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SOME COEFFICIENT ESTIMATES FOR POLYNOMIALS ON THE UNIT INTERVAL

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Communicated by G. Nikolov

ABSTRACT. In this paper we present some inequalities about the moduli of the coefficients of polynomials of the form $f(x) := \sum_{\nu=0}^n a_{\nu}x^{\nu}$, where $a_0, \dots, a_n \in \mathbb{C}$. They can be seen as generalizations, refinements or analogues of the famous inequality of P. L. Chebyshev, according to which $|a_n| \leq 2^{n-1}$ if $|\sum_{\nu=0}^n a_{\nu}x^{\nu}| \leq 1$ for $-1 \leq x \leq 1$.

1. Chebyshev Polynomials T_n . Using *de Moivre's theorem*, i.e. the formula

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n,$$

we readily see that $\cos n\theta$ can be expressed as a polynomial of degree n in $\cos \theta$. The polynomial T_n such that

$$(1) \quad T_n(\cos \theta) \equiv \cos n\theta$$

is called the Chebyshev polynomial of the first kind of degree n .

2000 *Mathematics Subject Classification*: 26C05, 26C10, 30A12, 30D15, 42A05, 42C05.

Key words: Polynomials, Inequality, Weighted L^p norm.

Formula (1) implies that $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$ and that

$$T_n \left(\cos \frac{\nu\pi}{n} \right) = (-1)^\nu \quad (\nu = 0, 1, \dots, n).$$

From (1), it also follows that the zeros of T_n are

$$\cos \frac{(2\nu - 1)\pi}{2n} \quad (\nu = 1, \dots, n).$$

Clearly, they all lie in the open interval $(-1, 1)$. Writing (1) in the form

$$T_n \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right) \equiv \frac{e^{in\theta} + e^{-in\theta}}{2} \quad (\theta \in \mathbb{R})$$

we see that

$$\tau(z) := z^n T_n \left(\frac{z + z^{-1}}{2} \right) - \frac{z^{2n} + 1}{2}$$

is a polynomial of degree not exceeding $2n$, which vanishes at each point of the unit circle. Since a polynomial of degree at most $2n$ cannot vanish at more than $2n$ points without being identically zero, we conclude that

$$(2) \quad T_n \left(\frac{z + z^{-1}}{2} \right) \equiv \frac{z^n + z^{-n}}{2} \quad (z \neq 0).$$

Let $T_n(z) := \sum_{\nu=0}^n t_{n,\nu} z^\nu$ be the Maclaurin expansion of T_n . Then, comparing the coefficients of z^n on the two sides of (2) we find that

$$(3) \quad t_{n,n} = 2^{n-1},$$

i.e. the leading coefficient in the Maclaurin expansion of T_n is 2^{n-1} . It is useful to know that

$$(4) \quad T_n(z) := \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2z)^{n-2k}.$$

2. The polynomial inequality of Chebyshev. The polynomials $\{T_n\}$ are known to have several remarkable extremal properties. They play an important role in the Theory of Approximation and related areas of mathematics. Let \mathcal{P}_m denote the class of all algebraic polynomials of degree at most m . For any function $\Phi \in C[a, b]$, let

$$E_n[\Phi] = \inf_{P \in \mathcal{P}_n} \{ \max_{x \in [a, b]} |\Phi(x) - P(x)| \}.$$

A polynomial $P^* \in \mathcal{P}_n$ is said to be a polynomial of best approximation to Φ if

$$E_n[\Phi] = \max_{a \leq x \leq b} |\Phi(x) - P^*(x)|.$$

By a classical result of P. L. Chebyshev, a polynomial $P^* \in \mathcal{P}_n$ is a polynomial of best approximation to Φ if there exist $n + 2$ points

$$a \leq x_0 < \dots < x_{n+1} \leq b$$

such that

$$\pm(-1)^\nu[\Phi(x_\nu) - P^*(x_\nu)] = \max_{a \leq x \leq b} |\Phi(x) - P^*(x)| \quad (\nu = 0, \dots, n + 1),$$

i.e. if $\Phi(x) - P^*(x)$ takes on the values $\pm \max_{a \leq x \leq b} |\Phi(x) - P^*(x)|$ in alternating succession at the points x_0, \dots, x_{n+1} . This alternation theorem shows, in particular, that P^* is unique. For a historical perspective of the alternation theorem, we refer the reader to an interesting article of Butzer and Jongemans [2].

As an example, let us find the best uniform approximation of the function $\Phi(x) := x^n$ on $[-1, 1]$ by polynomials in \mathcal{P}_{n-1} . Clearly, the polynomial of best approximation must be real. In view of (3), we can write

$$x^n = \frac{T_n(x)}{2^{n-1}} + \text{a polynomial in } \mathcal{P}_{n-1}.$$

Since $T_n(x)$ takes the value $+1$ and -1 at $n + 1$ points in $[-1, 1]$ and $n + 1$ can be written as $(n - 1) + 2$, the alternation theorem shows that the desired polynomial of best approximation to x^n , in \mathcal{P}_{n-1} , is the polynomial

$$x^n - \frac{1}{2^{n-1}}T_n(x).$$

Equivalently, the monic polynomial $P^*(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ of best approximation to the function $\Phi(x) \equiv 0$ on the interval $[-1, 1]$ is the Chebyshev polynomial $2^{1-n}T_n(x)$. The maximum deviation of this polynomial from zero is 2^{1-n} .

The above discussion may be summarized as follows.

Theorem A. *Let $T_n(x) := \sum_{\nu=0}^n t_{n,\nu}x^\nu$ be the Chebyshev polynomial of the first kind of degree n . Then, for any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree at most n , not identically zero, we have*

$$(5) \quad \frac{|a_n|}{\max_{-1 \leq x \leq 1} |f(x)|} \leq \frac{t_{n,n}}{\max_{-1 \leq x \leq 1} |T_n(x)|} = 2^{n-1}.$$

3. Upper bound for $|a_n|$ and the distribution of the zeros of f . In (5), equality holds if and only if $f(z) \equiv e^{i\gamma} T_n(z)$, $\gamma \in \mathbb{R}$. It was shown by Schur [16] that if the polynomial $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ vanishes at $c = 1$ or $c = -1$ and $|f(x)| \leq 1$ for $-1 \leq x \leq 1$, then

$$(6) \quad |a_n| \leq \left(\cos \frac{\pi}{4n} \right)^{2n} 2^{n-1},$$

where equality holds if and only if

$$f(z) := e^{i\gamma} T_n \left(z \cos^2 \frac{\pi}{4n} - c \sin^2 \frac{\pi}{4n} \right) \quad (\gamma \in \mathbb{R}).$$

In fact, the following more general result (see [9, Theorem 1]) holds.

Theorem B. *Let $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ be a polynomial of degree n such that $f(c) = 0$ for some c belonging to $[-1, -\cos(\pi/2n)] \cup (\cos(\pi/2n), 1]$. Furthermore, let $|f(x)| \leq 1$ at the points*

$$\xi_\nu = \xi_{\nu,c} := \frac{c(1+|c|)\cos(\nu\pi/n) + c - (\text{sign } c)\cos(\pi/2n)}{2\cos^2(\pi/4n)} \quad (1 \leq \nu \leq n).$$

Then

$$(7) \quad |a_n| \leq \left(\frac{1 + \cos(\pi/2n)}{1 + |c|} \right)^n 2^{n-1}.$$

The upper bound for $|a_n|$ given in (7) is attained for the polynomial

$$e^{i\gamma} T_n \left(\frac{(1 + \cos(\pi/2n))z - c + (\text{sign } c)\cos(\pi/2n)}{1 + |c|} \right) \quad (\gamma \in \mathbb{R}),$$

which satisfies the conditions of Theorem B.

The above mentioned result of Schur has also been generalized [12, Theorem 2] as follows.

Theorem C. *Inequality (6) holds for any polynomial $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ of degree n with real coefficients which has at most $n - 1$ distinct zeros in the open interval $(-1, 1)$.*

It was conjectured by K. Mahler and stated by him in some of his lectures (see [20, p. 15]) that if f is as above, then

$$(8) \quad \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\theta})| d\theta \right\} \leq 2^{n-1}.$$

It is well-known [18, p. 284d] that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |e^{i\theta} - \zeta| d\theta = \log^+ |\zeta| := \begin{cases} 0 & \text{for } |\zeta| \leq 1 \\ \log |\zeta| & \text{for } |\zeta| \geq 1. \end{cases}$$

Hence, if z_1, \dots, z_n are the zeros of f then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{i\theta})| d\theta = \log |a_n| + \sum_{\nu=1}^n \log^+ |z_\nu| = \log |a_n| + \log \prod_{|z_\nu|>1} |z_\nu|,$$

where the product on the right is to be replaced by 1 if $|z_\nu| \leq 1$ for $1 \leq \nu \leq n$. Thus (8) is equivalent to the inequality

$$(9) \quad \left(\prod_{|z_\nu|>1} |z_\nu| \right) |a_n| \leq 2^{n-1}.$$

Turán [20] settled the conjecture of Mahler by proving it in the form (9).

Inequality (9) gives a quantitative improvement on (5) only if f has some zeros outside the unit circle. However, as mentioned above, inequality (5) is strict unless $f(z) \equiv e^{i\gamma} T_n(z)$, $\gamma \in \mathbb{R}$, and so certainly if some of the zeros of f lie in $\mathbb{C} \setminus [-\cos(\pi/2n), \cos(\pi/2n)]$. Theorem B deals with the case where f has a zero in $[-1, -\cos(\pi/2n)) \cup (\cos(\pi/2n), 1]$. What if f has some zeros in $\mathbb{C} \setminus [-1, 1]$? An answer to this question is to be found in Corollary 1. We shall deduce it from the following theorem, which is a refined version of a result from [11, Theorem 1].

Theorem 1. *Let $\psi(\zeta) := \sum_{\mu=0}^m \gamma_\mu \zeta^\mu$ be a polynomial of degree m such that $\gamma_0 \bar{\gamma}_m > 0$ and $|\psi(e^{2\ell\pi i/m})| \leq 1$ for $\ell = 0, 1, \dots, m-1$. Furthermore, let ζ_1, \dots, ζ_J be amongst the zeros of ψ , where $1 \leq J \leq m$. Then*

$$(10) \quad \frac{|\gamma_0|}{\prod_{j=1}^J |\zeta_j|} + |\gamma_m| \prod_{j=1}^J |\zeta_j| \leq 1.$$

The following modified form of a famous inequality of C. Visser [21] plays an important role in the proof of Theorem 1.

Lemma 1. *Let $\phi(\zeta) := \sum_{\mu=0}^m c_\mu \zeta^\mu$ be a polynomial of degree m such that $c_0 \bar{c}_m > 0$. Furthermore, let $|\phi(e^{2\ell\pi i/m})| \leq 1$ for $\ell = 0, 1, \dots, m - 1$. Then*

$$|c_0| + |c_m| \leq 1.$$

Proof. Let $\omega := e^{2\ell\pi i/m}$, where $\ell \in \{1, \dots, m - 1\}$. Then $\omega^m = 1$ and because $\omega \neq 1$, we have

$$\sum_{\mu=0}^{m-1} \left(e^{2\ell\pi i/m}\right)^\mu = \sum_{\mu=0}^{m-1} \omega^\mu = \frac{1 - \omega^m}{1 - \omega} = 0 \quad (\ell = 1, \dots, m - 1).$$

Hence

$$\sum_{\ell=0}^{m-1} \left(e^{2\ell\pi i/m}\right)^\mu = \sum_{\ell=0}^{m-1} \left(e^{2\mu\pi i/m}\right)^\ell = 0 \quad (\mu = 1, \dots, m - 1),$$

and so

$$\sum_{\ell=0}^{m-1} \phi\left(e^{2\ell\pi i/m}\right) = \sum_{\ell=0}^{m-1} c_0 + \sum_{\mu=1}^{m-1} c_\mu \sum_{\ell=0}^{m-1} \left(e^{2\ell\pi i/m}\right)^\mu + \sum_{\ell=0}^{m-1} c_m = m(c_0 + c_m).$$

Since $c_0 \bar{c}_m > 0$, this implies that

$$|c_0| + |c_m| = |c_0 + c_m| \leq \frac{1}{m} \sum_{\ell=0}^{m-1} \left|\phi\left(e^{2\ell\pi i/m}\right)\right| \leq 1. \quad \square$$

Proof of Theorem 1. Let

$$\phi(\zeta) := \psi(\zeta) \prod_{j=1}^J \frac{\bar{\zeta}_j \zeta - 1}{\zeta - \zeta_j}.$$

Then $\phi(\zeta) := \sum_{\mu=0}^m c_\mu \zeta^\mu$, where

$$c_0 = \frac{\gamma_0}{\prod_{j=1}^J \zeta_j} \quad \text{and} \quad c_m = \gamma_m \prod_{j=1}^J \bar{\zeta}_j,$$

and so

$$c_0 \bar{c}_m = \frac{\gamma_0}{\prod_{j=1}^J \zeta_j} \bar{\gamma}_m \prod_{j=1}^J \zeta_j = \gamma_0 \bar{\gamma}_m > 0.$$

Furthermore, $|\phi(e^{2\ell\pi i/m})| \leq 1$ for $\ell = 0, 1, \dots, m - 1$ since

$$\left| \frac{\bar{\zeta}_j \zeta - 1}{\zeta - \zeta_j} \right| = 1 \quad (|\zeta| = 1).$$

Thus, Lemma 1 applies and so (10) holds. \square

We are ready to state and prove our refinement of Chebyshev’s inequality, which takes into account each and every zero that does not belong to $[-1, 1]$.

Corollary 1. *Let $f(z) := a_n \prod_{\nu=1}^n (z - z_\nu)$, $a_n \neq 0$, and suppose that*

$$(11) \quad \left| f\left(\cos \frac{k\pi}{n}\right) \right| \leq 1 \quad (k = 0, 1, \dots, n).$$

Furthermore, for any $z_\nu \notin [-1, 1]$ let R_ν denote the sum of the semi-axes of the ellipse whose foci lie at ± 1 and which passes through z_ν ; otherwise let $R_\nu = 1$. Then

$$(12) \quad \left(R_1 \dots R_n + \frac{1}{R_1 \dots R_n} \right) |a_n| \leq 2^n.$$

Proof. Let

$$\psi(\zeta) := \zeta^n f\left(\frac{\zeta + \zeta^{-1}}{2}\right).$$

Then $\psi(\zeta) := \sum_{\nu=0}^{2n} \gamma_\nu \zeta^\nu$, where

$$(13) \quad \gamma_0 = \gamma_{2n} = 2^{-n} a_n.$$

In particular, $\gamma_0 \bar{\gamma}_{2n} = 2^{-2n} |a_n|^2 > 0$. Furthermore,

$$\left| \psi\left(\exp\left(\frac{2\ell\pi i}{2n}\right)\right) \right| = \left| f\left(\cos \frac{\ell\pi}{n}\right) \right| \leq 1 \quad (\ell = 0, 1, \dots, 2n - 1)$$

since

$$\cos \frac{\ell\pi}{n} = \cos \frac{(2n - \ell)\pi}{n} \quad (\ell = n + 1, \dots, 2n - 1).$$

Thus, ψ satisfies the conditions of Theorem 1 with $m = 2n$.

For any given $R > 1$, let C_R denote the circle $\zeta = Re^{it}$, $0 \leq t \leq 2\pi$. Also, let \mathcal{E}_R be the ellipse whose foci lie at ± 1 and whose semi-axes are

$$\frac{R + R^{-1}}{2} \quad \text{and} \quad \frac{R - R^{-1}}{2}.$$

It may be noted that as ζ goes around the circle C_R , the point $z := (\zeta + \zeta^{-1})/2$ describes the ellipse \mathcal{E}_R once counterclockwise, starting at the point $(R + R^{-1})/2$. Hence, corresponding to a zero z_ν of f , not belonging to $[-1, 1]$, there is a number ζ_ν lying outside the closed unit disc such that

$$(14) \quad \frac{1}{2}(\zeta_\nu + \zeta_\nu^{-1}) = z_\nu.$$

In fact, $|\zeta_\nu|$ is equal to R_ν , the sum of the semi-axes of the ellipse whose foci lie at ± 1 and which passes through z_ν .

Note that

$$\frac{1}{2}(\zeta + \zeta^{-1}) = z_\nu$$

has a double root at $\zeta = 1$ if $z_\nu = 1$ and a double root at $\zeta = -1$ if $z_\nu = -1$. For any $z_\nu \in (-1, 1)$ there are two different values of ζ_ν , both of modulus 1, that satisfy (14). So, we take $R_\nu = |\zeta_\nu| = 1$ if z_ν is a zero of f that lies somewhere in $[-1, 1]$.

Amongst the zeros of ψ we can therefore count n zeros ζ_1, \dots, ζ_n with $|\zeta_\nu| = R_\nu \geq 1$ for $\nu = 1, \dots, n$. Applying Theorem 1 with $m = 2n$, $J = n$ and taking note of (13), by which $\gamma_0 \bar{\gamma}_{2n} > 0$, we see that (12) holds. \square

Remark 1. In [11, Corollary 4], inequality (12) was obtained under the assumption that $|f(x)| \leq 1$ for $-1 \leq x \leq 1$. Some other interesting results appear in [7]. The result, stated above as Corollary 1, first appeared in [9], but the proof presented here is different.

It is easily proved by induction that if R_1, \dots, R_n are as in Corollary 1, then

$$\prod_{\nu=1}^n \frac{R_\nu^2 + 1}{2} \leq \frac{\prod_{\nu=1}^n R_\nu^2 + 1}{2}.$$

This is clearly equivalent to the inequality

$$(15) \quad \prod_{\nu=1}^n \frac{R_\nu + R_\nu^{-1}}{2} \leq \frac{\prod_{\nu=1}^n R_\nu + \prod_{\nu=1}^n R_\nu^{-1}}{2},$$

which may be combined with (12) to obtain the following result.

Corollary 2. *Under the conditions of Corollary 1, we have*

$$(16) \quad \left(\prod_{\nu=1}^n \frac{R_\nu + R_\nu^{-1}}{2} \right) |a_n| \leq 2^{n-1}.$$

Since

$$|z_\nu| = \left| \frac{\zeta_\nu + \zeta_\nu^{-1}}{2} \right| \leq \frac{R_\nu + R_\nu^{-1}}{2},$$

we see that (12) is considerably stronger than (9). In addition, the hypothesis in Corollary 1 is quite a bit weaker. Instead of supposing $|f(x)|$ to be bounded by 1 on $[-1, 1]$ we only require this restriction on $|f(x)|$ to be satisfied at the extrema of T_n .

The following result is an immediate consequence of Corollary 1.

Corollary 3. *For any $R > 1$, let \mathcal{E}_R be the ellipse whose foci lie at ± 1 and the sum of whose semi-axes is R . Furthermore, let $f(z) := a_n \prod_{\nu=1}^n (z - z_\nu)$ satisfy (11), and have at most $n - \ell$ zeros inside \mathcal{E}_R . Then*

$$(17) \quad (R^\ell + R^{-\ell})|a_n| \leq 2^n.$$

Corollary 3 says in particular that if $f(z) \neq 0$ inside \mathcal{E}_R , then

$$(18) \quad (R^n + R^{-n})|a_n| \leq 2^n.$$

This estimate is sharp. In order to see this, let us consider the polynomial

$$f_R(z) := \frac{2}{R^n + R^{-n}} \left(T_n(z) + i \frac{R^n - R^{-n}}{2} \right).$$

Clearly,

$$|f_R(x)| \leq \frac{2}{R^n + R^{-n}} \sqrt{1 + \left(\frac{R^n - R^{-n}}{2} \right)^2} = 1 \quad (-1 \leq x \leq 1).$$

Using (2) we find that

$$\frac{R^n + R^{-n}}{2} f_R \left(\frac{\zeta + \zeta^{-1}}{2} \right) = \frac{\zeta^n + \zeta^{-n}}{2} + i \frac{R^n - R^{-n}}{2},$$

which is easily seen to vanish for

$$\zeta = \zeta_\nu = Re^{(4\nu-1)\pi i/2n} \quad (\nu = 1, \dots, n).$$

Hence, $f_R(z) = 0$ at each of the n points

$$z = z_\nu = \frac{1}{2} \left\{ Re^{(4\nu-1)\pi i/2n} + R^{-1}e^{-(4\nu-1)\pi i/2n} \right\} \quad (\nu = 1, \dots, n).$$

Since f_R is a polynomial of degree n , it has no other zeros. Setting

$$A := \frac{R + R^{-1}}{2} \quad \text{and} \quad B := \frac{R - R^{-1}}{2},$$

the points z_ν can also be written as

$$z_\nu = A \cos \frac{(4\nu - 1)\pi}{2n} + iB \sin \frac{(4\nu - 1)\pi}{2n} \quad (\nu = 1, \dots, n).$$

Thus, f_R has all its zeros on the ellipse \mathcal{E}_R . Since $f_R(z)$ is of the form

$$f_R(z) = \frac{2^n}{R^n + R^{-n}} z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu,$$

we conclude that the bound for $|a_n|$, given by (18), cannot be improved even if (11) is replaced by the stronger condition: “ $|f(x)| \leq 1$ for $-1 \leq x \leq 1$ ”.

4. Relevance of the points $\left\{ \cos \frac{k\pi}{n} \right\}$. Corollary 1 says that in (5) we may replace $\max_{-1 \leq x \leq 1} |f(x)|$ by the maximum of $|f(x)|$ at the extrema of the (extremal) polynomial T_n . One might wonder about the reason behind it. The following result (see [3, Theorems 4 and 5]) provides an explanation.

Theorem D. *Let $t_0 < \dots < t_n$ be an arbitrary set of $n + 1$ real numbers. Setting $P(z) := \prod_{\nu=0}^n (z - t_\nu)$ let $t_{0,k} < \dots < t_{n-k,k}$ be the zeros of $P^{(k)}$, the k -th derivative of P . In addition, let y_0, \dots, y_n be any set of $n + 1$ non-negative numbers, not all zero, and denote by π_n the unique polynomial of degree at most n such that $\pi_n(t_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, 1, \dots, n$. Then, for any real polynomial f of degree at most n , other than $f(z) \equiv \pm \pi_n(z)$, such that $|f(t_\nu)| \leq y_\nu$ for $\nu = 0, 1, \dots, n$, we have*

$$(19) \quad |f(z)| < |\pi_n(z)| \quad \left(\left| z - \frac{t_0 + t_n}{2} \right| \geq \left| \frac{t_n - t_0}{2} \right|, z \notin \{t_0, t_n\} \right),$$

and

$$(20) \quad |f^{(k)}(z)| < |\pi_n^{(k)}(z)| \quad (k = 1, \dots, n)$$

if z satisfies

$$\left| z - \frac{t_{0,k} + t_{n-k,k}}{2} \right| \geq \left| \frac{t_{n-k,k} - t_{0,k}}{2} \right|, \quad z \notin \{t_{0,k}, t_{n-k,k}\}.$$

Remark 2. The case $k = n$ of (20) says that if $\pi_n(z) := \sum_{\nu=0}^n \pi_{n,\nu} z^\nu$, then $|a_n| < |\pi_{n,n}|$ unless $f(z) \equiv \pm \pi_n(z)$. This is a good deal more than what Theorem A says. In the important case where $t_\nu = \cos(\nu\pi/n)$, inequality (19) goes back to S. Bernstein [1] and to Erdős [4]. One may consult [14, Chapter 12] for related results. The following application of (19) gives a meaningful refinement of (5) provided that n is odd.

Theorem 2. For any odd integer n , let $t_0 = -1 < \dots < t_n = 1$ be a set of $n + 1$ real numbers such that $t_\nu = -t_{n-\nu}$ for $\nu = 0, 1, \dots, n$. Also, let y_0, \dots, y_n be a set of $n + 1$ non-negative numbers, not all zero, satisfying the condition $y_\nu = y_{n-\nu}$ for $\nu = 0, 1, \dots, n$. Finally, let $\pi_n(z) := \sum_{\nu=0}^n \pi_{n,\nu} z^\nu$ be the unique polynomial of degree at most n for which $\pi_n(t_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, 1, \dots, n$. Then for any real polynomial $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ of degree at most n , other than $f(z) \equiv \pm \pi_n(z)$, satisfying $|f(t_\nu)| \leq y_\nu$ for $\nu = 0, 1, \dots, n$, we have

$$(21) \quad |a_n| + |a_0| < \pi_{n,n}.$$

Proof. Because of the restrictions on $\{t_\nu\}$ and $\{y_\nu\}$, the polynomial π_n must be odd, and so $\pi_{n,0} = 0$. Without loss of generality we may assume a_n to be positive. Applying (19) with $t_0 := -1$ and $t_n := 1$ we see that $|f(z)| < |\pi_n(z)|$ if $|z| \geq 1, z \neq \pm 1$. Hence, the polynomial $\pi_n(z) - f(z)$ has all its zeros in $\mathcal{E} := \{z \in \mathbb{C} : |z| < 1\} \cup \{-1, 1\}$. We claim that the zeros cannot all lie at ± 1 . In fact, if they did then $\pi_n(x) - f(x)$ would be either strictly positive or strictly negative on $(-1, 1)$. If it was strictly positive then in particular we would have

$$-y_{n-1} - f(t_{n-1}) = \pi_n(t_{n-1}) - f(t_{n-1}) > 0,$$

which is not possible since $|f(t_{n-1})| \leq y_{n-1}$ by hypothesis. Similarly, if $\pi_n(x) - f(x)$ was strictly negative on $(-1, 1)$ then for $x = t_{n-2}$ we would obtain

$f(t_{n-2}) > y_{n-2}$, which is a contradiction since $f(t_{n-2})$ is supposed to be bounded above by y_{n-2} . Thus, the polynomial

$$\pi_n(z) - f(z) = (\pi_{n,n} - a_n)z^n + \sum_{\nu=1}^{n-1} (\pi_{n,n-\nu} - a_{n-\nu})z^{n-\nu} - a_0,$$

which has all its zeros in \mathcal{E} , has at least one in $\{z \in \mathbb{C} : |z| < 1\}$. Since $\mathcal{E} \subset \{z \in \mathbb{C} : |z| \leq 1\}$ this implies that

$$|a_0| < \pi_{n,n} - a_n,$$

and so (21) holds. \square

5. Bounds for the other coefficients. Generalizing Theorem A, Wladimir Markoff ([8]; see also [5]) found sharp bounds for all the Maclaurin coefficients of a polynomial f of degree at most n in terms of its maximum modulus on $[-1, 1]$.

Theorem E. *Let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a polynomial of degree at most n such that $|f(x)| \leq 1$ for $-1 \leq x \leq 1$. Then, $|a_{n-2\mu}|$ is bounded above by the modulus of the corresponding coefficient of T_n for $\mu = 0, \dots, \lfloor n/2 \rfloor$, and $|a_{n-1-2\mu}|$ is bounded above by the modulus of the corresponding coefficient of T_{n-1} for $\mu = 0, \dots, \lfloor (n-1)/2 \rfloor$.*

Erdős [4, p. 1176] remarked that if the coefficients of $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ are all real, then $\sum_{\nu=0}^n |a_\nu|$ is maximal for $f(z) \equiv \pm T_n(z)$. According to him, Szegő proved that $|a_{2k}| + |a_{2k+1}|$ is maximal for $f(z) \equiv \pm T_n(z)$, which is stronger than the observation made by Erdős himself. Unfortunately, Szegő never published his proof. Erdős learned about the result orally (see [4, p. 1176]) from Szegő. It would be nice to know Szegő's proof of his result.

A few years ago (see [3, Theorem 3]) we proved the following result, which covers that of Szegő.

Theorem F. *Let $x_0 < \dots < x_n$ be $n+1$ real numbers, and let y_0, \dots, y_n be a sequence of $n+1$ non-negative numbers, where we suppose that $x_\nu = -x_{n-\nu}$ and $y_\nu = y_{n-\nu}$ for $\nu = 0, \dots, n$, and that $\sum_{\nu=0}^n y_\nu > 0$. In addition, let $F(x) := \sum_{\mu=0}^{\lfloor n/2 \rfloor} A_{n-2\mu} x^{n-2\mu}$ be the unique polynomial of degree n such that $F(x_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, \dots, n$. Furthermore, let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a real polynomial of degree at most n , whose modulus does not exceed that of F at the points $x_0 < \dots < x_n$, that is*

$$|f(x_\nu)| \leq y_\nu = |F(x_\nu)| \quad (\nu = 0, \dots, n).$$

Then,

$$(22) \quad |a_{n-2k}| + |a_{n-2k-1}| \leq |A_{n-2k}| \quad \left(k = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor\right).$$

Theorem F says, in particular, that if $T_n(x) := \sum_{\nu=0}^{\lfloor n/2 \rfloor} t_{n,\nu} x^\nu$ is the Chebyshev polynomial of the first kind of degree n then, for any real polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree n satisfying (11), we have

$$(23) \quad |a_{n-2k}| + |a_{n-2k-1}| \leq |t_{n,n-2k}| \quad \left(k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right).$$

Not only T_n but all the ultraspherical polynomials $P_n^{(\lambda)}$ have the properties the polynomial F of Theorem F is required to have for (22) to hold. This observation suggests an inequality, more general than (23), involving the coefficients of $P_n^{(\lambda)}$ for any $\lambda > -1/2$ (see Corollary 4). In order to present such a generalization of (23) we need to recall some basic facts about the polynomials $P_n^{(\lambda)}$. This will come in handy in connection with certain extensions (see Theorem 3 and Corollary 5) of the well-known L^2 analogues of Chebyshev’s inequality, which we also intend to discuss in this paper.

6. The polynomials $P_n^{(\lambda)}$ and an extension of (23). For $\lambda \in (-\frac{1}{2}, 0) \cup (0, \infty)$, the ultraspherical polynomials $P_n^{(\lambda)}$ are given ([17], see (4.7.1) on p. 81 and (4.7.31) on p. 85) by

$$(24) \quad \begin{aligned} P_n^{(\lambda)}(x) &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n - k + \lambda)}{\Gamma(\lambda)\Gamma(k + 1)\Gamma(n - 2k + 1)} (2x)^{n-2k}. \end{aligned}$$

Thus $P_n^{(\lambda)}$ is a multiple of the Jacobi polynomial $P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}$. The multiplicative factor

$$\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})},$$

which depends on λ and n , vanishes for $\lambda = 0$. It is known ([17, p. 82], see (4.7.8)) that

$$\lim_{\lambda \rightarrow 0} \lambda^{-1} P_n^{(\lambda)}(x) = \frac{2}{n} T_n(x) \quad (n = 1, 2, \dots),$$

where T_n is the Chebyshev polynomial of the first kind of degree n . Hence, we may define

$$(25) \quad P_n^{(0)}(x) := \frac{2}{n}T_n(x) \quad (n = 1, 2, \dots).$$

It is known ([17, p. 82], see (4.7.15)) that if $\lambda \in (-\frac{1}{2}, 0) \cup (0, \infty)$, then

$$(26) \quad \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)}(x) P_m^{(\lambda)}(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2^{1-2\lambda} \pi \Gamma(n+2\lambda)}{\{\Gamma(\lambda)\}^2 (n+\lambda) \Gamma(n+1)} & \text{if } m = n. \end{cases}$$

In view of (25), the corresponding formula for $\lambda = 0$ is

$$(26') \quad \int_{-1}^1 P_n^{(0)}(x) P_m^{(0)}(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2\pi}{n^2} & \text{if } m = n \geq 1. \end{cases}$$

We find it convenient to define

$$(27) \quad P_n^{(\lambda)*}(x) := \begin{cases} \Gamma(\lambda) \sqrt{\frac{(n+\lambda)\Gamma(n+1)}{2^{1-2\lambda} \pi \Gamma(n+2\lambda)}} P_n^{(\lambda)}(x) & \text{if } \lambda > -\frac{1}{2}, \lambda \neq 0, \\ \frac{n}{\sqrt{2\pi}} P_n^{(0)}(x) & \text{if } \lambda = 0. \end{cases}$$

Because of (26) and (26'), the polynomials $\{P_n^{(\lambda)*}\}$, $\lambda > -1/2$ are orthonormal in the sense that

$$(28) \quad \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)*}(x) P_m^{(\lambda)*}(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

For any given $\lambda > -1/2$ and any $n \geq 2$, let $x_{n,1}(\lambda) < \dots < x_{n,n-1}(\lambda)$ be the extrema of $P_n^{(\lambda)*}$ in $(-1, 1)$. Then, Theorem F, applied with

$$x_0 := -1, x_\nu := x_{n,\nu}(\lambda) \text{ for } \nu = 1, \dots, n-1, x_n := 1$$

and

$$y_\nu := \left| P_n^{(\lambda)*}(x_\nu) \right| \quad (\nu = 0, \dots, n),$$

gives us the following result.

Corollary 4. *For any given $\lambda > -1/2$ and any integer $n \geq 2$, let*

$$(29) \quad P_n^{(\lambda)*}(x) := \sum_{\nu=0}^n p_{n,\nu}^{(\lambda)*} x^\nu$$

be the ultraspherical polynomial of degree n , as defined in (27). Also let $x_1 := x_{n,1}(\lambda), \dots, x_{n-1} := x_{n,n-1}(\lambda)$ be the extrema of $P_n^{(\lambda)*}$ in $(-1, 1)$, arranged in increasing order. Furthermore, let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a real polynomial of degree at most n such that

$$(30) \quad |f(x_\nu)| \leq \left| P_n^{(\lambda)*}(x_\nu) \right| \quad (\nu = 0, \dots, n),$$

where $x_0 := -1$ and $x_n := 1$. Then,

$$(31) \quad |a_{n-2k}| + |a_{n-2k-1}| \leq |p_{n,n-2k}^{(\lambda)*}| \quad \left(k = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor \right).$$

7. A refined L^2 analogue of Chebyshev’s inequality. In Corollary 4 we require $|f(x)|$ to be dominated by $|P_n^{(\lambda)*}(x)|$ at the points x_0, x_1, \dots, x_n . How large can $|a_{n-2k}| + |a_{n-2k-1}|$ be if $|f(x)|$ is not necessarily bounded by $|P_n^{(\lambda)*}(x)|$ at any specific points of $[-1, 1]$ but in some average sense, like

$$\frac{\int_{-1}^1 w(x)|f(x)|^2 dx}{\int_{-1}^1 w(x) dx} \leq \frac{\int_{-1}^1 w(x)|P_n^{(\lambda)*}(x)|^2 dx}{\int_{-1}^1 w(x) dx}$$

for some appropriate weight function w ? We are able to shed some light on this question in the case where $w(x) := (1 - x^2)^{\lambda - \frac{1}{2}}$, the weight whose relevance is clear from (28).

As in (29), let $p_{n,n}^{(\lambda)*}$ denote the leading coefficient in the Maclaurin expansion of $P_n^{(\lambda)*}(x)$. Then, from (27), (24), (25) and (4), it follows that

$$(32) \quad p_{n,n}^{(\lambda)*} = 2^n \sqrt{\frac{\Gamma(n + \lambda + 1)\Gamma(n + \lambda)}{2^{1-2\lambda}\pi\Gamma(n + 1)\Gamma(n + 2\lambda)}} \quad \left(\lambda > -\frac{1}{2} \right),$$

where the formula proves to be in order for $\lambda = 0$ because $P_n^{(0)} := (2/n)T_n$.

Theorem 3. For any given $\lambda > -1/2$ and any integer $n \geq 2$, let $p_{n,n}^{(\lambda)*}$ denote the coefficient of x^n in the Maclaurin expansion of the ultraspherical polynomial $P_n^{(\lambda)*}(x)$, as defined in (27). Furthermore, let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a polynomial of degree n with coefficients in \mathbb{C} such that

$$(33) \quad \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} |f(x)|^2 dx \leq \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \left| P_n^{(\lambda)*}(x) \right|^2 dx = 1.$$

Then,

$$(34) \quad |a_n|^2 + 4 \frac{(n + \lambda)(n + \lambda - 1)}{n(n + 2\lambda - 1)} |a_{n-1}|^2 \leq \left\{ p_{n,n}^{(\lambda)*} \right\}^2.$$

The inequality is best possible in the sense that the coefficient of $|a_{n-1}|^2$ in (34) cannot be replaced by any number larger than

$$4 \frac{(n + \lambda)(n + \lambda - 1)}{n(n + 2\lambda - 1)}.$$

It is well-known that the sharp upper bound for each of the two terms

$$|a_n|^2 \quad \text{and} \quad 4 \frac{(n + \lambda)(n + \lambda - 1)}{n(n + 2\lambda - 1)} |a_{n-1}|^2,$$

appearing on the left-hand side of (34) is $\left\{ p_{n,n}^{(\lambda)*} \right\}^2$. So apparently, inequality (34) is not to be sneezed at. Nevertheless, in view of (31), it would be interesting to know the largest admissible value of ε for the inequality

$$|a_n| + \varepsilon |a_{n-1}| \leq p_{n,n}^{(\lambda)*}$$

to be true for any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ satisfying the hypothesis of Theorem 3. The following corollary of Theorem 3 gives the sharp upper bound for $|a_n| + \varepsilon |a_{n-1}|$ in terms of $\varepsilon > 0$. The upper bound is larger than $p_{n,n}^{(\lambda)*}$ for any $\varepsilon > 0$.

Corollary 5. For any given $\lambda > -1/2$ and any integer $n \geq 2$, let $p_{n,n}^{(\lambda)*}$ denote the coefficient of x^n in the Maclaurin expansion of the ultraspherical polynomial $P_n^{(\lambda)*}(x)$, as defined in (27). Furthermore, let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a polynomial of degree n , with coefficients in \mathbb{C} , satisfying (33). Then,

$$(35) \quad |a_n| + \varepsilon |a_{n-1}| \leq \sqrt{1 + \frac{\varepsilon^2}{4} \frac{n(n + 2\lambda - 1)}{(n + \lambda)(n + \lambda - 1)}} p_{n,n}^{(\lambda)*} \quad (\varepsilon > 0).$$

The inequality is sharp. It cannot be improved even if the coefficients of f are all real.

Proof of Theorem 3. For any given $\lambda > -1/2$ there exist constants b_ν such that

$$f(x) = \sum_{\nu=0}^n b_\nu P_\nu^{(\lambda)*}(x).$$

Since $P_\nu^{(\lambda)*}$ is odd or even according as n is odd or even, respectively, we note that $p_{n,n-1}^{(\lambda)*} = 0$. Because of this fact, comparing the coefficients in the two expansions of $f(x)$ and taking (32) into account, we see that

$$(36) \quad a_n = p_{n,n}^{(\lambda)*} b_n = 2^n \sqrt{\frac{\Gamma(n + \lambda + 1)\Gamma(n + \lambda)}{2^{1-2\lambda}\pi\Gamma(n + 1)\Gamma(n + 2\lambda)}} b_n$$

and

$$(37) \quad a_{n-1} = p_{n-1,n-1}^{(\lambda)*} b_{n-1} = 2^{n-1} \sqrt{\frac{\Gamma(n + \lambda)\Gamma(n + \lambda - 1)}{2^{1-2\lambda}\pi\Gamma(n)\Gamma(n + 2\lambda - 1)}} b_{n-1}.$$

The orthonormality of $\{P_0^{(\lambda)*}, P_1^{(\lambda)*}, P_2^{(\lambda)*}, \dots\}$, given in (28), implies that

$$(38) \quad \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} |f(x)|^2 dx = \sum_{\nu=0}^n |b_\nu|^2 \geq |b_n|^2 + |b_{n-1}|^2,$$

where, for $n \geq 3$, the inequality becomes an equality if and only if b_0, \dots, b_{n-2} are all zero, that is, if and only if $f(x) := b_n P_n^{(\lambda)*}(x) + b_{n-1} P_{n-1}^{(\lambda)*}(x)$. From (33) and (38) it follows that

$$(39) \quad |b_n|^2 + |b_{n-1}|^2 \leq 1.$$

By (37), we have

$$4 \frac{(n + \lambda)(n + \lambda - 1)}{n(n + 2\lambda - 1)} |a_{n-1}|^2 = 2^{2n} \frac{\Gamma(n + \lambda + 1)\Gamma(n + \lambda)}{2^{1-2\lambda}\pi\Gamma(n + 1)\Gamma(n + 2\lambda)} |b_{n-1}|^2,$$

which, in conjunction with (36), shows that

$$|a_n|^2 + 4 \frac{(n + \lambda)(n + \lambda - 1)}{n(n + 2\lambda - 1)} |a_{n-1}|^2 = (|b_n|^2 + |b_{n-1}|^2) \left\{ p_{n,n}^{(\lambda)*} \right\}^2.$$

This, in view of (39), gives us the desired inequality (34).

In order to see that (34) is sharp let us consider, for any $t \in [0, 1]$, the polynomial

$$(40) \quad f(x) := te^{i\alpha} P_n^{(\lambda)*}(x) + \sqrt{1 - t^2} e^{i\beta} P_{n-1}^{(\lambda)*}(x) \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R}).$$

Then

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} |f(x)|^2 dx &= t^2 \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} \left| P_n^{(\lambda)*}(x) \right|^2 dx \\ &\quad + (1-t^2) \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} \left| P_{n-1}^{(\lambda)*}(x) \right|^2 dx \\ &= 1 \end{aligned}$$

by (28). Considering the Maclaurin expansion $f(x) := \sum_{\nu=0}^n a_{\nu} x^{\nu}$ of the polynomial defined in (40), we see that

$$a_n = t e^{i\alpha} p_{n,n}^{(\lambda)*} \quad \text{and} \quad a_{n-1} = \sqrt{1-t^2} e^{i\beta} p_{n-1,n-1}^{(\lambda)*}.$$

Hence

$$\begin{aligned} |a_n|^2 + 4 \frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)} |a_{n-1}|^2 &= t^2 \left\{ p_{n,n}^{(\lambda)*} \right\}^2 + (1-t^2) 4 \frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)} \left\{ p_{n-1,n-1}^{(\lambda)*} \right\}^2 \\ &= t^2 \left\{ p_{n,n}^{(\lambda)*} \right\}^2 + (1-t^2) \left\{ p_{n,n}^{(\lambda)*} \right\}^2 = \left\{ p_{n,n}^{(\lambda)*} \right\}^2. \end{aligned}$$

The above calculations also show that (34) would fail if its left-hand side was replaced by $|a_n|^2 + c |a_{n-1}|^2$ with any

$$c = c(n, \lambda) > 4 \frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}. \quad \square$$

Proof of Corollary 5. From (34) it follows that $|a_n| \leq p_{n,n}^{(\lambda)*}$ and then

$$|a_{n-1}| \leq \frac{1}{2} \sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}} \sqrt{\left\{ p_{n,n}^{(\lambda)*} \right\}^2 - |a_n|^2}.$$

Hence $|a_n| + \varepsilon |a_{n-1}| \leq \varphi(|a_n|)$, where

$$\varphi(u) := u + \frac{\varepsilon}{2} \sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}} \sqrt{\left\{ p_{n,n}^{(\lambda)*} \right\}^2 - u^2}$$

for $0 \leq u \leq p_{n,n}^{(\lambda)*}$. The function φ has only one critical point

$$u = \frac{1}{\sqrt{1 + \frac{1}{4} \frac{n(n+2\lambda-1)}{(n+\lambda)((n+\lambda-1))} \varepsilon^2}} p_{n,n}^{(\lambda)*}$$

in $(0, p_{n,n}^{(\lambda)*})$ and there it has a local maximum. Simple calculations then lead us to the estimate for $|a_n| + \varepsilon|a_{n-1}|$ that is given in (35).

To see that (35) cannot be improved even if $f(x)$ is real for all real x , let

$$\delta := \frac{p_{n,n}^{(\lambda)*}}{p_{n-1,n-1}^{(\lambda)*}} = 2 \sqrt{\frac{(n + \lambda + 1)(n + \lambda)}{n(n + 2\lambda - 1)}}$$

and, for any $\varepsilon > 0$, consider the real polynomial

$$f(x; \varepsilon) := \frac{\delta}{\sqrt{\varepsilon^2 + \delta^2}} P_n^{(\lambda)*}(x) + \frac{\varepsilon}{\sqrt{\varepsilon^2 + \delta^2}} P_{n-1}^{(\lambda)*}(x) = \sum_{\nu=0}^n a_\nu x^\nu.$$

Then,

$$a_n = \frac{\delta}{\sqrt{\varepsilon^2 + \delta^2}} p_{n,n}^{(\lambda)*}$$

and

$$a_{n-1} = \frac{\varepsilon}{\sqrt{\varepsilon^2 + \delta^2}} p_{n-1,n-1}^{(\lambda)*} = \frac{\varepsilon}{\delta \sqrt{\varepsilon^2 + \delta^2}} p_{n,n}^{(\lambda)*}.$$

Clearly,

$$\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} |f(x; \varepsilon)|^2 dx = \frac{\delta^2}{\delta^2 + \varepsilon^2} + \frac{\varepsilon^2}{\delta^2 + \varepsilon^2} = 1$$

and

$$|a_n| + \varepsilon|a_{n-1}| = \frac{\delta}{\sqrt{\varepsilon^2 + \delta^2}} p_{n,n}^{(\lambda)*} + \frac{\varepsilon^2}{\delta \sqrt{\varepsilon^2 + \delta^2}} p_{n,n}^{(\lambda)*} = \sqrt{1 + \frac{\varepsilon^2}{\delta^2}} p_{n,n}^{(\lambda)*}.$$

This shows that (35) becomes an equality for the polynomial $f(x; \varepsilon)$. \square

8. The L^p mean of f with Chebyshev weight. Let $\Phi(z) := \sum_{\mu=0}^m c_\mu z^\mu$ be a polynomial of degree $m \geq 1$. In addition, let $A(z) := z^m + 1$. It was shown by one of us [10, Theorem 2] that for any $p \in [1, \infty)$, we have

$$(41) \quad |c_0| + |c_m| \leq \frac{2}{\left(\frac{1}{2\pi} \int_{-\pi}^\pi |A(e^{i\theta})|^p d\theta\right)^{1/p}} \left(\frac{1}{2\pi} \int_{-\pi}^\pi |\Phi(e^{i\theta})|^p d\theta\right)^{1/p}.$$

If $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ is a polynomial of degree n then

$$\Phi(z) := z^n f\left(\frac{z + z^{-1}}{2}\right) = \sum_{\nu=0}^{2n} c_\nu z^\nu$$

is a polynomial of degree $2n$, where

$$c_0 = c_{2n} = \frac{1}{2^n} a_n,$$

and (41) may be applied with $m = 2n$ to obtain

$$\begin{aligned} |a_n| &\leq \frac{2^{n-1}}{\left(\frac{1}{\pi} \int_0^\pi |\cos n\theta|^p d\theta\right)^{1/p}} \left(\frac{1}{\pi} \int_0^\pi |f(\cos \theta)|^p d\theta\right)^{1/p} \\ (42) \quad &= \frac{2^{n-1}}{\left(\frac{1}{\pi} \int_0^\pi |T_n(x)|^p \frac{dx}{\sqrt{1-x^2}}\right)^{1/p}} \left(\frac{1}{\pi} \int_{-1}^1 |f(x)|^p \frac{dx}{\sqrt{1-x^2}}\right)^{1/p} \end{aligned}$$

for any $p \in [1, \infty)$. Note that this result is a direct consequence of (41) and that the restriction imposed on p comes solely from the restriction on p in that inequality. Hence, (42) should hold for any p for which (41) may turn out to be true. In [13, Theorem 3], it was shown that (41) holds not only for $p \in [1, \infty)$ but also for $p \in (0, 1)$. Hence, so does (42). Thus, the following result holds. Other references for this result are [6, p. 183] and [15, p. 120].

Theorem G. *For any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree n and any $p \in (0, \infty)$, we have*

$$(43) \quad |a_n| \leq 2^{n-1} \left(\frac{\pi \Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \right)^{1/p} \left(\frac{1}{\pi} \int_{-1}^1 |f(x)|^p \frac{dx}{\sqrt{1-x^2}} \right)^{1/p}.$$

The inequality is sharp for all $p \in (0, \infty)$.

It may be noted that

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^\pi d\theta = \pi$$

and that

$$\left(\frac{1}{\pi} \int_{-1}^1 |f(x)|^p \frac{dx}{\sqrt{1-x^2}} \right)^{1/p} \rightarrow \max_{-1 \leq x \leq 1} |f(x)| \text{ as } x \rightarrow \infty.$$

Hence, (43) is a generalization of (5).

In looking for a generalization (of the case $\lambda = 0$) of Theorem 3 in the spirit of Theorem G we did not find anything of interest for $p \in (2, \infty)$ or for $p \in (0, 1]$. We do have the following result for values of $p \in (1, 2]$.

Theorem 4. For any $p \in (1, 2]$ let

$$p' := \frac{p}{p-1}.$$

Then, for any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree $n \geq 2$ with coefficients in \mathbb{C} and any $p \in (1, 2]$, we have

$$(44) \quad \left(|a_n|^{p'} + |2a_{n-1}|^{p'}\right)^{1/p'} \leq 2^{n-1} 2^{1/p} \left(\frac{1}{\pi} \int_{-1}^1 |f(x)|^p \frac{dx}{\sqrt{1-x^2}}\right)^{1/p}.$$

Proof. As the first step, we write $f(x) = \sum_{\nu=0}^n b_\nu T_\nu(x)$. Then, clearly

$$(45) \quad a_n = 2^{n-1}b_n \quad \text{and} \quad a_{n-1} = 2^{n-2}b_{n-1}.$$

Let

$$g(\theta) := f(\cos \theta) = \sum_{\nu=0}^n b_\nu \cos \nu\theta = \sum_{\nu=-n}^n c_\nu e^{i\nu\theta},$$

where

$$(46) \quad c_0 = b_0 \quad \text{and} \quad c_{-\nu} = c_\nu = \frac{1}{2}b_\nu \quad (\nu = 1, \dots, n).$$

By a well-known result due to Hausdorff–Young (see [22, p. 101]), if g belongs to $L^p(0, 2\pi)$ for some $p \in (1, 2]$ and

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-ik\theta} d\theta \quad (k = 0, \pm 1, \pm 2, \dots),$$

then with $p' := p/(p-1)$, we have

$$\left(\sum_{k=-\infty}^{\infty} |c_k|^{p'}\right)^{1/p'} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |g(\theta)|^p d\theta\right)^{1/p}.$$

Hence, taking (46) into account, we obtain

$$\begin{aligned} \frac{1}{2^{1/p}} \left(|b_n|^{p'} + |b_{n-1}|^{p'} \right)^{1/p'} &= \left\{ 2 \left(\frac{|b_n|}{2} \right)^{p'} + 2 \left(\frac{|b_{n-1}|}{2} \right)^{p'} \right\}^{1/p'} \\ &\leq \left(\sum_{\nu=-n}^n |c_\nu|^{p'} \right)^{1/p'} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} |g(\theta)|^p d\theta \right)^{1/p} = \left(\frac{1}{\pi} \int_0^\pi |g(\theta)|^p d\theta \right)^{1/p} \\ &= \left(\frac{1}{\pi} \int_{-1}^1 |f(x)|^p \frac{dx}{\sqrt{1-x^2}} \right)^{1/p}. \end{aligned}$$

In view of (45), this is equivalent to (44). \square

9. Polynomials satisfying $\int_{-\infty}^\infty |f(x)|^2 e^{-x^2} dx \leq 1$. In this last section we wish to present the analogue of (34) for polynomials $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree n for which $\int_{-\infty}^\infty |f(x)|^2 e^{-x^2} dx \leq 1$. For this we need to recall certain facts about Hermite polynomials [19, p. 52].

There is a unique polynomial $\sum_{\nu=0}^n a_\nu x^\nu$ of degree n , with prescribed $a_n \neq 0$, that satisfies the differential equation

$$y'' - 2xy' + 2ny = 0.$$

The one whose leading term is $2^n x^n$ is denoted by H_n and is called the Hermite polynomial of degree n . Hermite polynomials of odd degree are odd and those of even degree are even. The first four Hermite polynomials are

$$H_0 = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x,$$

and for $n = 4, 5, 6, \dots$ the Maclaurin expansion of H_n is

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!} (2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} - \dots.$$

The Hermite polynomials are orthogonal, with

$$\int_{-\infty}^\infty H_n(x) H_m(x) e^{-x^2} dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2^n \times n! \sqrt{\pi} & \text{if } m = n. \end{cases}$$

Hence, the polynomials

$$(47) \quad H_n^*(x) := \frac{1}{\sqrt{2^n n! \pi^{1/4}}} H_n(x) \quad (n = 0, 1, 2, \dots),$$

are orthonormal in the sense that

$$(48) \quad \int_{-\infty}^{\infty} H_n^*(x)H_m^*(x)e^{-x^2} dx = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Let $H_m^*(x) := \sum_{\mu=0}^m h_{m,\mu}^* x^\mu$ be the Maclaurin expansion of $H_m^*(x)$, and note that

$$(49) \quad h_{m,m}^* = \frac{1}{\pi^{1/4}} \sqrt{\frac{2^m}{m!}} \quad (m = 0, 1, 2, \dots).$$

Now, we are ready to prove the following analogue of Theorem 3.

Theorem 5. *Let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a polynomial of degree n with coefficients in \mathbb{C} such that*

$$(50) \quad \int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx \leq 1.$$

Then,

$$(51) \quad |a_n|^2 + \frac{2}{n} |a_{n-1}|^2 \leq \frac{2^n}{n! \sqrt{\pi}}.$$

The inequality is best possible in the sense that the coefficient of $|a_{n-1}|^2$ in (51) cannot be replaced by any number larger than $2/n$.

Here it may be mentioned that the sharp upper bound for each of the two terms

$$|a_n|^2 \quad \text{and} \quad \frac{2}{n} |a_{n-1}|^2,$$

appearing on the left-hand side of (51) is also $2^n/(n! \sqrt{\pi})$.

Proof of Theorem 5. Let $f(x) = \sum_{m=0}^n \beta_m H_m^*(x)$ be the Hermite-Fourier expansion of f in terms of H_0^*, \dots, H_n^* . Then

$$(52) \quad \sum_{\nu=0}^n a_\nu x^\nu \equiv \sum_{m=0}^n \beta_m \sum_{\mu=0}^m h_{m,\mu}^* x^\mu.$$

Because of the fact that $h_{n,n-1}^* = 0$, when we compare the coefficients of x^n and of x^{n-1} on the two sides of (52), and take (49) into account, we obtain

$$(53) \quad a_n = h_{n,n}^* \beta_n = \frac{1}{\pi^{1/4}} \sqrt{\frac{2^n}{n!}} \beta_n$$

and

$$(54) \quad a_{n-1} = h_{n-1,n-1}^* \beta_{n-1} = \frac{1}{\pi^{1/4}} \sqrt{\frac{2^{n-1}}{(n-1)!}} \beta_{n-1}.$$

Formula (48), applied in conjunction with condition (50), which f satisfies, implies that

$$(55) \quad |\beta_n|^2 + |\beta_{n-1}|^2 \leq 1.$$

Using (53) and (54) in (55), we obtain

$$|a_n|^2 + \frac{2}{n} |a_{n-1}|^2 = (|\beta_n|^2 + |\beta_{n-1}|^2) \frac{2^n}{n! \sqrt{\pi}} \leq \frac{2^n}{n! \sqrt{\pi}},$$

which proves (51).

It is easily checked that (51) becomes an equality for any polynomial of the form

$$(56) \quad f_t(x) := te^{i\alpha} H_n^*(x) + \sqrt{1-t^2} e^{i\beta} H_{n-1}^*(x) \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R}),$$

where t can be any number in $[0, 1]$.

For any t in $[0, 1)$, the polynomial f_t appearing in (56) shows that the coefficient of $|a_{n-1}|^2$ in (51) cannot be replaced by any number larger than $2/n$. \square

By Schwarz's inequality,

$$|a_n| + \varepsilon |a_{n-1}| \leq \sqrt{1 + \varepsilon^2 \frac{n}{2}} \sqrt{|a_n|^2 + \frac{2}{n} |a_{n-1}|^2},$$

and so Theorem 5 readily implies the following result

Corollary 6. *Let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be a polynomial of degree n , with coefficients in \mathbb{C} , satisfying (50). Furthermore, let $h_{n,n}^*$ denote the coefficient of x^n in the Maclaurin expansion of the polynomial $H_n^*(x)$ defined in (47). Then*

$$(57) \quad |a_n| + \varepsilon |a_{n-1}| \leq \sqrt{1 + \varepsilon^2 \frac{n}{2}} \frac{1}{\pi^{1/4}} \sqrt{\frac{2^n}{n!}}.$$

The example

$$f(x) := \frac{\delta}{\sqrt{\varepsilon^2 + \delta^2}} H_n^*(x) + \frac{\varepsilon}{\sqrt{\varepsilon^2 + \delta^2}} H_{n-1}^*(x), \quad \delta := \sqrt{\frac{2}{n}}$$

shows that inequality (57) is sharp, and that it cannot be improved even if the coefficients of f are all real.

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Received May 29, 2007