the function. In particular, a characterization of various weighted K-functionals by unweighted fixed-step moduli of smoothness is derived. This is applied in estimating the rate of convergence of several approximation processes.

1. *K*-functionals in measuring the approximation error.

Finding a good estimate of the error of a given approximation process is a basic problem in approximation theory. The so called K-functional turned out to be

2000 *Mathematics Subject Classification:* 46B70, 41A10, 41A25, 41A27, 41A35, 41A36, 42A10.

Key words: K-functional, modulus of smoothness, rate of convergence, best approximation, linear operator.

*Partially supported by grant No. 103/2007 of the National Science Fund of the Sofia University.

very useful in this respect. It has been introduced by Lions and by Peetre and in its present form by Peetre [28], as a basis of the theory of interpolation of operators (e.g. [2, Ch. 5]). Later Butzer and Berens [4] clarified its importance in approximation theory.

Let X be a Banach space with norm $\|\cdot\|_X$ and Y be another space with a semi-norm $|\cdot|_Y$. For $f \in X$ and $t > 0$ we define the K-functional between the spaces X and Y by

$$K(f,t;X,Y) = \inf\{\|f-g\|_X + t|g|_Y : g \in Y \cap X\}. \tag{1.1}$$

Usually Y is a dense subspace of X and consists of elements that possess certain additional properties as, for example, high smoothness. As it is seen from its definition the K-functional measures how well a given function $f \in X$ can be approximated by elements $g \in Y$ with control of their semi-norm $|g|_Y$. For given $f \in X$ the K-functional is a non-negative and non-decreasing function of t. Thus, $\lim_{t\to 0+} K(f,t;X,Y)$ always exists. Moreover, this limit is 0 for every $f \in X$ iff Y is dense in X.

For many applications in approximation theory X is a weighted L_p-space $L_p(w)(I) = \{f \in L_{1,loc}(I) : wf \in L_p(I)\}$, $1 \le p \le \infty$, with a norm $\|f\|_X = \|wf\|_{p(I)}$ and Y is a weighted Sobolev space $W_p^r(\phi)(I) = \{g \in AC_{loc}^{r-1}(I) : \phi g^{(r)} \in L_p(I)\}$ with a semi-norm $|g|_Y = \|\phi g^{(r)}\|_{p(I)}$, where I is an interval on the real line and w, ϕ are measurable on it with singularities only at its ends.

A standard application of the K-functional is the direct theorem for the error of an approximation process. If $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ is a family of linear operators that maps the Banach space X into itself such that

a) $\|\mathcal{L}_\alpha f\|_X \le c\,\|f\|_X$ for every $f \in X$ and $\alpha \in A$,

b) $\|g - \mathcal{L}_\alpha g\|_X \le c\,\theta(\alpha)\,|g|_Y$ for every $g \in Y$ and $\alpha \in A$ with $\theta : A \to \mathcal{R}_+$,

then for every $f \in X$ and $\alpha \in A$ there holds the estimate

$$\|f - \mathcal{L}_\alpha f\|_X \le c\,K(f,\theta(\alpha);X,Y). \tag{1.2}$$

Above and in what follows we denote by c positive constants not necessarily the same on each occurrence that do not depend on the function f and the parameter t in the K-functional.

Usually the estimates converse to (1.2) are of weak type but in a number of cases connected with saturated approximation processes strong type inverse inequalities can be established. Let $\mathcal{D}$ be a differential operator, $Y = \{g \in X :$

$\mathcal{D}g \in X\}$ and $|g|_Y = \|\mathcal{D}g\|_X$. Ditzian and Ivanov proved in [9] that the inverse inequality to (1.2)

$$K(f, \theta(\alpha); X, Y) \leq c\, \|f - \mathcal{L}_\alpha f\|_X \tag{1.3}$$

follows from a) and the inequalities:

c) $\|g - \mathcal{L}_\alpha g - \theta(\alpha)\mathcal{D}g\|_X \leq \psi(\alpha)\, |g|_Z$ for every $g \in Z$,

d) $\|\mathcal{D}\mathcal{L}_\alpha^k f\|_X \leq \dfrac{c}{\theta(\alpha)}\, \|f\|_X$ for every $f \in X$,

e) $|\mathcal{L}_\alpha^\ell g|_Z \leq M\, \dfrac{\theta(\alpha)}{\psi(\alpha)}\, \|\mathcal{D}\mathcal{L}_\alpha^k g\|_X$ for every $g \in X$,

where $Z \subset X$, $\theta : A \to \mathcal{R}_+$ is bounded, $\psi : A \to \mathcal{R}_+$, $k, \ell \in \mathcal{N}$, $k \leq \ell$, and $M < 1$ is a constant.

Thus, inequalities (1.2) and (1.3) imply that the approximation error and the K-functional are equivalent, which we denote by

$$\|f - \mathcal{L}_\alpha f\|_X \sim K(f, \theta(\alpha); X, Y). \tag{1.4}$$

So, using K-functionals we can derive an estimate of the approximation error from several inequalities. However, let us note that in general it is difficult to establish some of inequalities c)-e), especially e) with a *small* constant on the right-hand side. The error estimates through K-functionals are of high importance in approximation theory but they solely are of little effectiveness because it is difficult to evaluate for a given f and every $t \in (0, t_0]$ the infimum *on the wide space* Y. This shortcoming can be overcome by defining a new functional characteristic $\Omega(f, t)$, called modulus of smoothness, which is equivalent to the K-functional. The modulus of smoothness depends on f more directly and can be more easily estimated being a supremum or an average on a neighbourhood of the origin *in a finite dimensional space.*

2. Moduli of smoothness. Let $X = L_p(I)$, $I \subseteq \mathcal{R}$ is an interval, $1 \leq p \leq \infty$, with the usual L_p-norm on the interval I, and $Y = W_p^r(I) = \{g \in AC_{loc}^{r-1}(I) : g^{(r)} \in L_p(I)\}$ – the Sobolev space with the semi-norm $|g|_{W_p^r(I)} = \|g^{(r)}\|_{p(I)}$. It is well known (see [24, 25]) that for every $f \in L_p(I)$ and $0 < t \leq t_0$ we have

$$K(f, t^r; L_p(I), W_p^r(I)) \sim \omega_r(f, t)_{p(I)}, \tag{2.1}$$

where $\omega_r(f,t)_{p(I)}$ is the classical unweighted fixed-step modulus of smoothness of order r of the function f, namely,

$$\omega_r(f,t)_{p(I)} = \sup_{0<h\le t} \|\Delta_h^r f\|_{p(I)}$$

and $\Delta_h^r f(x)$ is the finite difference of the function f of order r and step h. We assume that $\Delta_h^r f(x) = 0$ if the argument of any of the summands of the finite differences $\Delta_h^r f(x)$ is outside I. Thus, if we consider, for example, symmetric finite differences:

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h) \tag{2.2}$$

of functions $f \in L_p[a,b]$, where $[a,b]$ is a finite interval, we have

$$\omega_r(f,t)_{p[a,b]} = \sup_{0<h\le t} \|\Delta_h^r f\|_{p[a+rh/2,b-rh/2]}.$$

We also set $\omega_0(f,t)_{p(I)} = \|f\|_{p(I)}$. Let us note that for 2π-periodic functions the r-th modulus $\omega_r(f,t)_{p,2\pi}$ is defined in a slightly different way – the norm is taken on an arbitrary period but the convention for vanishing of the finite difference is not applied.

Let w and φ be power-type weights with singularities only at the ends of the interval $I \subseteq \mathcal{R}$. Ditzian and Totik [10, pp. 56, 218] introduced the varying-step moduli of smoothness, which for a finite interval $I = [a,b]$ are defined by

$$\begin{aligned} \omega_\varphi^r(f,t)_{w,p[a,b]} = \sup_{0<h\le t} \|w\Delta_{h\varphi}^r f\|_{p[a+t_a^*,b-t_b^*]} \\ + \sup_{0<h\le t_a^*} \|w\overrightarrow{\Delta}_h^r f\|_{p[a,a+12t_a^*]} + \sup_{0<h\le t_b^*} \|w\overleftarrow{\Delta}_h^r f\|_{p[b-12t_b^*,b]}, \end{aligned} \tag{2.3}$$

where $\overrightarrow{\Delta}_h^r$ and $\overleftarrow{\Delta}_h^r$ denote forward and backward r-th finite differences respectively and t_a^*, t_b^* are functions of t and r depending on the behavior of φ at the respective end-points.

Under certain conditions, the most important of which are the "finite overlapping condition" on φ and the boundness of w at a finite end-point of I, Ditzian and Totik proved in [10, Ch. 2 and Ch. 6] the equivalence

$$K(f,t^r; L_p(w)(I), W_p^r(w\varphi^r)(I)) \sim \omega_\varphi^r(f,t)_{w,p(I)},$$

where the K-functional is defined for $X = L_p(w)(I)$, $1 \le p \le \infty$, and $Y = W_p^r(w\varphi^r)(I)$ with the semi-norm $|g|_{W_p^r(w\varphi^r)(I)} = \|w\varphi^r g^{(r)}\|_{p(I)}$.

On the other hand, the second author introduced the following moduli of smoothness

$$\tau_r(f;\psi(t))_{q,p(I)} = \|\omega_r(f,\cdot;\psi(t,\cdot))_q\|_{p(I)}, \tag{2.4}$$

where the local moduli are given by

$$\omega_r(f,x;\psi(t,x))_q = \left((2\psi(t,x))^{-1}\int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^r f(x)|^q\,dh\right)^{1/q}, \ 1\le q<\infty,$$

$$\omega_r(f,x;\psi(t,x))_\infty = \sup\{|\Delta_h^r f(x)| : |h|\le\psi(t,x)\}$$

and ψ is a continuous function connected with φ in a certain way (see [21, 22]). Under certain conditions on φ (see [22]), Ivanov proved the relation

$$K(f,t^r;L_p(I),W_p^r(\varphi^r)(I)) \sim \tau_r(f;\psi(t))_{p,p(I)}.$$

The power and logarithmic-type weights φ are covered. The first characterization, based on the standard translation operator, of the best approximations in $L_p[a,b]$, $1\le p\le\infty$, by algebraic polynomials was established in the terms of τ-moduli [21].

Let us also mention that Ky [27] defined moduli through which he characterized K-functionals of the type (1.1) with $X = L_p(w)(I)$, $I\subseteq\Re$ is an interval, $1\le p\le\infty$, and $Y = W_p^r(w)(I)$, where the weight w is bounded and satisfies certain monotonicity requirements near the end-points of the interval.

M. K. Potapov (see [29] and the references cited there) characterized the K-functional (1.1) with $X = L_p(w)[-1,1]$ with Jacobean weight w, and $Y = \{g\in AC_{loc}^{2r-1}[-1,1] : D_{\nu,\mu}^r g\in L_p[-1,1]\}$ with the semi-norm $|g|_Y = \|D_{\nu,\mu}^r g\|_{p[-1,1]}$, where

$$D_{\nu,\mu} = (1-x)^{-\nu}(1+x)^{-\mu}\frac{d}{dx}(1-x)^{\nu+1}(1+x)^{\mu+1}\frac{d}{dx}$$

is the Jacobean differential operator. The moduli introduced by Potapov are based on generalized translation operators.

Also Butzer, Stens and Wehrens [3, 5, 6] defined moduli of smoothness by means of multipliers and generalized translations to characterize the best weighted algebraic approximation with Jacobean weights.

3. Equivalence between K-functionals. The main purpose of this paper is to present a general approach to establishing equivalence between two K-functionals of the type:

$$K(f,t;X_1,Y_1) \sim K(\mathcal{A}f,t;X_2,Y_2),$$

where $\mathcal{A} : X_1 \to X_2$ is a bounded linear operator. Hence, in view of (2.1) for $X_2 = L_p(I)$, $Y_2 = W_p^r(I)$, $|G|_{W_p^r(I)} = \|G^{(r)}\|_{p(I)}$, $I \subseteq \mathcal{R}$ is an interval, we get the following characterization of the K-functional $K(f, t; X_1, Y_1)$ by means of the unweighted fixed step-modulus of smoothness:

$$K(f, t^r; X_1, Y_1) \sim \omega_r(\mathcal{A}f, t)_{p(I)}.$$

We have (cf. [16, Definition 2.1 and Proposition 2.1] and [13, Definition 1.1 and Proposition 2.1])

Theorem 3.1. *If there exists a linear operator $\mathcal{B} : X_2 \to X_1$, related to $\mathcal{A} : X_1 \to X_2$, and both operators satisfy the conditions:*

a) $\|\mathcal{A}f\|_{X_2} \le c\,\|f\|_{X_1}$ *for every* $f \in X_1$*;*

b) $\mathcal{A}g \in Y_2 \cap X_2$ *and* $|\mathcal{A}g|_{Y_2} \le c\,|g|_{Y_1}$ *for every* $g \in Y_1 \cap X_1$*;*

c) $\|\mathcal{B}F\|_{X_1} \le c\,\|F\|_{X_2}$ *for every* $F \in X_2$*;*

d) $\mathcal{B}G \in Y_1 \cap X_1$ *and* $|\mathcal{B}G|_{Y_1} \le c\,|G|_{Y_2}$ *for any* $G \in Y_2 \cap X_2$*;*

e) $|f - \mathcal{B}\mathcal{A}f|_{Y_1} = 0$ *for every* $f \in X_1$*;*

f) $|F - \mathcal{A}\mathcal{B}F|_{Y_2} = 0$ *for every* $F \in X_2$*.*

Then

$$K(f, t; X_1, Y_1) \sim K(\mathcal{A}f, t; X_2, Y_2) \tag{3.1}$$

and

$$K(F, t; X_2, Y_2) \sim K(\mathcal{B}F, t; X_1, Y_1).$$

Remark 3.1. Let us note that to get only (3.1) it is sufficient condition f) to be fulfilled only for $F \in \mathcal{A}(X_1)$.

In some cases in order to apply Theorem 3.1 more effectively in establishing a characterization of K-functionals $K(f, t^r; L_p(w)(I), W_p^r(w\varphi^r)(I))$ by the unweighted fixed-step moduli we shall split the singularities of the weights. More precisely, let $-\infty \le \bar{a} < a_1 < b_1 < \bar{b} \le \infty$, $I = (\bar{a}, \bar{b})$, $I_1 = (\bar{a}, b_1)$ and $I_2 = (a_1, \bar{b})$. Let w and φ be non-negative measurable on I weights such that $w \sim 1$ and $\varphi \sim 1$

on $[a_1, b_1]$. Then for $r \in \mathcal{N}$, $1 \le p \le \infty$, $0 < t \le b_1 - a_1$ and $f \in L_p(w)(I)$ we have (cf. [8, p. 176, Lemma 2.3] and [16, Lemma 7.1]):

$$\begin{aligned} K(f,t;L_p(w)(I),W_p^r(w\varphi^r)(I)) &\sim K(f,t;L_p(w)(I_1),W_p^r(w\varphi^r)(I_1)) \\ &+ K(f,t;L_p(w)(I_2),W_p^r(w\varphi^r)(I_2)). \end{aligned} \tag{3.2}$$

Above we have used one and the same notation for the function f and for its restrictions on subdomains.

Let us note that in the applications we require operator $\mathcal{A}$ from Theorem 3.1 to be constructed explicitly and the computations of $\omega_r(\mathcal{A}f,t)_p$ and $\omega_r(f,t)_p$ to be equally difficult. The latter means that the operations in constructing $\mathcal{A}$ are addition and multiplication by elementary functions, change of the variable by elementary functions and integration.

4. Application. Now, we give several applications of Theorem 3.1 in establishing characterizations of K-functionals in terms of the unweighted fixed-step moduli of smoothness.

Let us consider the K-functional

$$\begin{aligned} &K(f,t;L_p(w)(I),W_p^r(w\varphi^r)(I)) \\ &\qquad = \inf\left\{\|w(f-g)\|_{p(I)} + t\|w\varphi^r g^{(r)}\|_{p(I)} \;:\; g \in AC_{loc}^{r-1}(I)\right\}. \end{aligned} \tag{4.1}$$

The domain I may be finite, semi-infinite or infinite, represented respectively by $I = [a,b]$, $I = [a,\infty)$ and $I = (-\infty,\infty)$. To define the weights w and φ we set $\chi_\xi(x) = |x-\xi|$ for a real number ξ. For a finite domain $I = [a,b]$ we consider the weights $w = \chi_a^{\gamma_a}\chi_b^{\gamma_b}$ and $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ with $\gamma_a, \gamma_b, \lambda_a, \lambda_b \in \mathcal{R}$. For a semi-infinite domain $I = [a,\infty)$ we consider the weights $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ (note that $w(x)/x^{\gamma_\infty} \to 1$ for $x \to \infty$) and $\varphi = \chi_a^{\lambda_a}\chi_{a-1}^{\lambda_\infty-\lambda_a}$ with $\gamma_a, \gamma_\infty, \lambda_a, \lambda_\infty \in \mathcal{R}$. For the infinite domain $I = (-\infty,\infty)$ we apply (3.2) to reduce the case to semi-infinite domain.

It is demonstrated in [16] that the case $\lambda_a > 1$ with a finite end-point a of the domain I is equivalent to the case $\lambda_\infty < 1$ (transfer to infinite end-point), as well as the case $\lambda_\infty > 1$ is equivalent to the case $\lambda_a < 1$. So the two main cases in characterization of the K-functional (4.1) are i) $\lambda_a < 1$ and $\lambda_\infty < 1$ (see Subsection 4.1) and ii) $\lambda_a = 1$ and $\lambda_\infty = 1$ (see Subsection 4.2). In order to solve the first case we apply power change of the variable and for the second case – exponential change of the variable. In the remaining subsections we consider applications in which the differential operator $\varphi^r D^r$ determining the second term of the K-functional is replaced by a linear differential operator of the form $P(D)$, where P is a polynomial with constant coefficients in Subsection 4.3 or with varying coefficients in Subsection 4.4.

Also to describe the restrictions on the powers γ_a, γ_b or γ_∞ of the weight w we set for $r \in \mathcal{N}$ and $1 \le p \le \infty$

$$\begin{aligned}
&\Gamma_+(p) = (-1/p, \infty),\ p < \infty,\ \text{and } \Gamma_+(\infty) = [0, \infty);\\
&\Gamma_0(p) = (-1/p, \infty);\\
&\Gamma_i(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 1, \dots, r-1;\\
&\Gamma_r(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1/p\}.
\end{aligned}$$

4.1. Power change of the variable. K-functionals with $\lambda_a < 1$ and/or $\lambda_\infty < 1$ are related to the best approximation by algebraic polynomials on a finite interval, to the approximation error of the Bernstein, Szasz-Mirakian and Baskakov operators, etc. [10, 21, 26, 31].

Let $r \in \mathcal{N}$ and $\xi \in (a, b)$. Let s be one of the ends of the finite interval $[a, b]$ and e – the other.

For $\rho \in \mathcal{R}$, $i \in \mathcal{N}_0$, $i \le r$, $x \in (a, b)$ and $f \in L_{1,loc}[a, b]$, satisfying the additional requirement $\chi_s^{-i+\rho} f \in L_1[s, (s+e)/2]$ if $i > 0$, we set

$$\begin{aligned}
(A_i(\rho; s, e; \xi) f)(x) &= \left(\frac{x-s}{e-s}\right)^{\rho} f(x)\\
&\quad + \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho) \left(\frac{x-s}{e-s}\right)^{k-1} \int_s^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\, dy\\
&\quad + \frac{1}{e-s} \sum_{k=i+1}^{r} \alpha_{r,k}(\rho) \left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\, dy,
\end{aligned}$$

where

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!} \binom{r-1}{k-1} \prod_{\nu=0}^{r-1} (\rho + r - k - \nu), \quad k = 1, 2, \dots, r.$$

As usual, above and in what follows we assume that the sum is 0 if the upper bound is smaller than the lower.

For $\sigma > 0$, $i \in \mathbb{N}$, $i \le r$, $x \in (a,b)$ and $f \in L_{1,loc}[a,b]$, satisfying the additional requirement $\chi_s^{(1-i)/\sigma-1} f \in L_1[s,(s+e)/2]$ if $i > 1$, we set

$$\begin{aligned}
&(B_i(\sigma;s,e;\xi)f)(x) = f\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^{\sigma}\right)\\
&+ \frac{1}{e-s}\sum_{k=2}^{i} \beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_s^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s+(e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right)dy,\\
&+ \frac{1}{e-s}\sum_{k=i+1}^{r} \beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s+(e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right)dy,
\end{aligned}$$

where

$$\beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!}\binom{r-2}{k-2}\prod_{i=1}^{r-1}(k-1-i\sigma), \quad k = 2,3,\dots,r.$$

By means of these operators and their modifications we can construct operators that satisfy Theorem 3.1 above for K-functionals $K(f,t;L_p(\chi_a^{\gamma_a}\chi_b^{\gamma_b})[a,b], W_p^r(\chi_a^{\gamma_a+r\lambda_a}\chi_b^{\gamma_b+r\lambda_b})[a,b])$ with $\lambda_a,\lambda_b \neq 1$. For example, by Propositions 3.9 and 6.2 in [16] (cf. Theorem 6.1 there) we establish

Theorem 4.1. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_a) > 0$, $(1-\lambda_b)(1-\nu_b) > 0$. Let also $\kappa_a,\mu_a,\mu_b \notin \Gamma_{exc}(p)$ and $\kappa_b \in \Gamma_0(p)$. We set $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$, $\tilde{w} = \chi_a^{\mu_a}\chi_b^{\mu_b}$, $\tilde{\varphi} = \chi_a^{\nu_a}\chi_b^{\nu_b}$ and*

$$\begin{aligned}
\mathcal{A} &= A_{i_1}(-\rho;a,b;\xi)A_{i_2}(-\rho_b;b,a;\xi)B_1(\sigma_b^{-1};b,a;\xi)B_1(\sigma_a;a,b;\xi)A_0(\rho_a;a,b;\xi)\\
\mathcal{B} &= A_{i'}(-\rho_a;a,b;\eta)B_1(\sigma_a^{-1};a,b;\eta)B_1(\sigma_b;b,a;\eta)A_0(\rho_b;b,a;\eta)A_0(\rho;a,b;\eta),
\end{aligned}$$

where $\xi,\eta \in (a,b)$, $\rho < \mu_a + 1/p$, the integers i_1,i_2,i' are such that $\Gamma_{i_1}(p) \ni \mu_a$, $\Gamma_{i_2}(p) \ni \mu_b$, $\Gamma_{i'}(p) \ni \kappa_a$, and

$$\begin{aligned}
&\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\lambda_b}{1-\nu_b},\\
&\rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a - \rho + 1/p}{\sigma_a}, \quad \rho_b = \mu_b + \frac{1}{p} - \frac{\kappa_b + 1/p}{\sigma_b}.
\end{aligned}$$

Then for $f \in L_p(w)[a,b]$ and $t > 0$ we have

$$K(f,t;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]) \sim K(\mathcal{A}f,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\tilde{\varphi}^r)[a,b])$$

and for $F \in L_p(\tilde{w})[a,b]$ *and* $t > 0$ *we have*

$$K(F,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\tilde{\varphi}^r)[a,b]) \sim K(\mathcal{B}F,t;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]).$$

Remark 4.1. Interchanging a and b in the definition of $\mathcal{A}$ and $\mathcal{B}$ in the theorem above we get a similar relation between the K-functionals under the hypothesis that $\kappa_a \in \Gamma_0(p)$ and $\kappa_b, \mu_a, \mu_b \notin \Gamma_{exc}(p)$.

Theorem 4.1 and (2.1) imply directly a characterization of the considered K-functional by the ordinary modulus of smoothness but here we present another one given in [16, Theorem 6.2], which is simpler to state.

Theorem 4.2. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$ *and* $\lambda_a, \lambda_b \in (-\infty, 1)$. *For* $p < \infty$ *we assume that* $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as at least one of them is in* $\Gamma_0(p)$, *and for* $p = \infty$ *we assume that* $\kappa_a = \kappa_b = 0$. *We set* $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ *and*

$$\mathcal{A} = B_1(\sigma_b;b,a;\xi)B_1(\sigma_a;a,b;\xi)A_0(\rho_b;b,a;\xi)A_0(\rho_a;a,b;\xi),$$

where $\xi \in (a,b)$ *and*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p}.$$

Then for $f \in L_p(w)[a,b]$ *and* $t > 0$ *we have*

$$K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]) \sim \omega_r(\mathcal{A}f,t)_{p[a,b]}.$$

The operators A and B defined in the beginning of this subsection can also be used when the weight exponent at the end s takes an exceptional value from $\Gamma_{exc}(p)$ but then they change the exponent into one that belongs to $\Gamma_{exc}(p)$ again. More precisely, the following assertion holds for the A-operators.

Proposition 4.1. *Let* $i, i' \in \mathbb{N}_0$, $r \in \mathbb{N}$, *as* $i, i' < r$, $1 \le p \le \infty$, $\gamma \in \Gamma_+(p)$, $\xi, \eta \in (a,b)$, *and* s *be one of the points* a *or* b *and* e *be the other one. We set* $w = \chi_s^{-i-1/p}\chi_e^{\gamma}$ *and* $\tilde{w} = \chi_s^{-i'-1/p}\chi_e^{\gamma}$. *Finally, let* ϕ *be measurable and non-negative on* (a,b). *Then we have*

$$K(f,t;L_p(w)[a,b],W_p^r(w\phi)[a,b]) \sim K(A_{i'}(i'-i;s,e;\xi)f,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\phi)[a,b])$$

and

$$K(F,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\phi)[a,b]) \\ \sim K(A_i(i-i';s,e;\eta)F,t;L_p(w)[a,b],W_p^r(w\phi)[a,b]).$$

P r o o f. Just similarly as in the proof of [16, Proposition 3.2] we verify that the operators $\mathcal{A} = A_{i'}(i'-i;s,e;\xi)$ and $\mathcal{B} = A_i(i-i';s,e;\eta)$ satisfy the hypotheses of Theorem 3.1 with $X_1 = L_p(w)[a,b]$, $X_2 = L_p(\tilde{w})[a,b]$, $Y_1 = W_p^r(w\phi)[a,b]$ and $Y_2 = W_p^r(\tilde{w}\phi)[a,b]$. In establishing properties a) and b) we also take into consideration that $\alpha_{r,i'+1}(i'-i) = 0$, $\alpha_{r,i+1}(i-i') = 0$ and hence Hardy's inequalities are applicable. $\square$

If we separate the singularities of the weights w and φ beforehand, using (3.2), we can get a similar characterization of the K-functional with simpler transforms of the function but by a sum of two moduli ω_r. Moreover, the requirement that the exponent of the weight w on at least one of the ends of the interval is greater than $-1/p$ for $p<\infty$ is trivially satisfied and hence relaxed. In addition, Proposition 4.1 allows us to characterize the K-functional in the case $p=\infty$ not only for $\kappa_a,\kappa_b = 0$ but for all $\kappa_a,\kappa_b \in \Gamma_{exc}(\infty) = \{1-r,\dots,-1,0\}$. Thus, (3.2), Theorem 4.2 (in the case $p<\infty$), and Proposition 4.1, [13, Theorem 5.4] (in the case $p=\infty$) yield the following relation (cf. [16, Theorem 7.1]).

Theorem 4.3. *Let $r\in\mathbb{N}$, $1\le p\le\infty$ and $\lambda_a,\lambda_b\in(-\infty,1)$. For $p<\infty$ we assume that $\kappa_a,\kappa_b\notin\Gamma_{exc}(p)$, and for $p=\infty$ we assume that $\kappa_a,\kappa_b\in\Gamma_{exc}(\infty)$. We set $w=\chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi=\chi_a^{\lambda_a}\chi_b^{\lambda_b}$ and*

$$\mathcal{A}_1 = B_1(\sigma_a;a,b_1;\xi_1)A_0(\rho_a;a,b_1;\xi_1),$$
$$\mathcal{A}_2 = B_1(\sigma_b;b,a_1;\xi_2)A_0(\rho_b;b,a_1;\xi_2),$$

where $a<a_1<b_1<b$, $\xi_1\in(a,b_1)$, $\xi_2\in(a_1,b)$ and

$$\sigma_a=\frac{1}{1-\lambda_a},\quad \sigma_b=\frac{1}{1-\lambda_b},\quad \rho_a=\kappa_a+\frac{\lambda_a}{p},\quad \rho_b=\kappa_b+\frac{\lambda_b}{p}.$$

Then for $f\in L_p(w)[a,b]$ and $t>0$ we have

$$K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]) \sim \omega_r(\mathcal{A}_1 f,t)_{p[a,b_1]} + \omega_r(\mathcal{A}_2 f,t)_{p[a_1,b]}.$$

Remark 4.2. The K-functional $K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b])$ for $\kappa_a,\kappa_b\in\Gamma_{exc}(p)$, $p<\infty$, and $\kappa_a,\kappa_b\notin\Gamma_{exc}(\infty)$, $p=\infty$, can also be characterized in a similar way but that involves new elements. Some initial comments on that are given in [15, Sections 3 and 4].

Results similar to those given in Theorems 4.1–4.3 are valid in the cases when one or both of the expressions $(1-\lambda_a)(1-\nu_a)$, $(1-\lambda_b)(1-\nu_b)$ is negative and/or the interval is (semi-)infinite (see [16]).

4.2. Exponential change of the variable. K-functionals with $\lambda_a=1$ and/or $\lambda_\infty=1$ are related to the approximation error of the Post-Widder, Gamma and Baskakov operators (see [17] and the references cited there, and also [31]). In [18] we show that

$$\|w(f-P_{1/t}f)\|_{p[0,\infty)}\sim\|w(f-G_{1/t}f)\|_{p[0,\infty)}\\ \sim K(f,t;L_p(w)[0,\infty),W_p^2(w\chi_0^2)[0,\infty)),$$

where $P_{1/t}$ and $G_{1/t}$ denote respectively the Post-Widder and the Gamma operators, $f\in L_p(w)[0,\infty)$, $w(x)=x^{\gamma_0}(1+x)^{\gamma_\infty-\gamma_0}$ with arbitrary $\gamma_0,\gamma_\infty\in\mathcal{R}$, and $1\le p\le\infty$ (the case $\gamma_0=\gamma_\infty$ was considered in [17]).

Following the ideas of the previous subsection for $r\in\mathcal{N}$, $\gamma\in\mathcal{R}$, $F\in L_{1,loc}(\mathcal{R})$, $f\in L_{1,loc}[a,\infty)$ and $x\in\mathcal{R}$ we define the operators

$$(A_\gamma F)(x)=e^{(\gamma+1/p)x}F(x)\\ +\sum_{k=1}^{r}(-1)^k\binom{r}{k}\frac{(\gamma+1/p)^k}{(k-1)!}\int_0^x(x-y)^{k-1}e^{(\gamma+1/p)y}F(y)\,dy,$$

$$(Bf)(x)=f(a+e^x)+\sum_{i=1}^{r-1}\frac{s(r,r-i)}{(i-1)!}\int_0^x(x-y)^{i-1}f(a+e^y)\,dy,$$

where $s(r,k)$ are the Stirling numbers of the first kind defined by

$$x(x-1)\dots(x-r+1)=\sum_{k=0}^{r}s(r,k)\,x^k$$

for $k=0,1,\dots,r$ and $s(r,k)=0$ for $k>r$. Then the following one-term characterization is valid [14, 19].

Theorem 4.4. *Let $r\in\mathcal{N}$, $1\le p\le\infty$, $0<t\le t_0$, $\gamma\in\mathcal{R}$ and $f\in L_p(\chi_a^\gamma)[a,\infty)$.*

a) If $\gamma \notin \Gamma_{exc}(p)$, then

$$K(f, t^r; L_p(\chi_a^\gamma)[a,\infty), W_p^r(\chi_a^{\gamma+r})[a,\infty)) \sim \omega_r(A_\gamma Bf, t)_{p(\mathcal{R})}.$$

b) If $\gamma \in \Gamma_{exc}(p)$, then

$$K(f, t^r; L_p(\chi_a^\gamma)[a,\infty), W_p^r(\chi_a^{\gamma+r})[a,\infty)) \sim \omega_r(B(\chi_a^{\gamma+1/p} f), t)_{p(\mathcal{R})}.$$

By means of the method of 3.1 the operators A_γ and B in the above theorem can be further simplified if we use two fixed step moduli of different order. To treat the more general weight $w(x) = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ with $\gamma_a, \gamma_\infty \in \mathcal{R}$ in some cases we shall also apply (3.2), which increases the number of fixed step moduli to four. For $r \in \mathcal{N}$, $i, j \in \mathcal{N}_0$, $j \leq r$, distinct points $x_0, \ldots, x_r \in (a, \infty)$ and a weight $\bar{w}$ we define the linear operator $\mathcal{A}_{i,j-1}(\bar{w}) : L_{1,loc}[a,\infty) \to L_{1,loc}(\mathcal{R})$ by

$$\mathcal{A}_{i,j-1}(\bar{w}) f = (\bar{w}(f - \mathcal{L}_{i,j-1} f)) \circ \mathcal{E},$$

where $\mathcal{E}(x) = e^x$ and

$$(\mathcal{L}_{i,j-1} f)(x) = \sum_{n=i}^{j-1} \frac{1}{n!} \left(\sum_{\ell=1}^{r} \frac{\Phi_\ell^{(n+1)}(a)}{\Phi_\ell(x_\ell)} \int_{x_0}^{x_\ell} f(y)\, dy \right) (x-a)^n,$$

$$\Phi_\ell(x) = \prod_{\substack{m=0 \\ m \neq \ell}}^{r} (x - x_m), \quad \ell = 1, \ldots, r.$$

We have the following characterization [18, Theorem 1.2].

Theorem 4.5. *Let $r \in \mathcal{N}$, $i, j \in \mathcal{N}_0$, $i, j \leq r$, $1 \leq p \leq \infty$ and $t_0 > 0$. Let also $w = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ with $\gamma_a \in \Gamma_i(p)$, $\gamma_\infty \in \Gamma_j(p)$. Then for every $f \in L_p(w)[a,\infty)$ and $0 < t \leq t_0$ there holds*

$$\begin{aligned} K(f, t^r; L_p(w)[a,\infty), W_p^r(w\chi_a^r)[a,\infty)) \\ \sim \omega_r(\mathcal{A}_{i,j-1}(\chi_a^{1/p} w) f, t)_{p(\mathcal{R})} + t^r \|\mathcal{A}_{i,j-1}(\chi_a^{1/p} w) f\|_{p(\mathcal{R})}. \end{aligned}$$

Proof. We shall show that the operator $\mathcal{A} = \mathcal{A}_{i,j-1}(\chi_a^{1/p} w)$ satisfies the hypotheses of Theorem 3.1 with $X_1 = L_p(w)[a,\infty)$, $Y_1 = W_p^r(w\chi_a^r)[a,\infty)$ as $|g|_{Y_1} = \|w \chi_a^r g^{(r)}\|_{p[a,\infty)}$, $X_2 = L_p(\mathcal{R})$, $Y_2 = W_p^r(\mathcal{R})$ as $|G|_{Y_2} = \|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}$ and $\mathcal{B} : X_2 \to X_1$, defined by $\mathcal{B}F = \chi_a^{-1/p} w^{-1} (F \circ \log)$. Since $\mathcal{L}_{i,j-1}$:

$X_1 \to X_2$ is bounded we verify that $\mathcal{A}$ and $\mathcal{B}$ satisfy respectively conditions a) and c) of Theorem 3.1 just by a change of the variable. In [18, Proposition 4.3 and 4.4.e] we establish the inequalities

$$\|w\chi_a^k(g - \mathcal{L}_{i,j-1}g)^{(k)}\|_{p[a,\infty)} \le c\,\|w\chi_a^r g^{(r)}\|_{p[a,\infty)}, \quad k = 0,\dots,r.$$

provided that $g \in W_p^r(w\chi_a^r)[a,\infty)$, $\gamma_0 \in \Gamma_i(p)$, $\gamma_\infty \in \Gamma_j(p)$. Hence condition b) of Theorem 3.1 follows. Similarly, by the well-known inequalities

$$\|G^{(k)}\|_{p(\mathcal{R})} \le c\left(\|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right), \quad k = 0,\dots,r,$$

we get d). Finally, we directly verify that $f - \mathcal{B}\mathcal{A}f = \mathcal{L}_{i,j-1}f \in \pi_{r-1} \cap Y_1$ for any $f \in X_1$, which implies e), and since $\mathcal{L}_{i,j-1}$ preserves the polynomials of the form $c_i\chi_a^i + \dots + c_{j-1}\chi_a^{j-1}$ we have $\mathcal{A}\mathcal{B}F = F$ for any $F \in \mathcal{A}(X_1)$, which implies f) for $F \in \mathcal{A}(X_1)$.

Now, Theorem 3.1 in view of Remark 3.1 yields

$$\begin{aligned} &K(f,t^r; L_p(w)[a,\infty), W_p^r(w\chi_a^r)[a,\infty)) \\ &\qquad \sim \inf\left\{\|\mathcal{A}f - G\|_{p(\mathcal{R})} + t^r\left(\|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right) : G \in W_p^r(\mathcal{R})\right\}. \end{aligned}$$

To complete the proof we just need to observe that for $F \in L_p(\mathcal{R})$, $1 \le p \le \infty$, and $0 < t \le t_0$ there holds (cf. [17, Lemma 5.2])

$$\begin{aligned} &\inf\left\{\|F - G\|_{p(\mathcal{R})} + t^r\left(\|G^{(\ell)}\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right) : G \in W_p^r(\mathcal{R})\right\} \\ &\qquad \sim \omega_r(F,t)_{p(\mathcal{R})} + t^{r-\ell}\omega_\ell(F,t)_{p(\mathcal{R})}, \quad \ell = 0,\dots,r-1. \end{aligned}$$

□

Let us explicitly note that for $j \le i$ we have $\mathcal{A}_{i,j-1}(\chi_a^{1/p}w)f = (\chi_a^{1/p}wf)\circ\mathcal{E}$.

Similarly, the following assertion can be established

Theorem 4.6 [18, Theorem 1.3]**.** *Let $r \in \mathbb{N}$, $1 \le p \le \infty$ and $b, t_0 > 0$. Let also $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ with $\gamma_a, \gamma_\infty \in \mathcal{R}$ and the integers i, j be determined by $\Gamma_i(p)\cup\{1-i-1/p\} \ni \gamma_a$, $\Gamma_j(p)\cup\{-j-1/p\} \ni \gamma_\infty$. We set $\ell_a = 1$ if $\gamma_a \in \Gamma_{exc}(p)$, and $\ell_a = 0$ otherwise. We set $\ell_\infty = 1$ if $\gamma_\infty \in \Gamma_{exc}(p)$, and $\ell_\infty = 0$ otherwise. Let the integers i', j' be such that $0 \le i' \le i - \ell_0$ and $j + \ell_\infty \le j' \le r$. Then for* *every* $f \in L_p(w)[a,\infty)$ *and* $0 < t \le t_0$ *there holds*

$$\begin{aligned}
&K(f,t^r;L_p(w)[a,\infty),W_p^r(w\chi_a^r)[a,\infty))\\
&\quad\sim \omega_r(\mathcal{A}_{i,j'-1}(\chi_a^{\gamma_a+1/p})f,t)_{p(-\infty,b]} + t^{r-\ell_a}\omega_{\ell_a}(\mathcal{A}_{i,j'-1}(\chi_a^{\gamma_a+1/p})f,t)_{p(-\infty,b]}\\
&\quad+\omega_r(\mathcal{A}_{i',j-1}(\chi_a^{\gamma_\infty+1/p})f,t)_{p[-b,\infty)} + t^{r-\ell_\infty}\omega_{\ell_\infty}(\mathcal{A}_{i',j-1}(\chi_a^{\gamma_\infty+1/p})f,t)_{p[-b,\infty)}.
\end{aligned}$$

Similar characterizations hold for the K-functionals $K(f,t;L_p(\chi_a^{\gamma_a})[a,b], W_p^r(\chi_a^{\gamma_a+r})[a,b])$ and $K(f,t;L_p(\chi_a^{\gamma_\infty})[a+1,\infty),W_p^r(\chi_a^{\gamma_\infty+r})[a+1,\infty))$.

4.3. A *K*-functional associated with the best approximation by trigonometric polynomials. Let $L_{p,2\pi}$ denote the set of the 2π-periodic functions in L_p. The best trigonometric approximation of a function $f \in L_{p,2\pi}$ is given by

$$E_n^T(f)_p = \inf_{g\in T_n} \|f-g\|_{p[-\pi,\pi]},$$

where T_n is the set of trigonometric polynomials of degree at most $n \in \mathbb{N}_0$. As it is known the rate of best approximation by trigonometric polynomials can be estimated by the periodic modulus of smoothness as follows (see e.g. [8, Ch. 7]):

$$\begin{aligned}
E_n^T(f)_p &\le c\,\omega_r(f,n^{-1})_{p,2\pi}, \quad n\in\mathbb{N},\\
\omega_r(f,t)_{p,2\pi} &\le c\,t^r \sum_{0\le k\le 1/t}(k+1)^{r-1}E_k^T(f)_p, \quad 0<t\le t_0.
\end{aligned} \tag{4.2}$$

However, $\omega_r(f,t)_{p,2\pi} \equiv 0$ iff $f \in T_0$, whereas $E_n^T(f)_p = 0$ for any $f \in T_n$ and thus the direct estimate (4.2) contains a gap. This discrepancy can be overcome by defining another periodic modulus which is zero iff f is trigonometric polynomial of a given degree.

Let Π_n denote the set of the *algebraic* polynomials of degree $n \in \mathbb{N}_0$. For $r \in \mathbb{N}$ let us define the linear operator $\mathcal{A}_{r-1} : L_{p,2\pi} \to L_{p,2\pi} + \Pi_{2r-2}$ by

$$\mathcal{A}_{r-1}(f,x) = f(x) + \sum_{j=1}^{r-1}\frac{a_{r-1,j}}{(2j-1)!}\int_0^x (x-t)^{2j-1}f(t)\,dt,$$

where $a_{r-1,j}$ are given by the Stirling numbers of the first kind with

$$a_{r-1,j} = \sum_{k=1}^{2r-2j-1}(-1)^{r-j-k}s(r,k)\,s(r,2r-2j-k).$$

The first author introduced in [12] the following periodic modulus of smoothness:

$$\omega_r^T(f,t)_{p,2\pi} = \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathcal{A}_{r-1}f\|_{p[-\pi,\pi]}. \tag{4.3}$$

Let us note that although $\mathcal{A}_{r-1}f$ is not generally a 2π-periodic function for $f \in L_{p,2\pi}$, its finite difference $\Delta_h^{2r-1}\mathcal{A}_{r-1}f$ is. It was established in [12] that

$$\begin{aligned} E_n^T(f)_p &\le c\,\omega_r^T(f,1/n)_{p,2\pi}, \quad n \ge r-1,\\ \omega_r^T(f,t)_{p,2\pi} &\le c\,t^{2r-1} \sum_{r-1\le k\le 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0<t\le 1/r, \end{aligned} \tag{4.4}$$

as $\omega_r^T(f,t)_{p,2\pi} \equiv 0$ iff $f \in T_{r-1}$. Let us note that (4.4) for $n = r-1$ is a trigonometric analogue of Whitney's theorem.

A substantial element of the proof of (4.4) is the following relation between K-functionals, which is established by the method given in Theorem 3.1:

$$\begin{aligned} K_{r,\ell}^T(f,t)_p = \inf\{\|f-g\|_{p[-\pi,\pi]} + t\|\widetilde{D}_r D^\ell g\|_{p[-\pi,\pi]} : g \in W_{p,2\pi}^{2r+\ell-1}\}\\ \sim K(\mathcal{A}_{r-1}f, t; L_{p,2\pi} + \Pi_{2r-2}, W_{p,2\pi}^{2r+\ell-1}), \quad \ell = 0,1,\dots, \end{aligned}$$

with $Dg = g'$, $\widetilde{D}_r g = (D^2+(r-1)^2)\cdots(D^2+1)Dg$ and $W_{p,2\pi}^m = \{g \in AC_{loc}^m(\mathcal{R}) : D^m g \in L_{p,2\pi}\}$. Let us recall that $\widetilde{D}_r g = 0$ iff $g \in T_{r-1}$ and hence $K_{r,0}^T(f,t)_p \equiv 0$ iff $g \in T_{r-1}$.

Another modulus which is equivalent to zero for the trigonometric polynomials up to a given degree was considered by A.G. Babenko, N.I. Chernykh and V.T. Shevaldin. Through it they proved an upper estimate just like (4.2) for $p = 2$ and $r \in \mathbb{N}$ in [1], and Shevaldin [30] proved it for $p = \infty$ and $r = 2$.

4.4. The K-functional associated with the approximation error of the Kantorovich and the Durrmeyer operators.

Theorem 3.1 can be also applied for characterizing K-functionals with the second term generated by a linear differential operator of the form $P(D)$, where P is a polynomial with varying coefficients. In such cases the application is more complicated but the arising problems can be overcome, for example, by a varying sets technique. In order to demonstrate the approach let us consider the K-functional associated with the approximation error of the Kantorovich and the Durrmeyer operators.

Consider the space $L_p[0,1]$, $1 \le p \le \infty$, as for $p = \infty$ we identify $L_\infty[0,1]$ in this subsection with $C[0,1]$. Let $\tilde{Y}_p$ be $C^2[0,1]$ equipped with the semi-norm

$|g|_{\tilde{Y}_p} = \|(\phi g')'\|_{p[0,1]}$, where $\phi(x) = x(1-x)$. In this case the differential operator is $\phi D^2 + \phi' D$. As it was shown by Chen, Ditzian and Ivanov [7] for the Durrmeyer operator M_n and by Gonska and Zhou [20] for the Kantorovich operator K_n, we have for $1 \le p \le \infty$ the equivalence

$$\|f - K_n f\|_{p[0,1]} \sim \|f - M_n f\|_{p[0,1]} \sim K(f, 1/n; L_p[0,1], \tilde{Y}_p). \tag{4.5}$$

Further, Gonska and Zhou [20] proved for $f \in L_p[0,1]$, $1 < p \le \infty$ that

$$K(f, t^2; L_p[0,1], \tilde{Y}_p) \sim \omega^2_{\sqrt{\phi}}(f,t)_{p[0,1]} + \omega_1(f, t^2)_{p[0,1]},$$

where $\omega^2_{\sqrt{\phi}}$ is given by (2.3) with $w \equiv 1$. The above equivalence is not valid in the case $p = 1$. For the characterization of the K-functional in (4.5) for $p = 1$ the second author [23] used the scheme

$$\begin{aligned} K(f, t; L_1[0,1], \tilde{Y}_1) &= K(f, t; L_1[0,1], Z_1) \sim K(\mathcal{A}f, t; L_1[0,1], Z_2) \\ &\sim K(\mathcal{A}f, t; L_1[0,1], W_1^2(\phi)[0,1]) + t\,\omega_1(f,1)_{1[0,1]}, \end{aligned} \tag{4.6}$$

where the operator $\mathcal{A}$ is given by

$$(\mathcal{A}f)(x) = f(x) + \int_{1/2}^{x} \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2} \right) f(y)\, dy.$$

Theorem 3.1 is applied in (4.6) with $X_1 = X_2 = L_1[0,1]$, $Y_1 = Z_1$ and $Y_2 = Z_2$, where

$$Z_1 = \left\{ f \in C^2[0,1] : f'(0) = 0,\ f'(1) = 0 \right\},$$
$$Z_2 = \left\{ f \in C^2[0,1] : f(0) = 2\int_0^{1/2} f(y)\, dy,\ f(1) = 2\int_{1/2}^{1} f(y)\, dy \right\}$$

and the semi-norms in Z_1 and Z_2 are given by $\|(\phi g')'\|_{1[0,1]}$ and $\|\phi g''\|_{1[0,1]}$ respectively. Note that Theorem 3.1 cannot be applied directly with $\mathcal{A}$ and the subspaces $Y_1 = \tilde{Y}_1$, $Y_2 = W_1^2(\phi)[0,1]$ because items b) and d) (with $\mathcal{B} = \mathcal{A}^{-1}$) are not fulfilled. Moreover, $\mathcal{A}(Z_1) = Z_2$ but $\mathcal{A}(\tilde{Y}_1) \ne W_1^2(\phi)[0,1]$. Using (4.5), (4.6) and Theorem 4.2 we get for every $n \in \mathbb{N}$ and every $f \in L_1[0,1]$

$$\|f - P_n f\|_{1[0,1]} \sim \|f - M_n f\|_{1[0,1]} \sim \omega_2(\tilde{\mathcal{A}}\mathcal{A}f, n^{-1/2})_{1[0,1]} + n^{-1}\omega_1(f,1)_{1[0,1]},$$

where $\tilde{\mathcal{A}}$ stays for the operator $\mathcal{A}$ from Theorem 4.2 with $r = 2$, $p = 1$, $a = 0$, $b = 1$, $\lambda_0 = \lambda_1 = 1/2$, $\kappa_0 = \kappa_1 = 0$, $\xi = 1/2$.

Following the approach sketched in this subsection Zapryanova [32] characterized the K-functional related to the $L_p[0,1]$, $1 \le p \le 2$, error of the algebraic version of the integral Jackson operator.

REFERENCES

[1] A. G. Babenko, N. I. Chernykh, V. T. Shevaldin. The Jackson-Stechkin inequality in L_2 with a trigonometric modulus of continuity. *Mat. Zametki* 65 (1999), 928–932 (in Russian); English translation: *Math. Notes* **65** (1999), 777–781.

[2] C. Bennett, R. Sharpley. Interpolation of operators. Academic Press, New York, 1988.

[3] P. L. Butzer. Legendre transform methods in the solution of basic problems in algebraic approximation. *Coll. Math. Soc. János Bolyai* **35** (1980), 277–301.

[4] P. L. Butzer, H. Berens. Semi-groups of operators and approximation. Springer-Verlag, Berlin, 1967.

[5] P. L. Butzer, R. L. Stens, M. Wehrens. Approximation by algebraic convolution integrals. *Proc. Approx. Theory Func. Anal.* (Ed. J. B. Prolla) North Holland Publ. Co., 1979, 71–120.

[6] P. L. Butzer, R. L. Stens, M. Wehrens. Higher order moduli of continuity based on the Jacobi translation operator and best approximation. *C. R. Math. Rep. Acad. Sci. Canada* **2** (1980), 83–87.

[7] W. Chen, Z. Ditzian, K. G. Ivanov. Strong converse inequality for the Bernstein-Durrmeyer operator. *J. Approx. Theory* **75** (1993), 25–43.

[8] R. A. DeVore, G. G. Lorentz. Constructive Approximation. Springer-Verlag, Berlin, 1993.

[9] Z. Ditzian, K. G. Ivanov. Strong converse inequalities. *J. Anal. Math.* **61** (1993), 61–111.

[10] Z. Ditzian, V. Totik. Moduli of smoothness. Springer Verlag, New York, 1987.

[11] Z. Ditzian, V. Totik. K-functionals and weighted moduli of smoothness. *J. Approx. Theory* **63** (1990), 3–29.

[12] B. R. Draganov. A new modulus of smoothness for trigonometric polynomial approximation. *East J. Approx.* **8** (2002), 465–499.

[13] B. R. Draganov, K. G. Ivanov. A new characterization of weighted Peetre K-functionals. *Constr. Approx.* **21** (2005), 113–148.

[14] B. R. Draganov, K. G. Ivanov. A characterization of weighted L_p approximations by the Gamma and the Post-Widder operators. In: Constructive Theory of Functions, Varna 2005 (Ed. B. Bojanov) Marin Drinov Academic Publishing House, Sofia, 2006, 80–87.

[15] B. R. Draganov, K. G. Ivanov. Characterizations of weighted K-functionals and their application. In: Constructive Theory of Functions, Varna 2005 (Ed. B. Bojanov) Marin Drinov Academic Publishing House, Sofia, 2006, 88–97.

[16] B. R. Draganov, K. G. Ivanov. A new characterization of weighted Peetre K-functionals (II). *Serdica Math. J.* **33** (2007), 59–124.

[17] B. R. Draganov, K. G. Ivanov. A Characterization of Weighted Approximations by the Post-Widder and the Gamma Operators. *J. Approx. Theory* **146** (2007), 3–27.

[18] B. R. Draganov, K. G. Ivanov. A Characterization of Weighted Approximations by the Post-Widder and the Gamma Operators (II), 2007, manuscript.

[19] B. R. Draganov, K. G. Ivanov. Another characterization of the K-functional associated with the Post-Widder and the Gamma operators, 2005, manuscript.

[20] H. Gonska, X.-l. Zhou. The strong converse inequality for Bernstein-Kantorovich operators. *Computers Math. Applic.* **30** (1995), No. 3-6, 103–128.

[21] K. G. Ivanov. Some characterizations of the best algebraic approximation in $L_p[-1,1]$ $(1 \le p \le \infty)$. *C. R. Acad. Bulgare Sci.* **34** (1981), 1229–1232.

[22] K. G. Ivanov. A characterization of weighted Peetre K-functionals. *J. Approx. Theory* **56** (1989), 185–211.

[23] K. G. Ivanov. A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators. In: Approximation Theory, a volume dedicated to Borislav Bojanov (Eds. D. K. Dimitrov, G. Nikolov, and R. Uluchev) Marin Drinov Academic Publishing House, Sofia, 2004, 112–125.

[24] H. Johnen. Inequalities connected with moduli of smoothness. *Mat. Vesnik* **3** (1972), 289–303.

[25] H. Johnen, K. Scherer. On equivalence of K-functional and moduli of continuity and some applications. *Lecture Notes in Math.* **571** (1976), 119–140.

[26] H.-B. Knoop, X.-l. Zhou. The lower estimate for linear positive operators II. *Results in Math.* **25** (1994), 315–330.

[27] N. X. Ky. A method for characterization of weighted K-functionals. *Annales Univ. Sci. Budapest.* **38** (1995), 147–152.

[28] J. Peetre. A theory of interpolation of normed spaces. Notes de Universidade de Brasilia, 1963.

[29] M. K. Potapov. On interrelation between generalized moduli of smoothness of differentiable functions and their best approximations by algebraic polynomials. *Vestnik Moskov. Univ., Ser. 1, Mat. Mekh.* No. 6 (2005), 10–17.

[30] V. T. Shevaldin. The Jackson-Stechkin inequality in the space $C(\mathbb{T})$ with trigonometric continuity modulus annihilating the first harmonics. *Proc. Steklov Inst. Math., Approximation Theory, Asymptotic expansions*, suppl. 1, 2001, 206–213.

[31] V. Totik. Approximation by Bernstein polynomials. *Amer. J. Math.* **116** (1994), 995–1018.

[32] T. Zapryanova. A characterization theorem for the K-functional associated with the algebraic version of trigonometric Jackson integrals. *Serdica Math. J.* **32** (2006), 303–322.

B. R. Draganov
Dept. of Mathematics and Informatics
The Sofia University
5, James Bourchier Blvd.
1164 Sofia
Bulgaria
e-mail: `bdraganov@fmi.uni-sofia.bg`
and
Inst. of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., bl. 8
1113 Sofia
Bulgaria

K. G. Ivanov
Inst. of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria
e-mail: `kamen@math.bas.bg`

Received June 6, 2007