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SOBOLEV TYPE DECOMPOSITION OF PALEY-WIENER-SCHWARTZ SPACE WITH APPLICATION TO SAMPLING THEORY

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Communicated by G. Nikolov

Dedicated to Academician Blagovest Sendov
on the occasion of his seventy-fifth anniversary

Abstract. We characterize Paley-Wiener-Schwartz space of entire functions as a union of three-parametric linear normed subspaces determined by the type of the entire functions, their polynomial asymptotic on the real line, and the index $p \geq 1$ of a Sobolev type L_p-summability on the real line with an appropriate weight function. An entire function belonging to a sub-space of the decomposition is exactly recovered by a sampling series, locally uniformly convergent on the complex plane. The sampling formulas obtained extend the Shannon sampling theorem, certain representation formulas due to Bernstein, and a transcendental interpolating theory due to Levin.

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Key words: Paley-Wiener-Schwartz space, Shannon sampling theorem, Tschakaloff-Bernstein representation formulas, Levin transcendental interpolating theory.

1. Introduction. We say that an entire function $f(z)$ is slowly increasing if it belongs to Paley-Wiener-Schwartz (PWS) space, i.e., if for some constants $c > 0$, m a non-negative integer, and $\sigma \geq 0$

$$|f(z)| \leq c(1 + |z|)^m \, e^{\sigma|\Im(z)|}, \quad z \in \mathbb{C}.$$

According to a result of Hörmander [10], [12], [17] an entire function belongs to PWS space if and only if its Fourier transform is a tempered distribution with a compact support in $[-\sigma, \sigma]$. In Section 2 of the present article we give a representation of the PWS space in terms of a two-parametric decomposition into sub-spaces with respect to the degree of the polynomial asymptotic on the real line and the type of the entire functions. We show (see Lemma 1) that a function is from a sub-space of our decomposition if and only if it is of given exponential type and has a prescribed asymptotic on the real line. In Section 3 we characterize the PWS space as **a** union of three-parametric linear normed subspaces determined by the type of the entire functions, their polynomial asymptotic on the real line, and the index $p \geq 1$ of Sobolev type L_p-summability of the entire functions on the real line with an appropriate weight function (see Theorem 1 and Theorem 2). Each entire function belonging to a sub-space of the decomposition is representable by a locally uniformly convergent sampling series. The sampling formula obtained extends the Shannon sampling theorem [8], [13], [18], [20] to band-limited signals having a polynomial time asymptotic preserving the optimal Nyquist rate of sampling. The Shannon sampling theorem is a fundamental result in the field of information theory, in particular telecommunications. Sampling is a process of converting a signal (a function of continuous time) into a numeric sequence (a function of discrete time). The process is called also analog-to-digital conversion or digitizing. A sampling procedure gives a digital sequence (sequence of numbers, functional values) which permits a complete recovery of the signal with bounded highest frequency. We obtain Shannon type sampling formulas that are of practical usefulness when digitize signals with polynomial growth on the time axis. They can be used to accelerate the convergence of sampling series even in the classical Shannon case. We extend also certain representation formulas due to Bernstein [2], [3] (see Example 1) and a transcendental interpolating theory due to Levin [14] (see Corollary 2). By using a comparison with similar results in the sampling theory [6], [5], [16] we attempt to see the usefulness of the results obtained in the present article.

2. Decomposition of Paley-Wiener-Schwartz Space.

Definition. *An entire function f is said to be of exponential type $\sigma > 0$ if for every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that*

$$|f(z)| \leq c(\varepsilon)\, e^{(\sigma+\varepsilon)\,|z|} \qquad (z \in \mathbb{C})\ .$$

Thus, any entire function of order $\rho < 1$ is of exponential type σ and so is any entire function of order 1 and type at most σ. Let us mention that the representation theory for entire functions of exponential type is a useful tool for establishing results on approximate recovery of functions on the real line. Bernstein [1] started the subject by proving an analog of Weierstrass approximation theorem: A function defined on $\mathbb{R}$ can be approximated arbitrary closely by entire functions of exponential type in $C(\mathbb{R})$-metric if and only if it is uniformly continuous. An interesting recent contribution to the theory of approximation by entire functions of exponential type can be found in [15].

For our study will be more convenient to give the following definition for the linear space of PWS functions that is obviously equivalent to the usual one.

Definition. *Let $\sigma \geq 0$ be fixed. An entire function $f(z)$ is from the linear space PWS_σ if for some non-negative integer m*

$$|f(z)| = o(|z|^m\, e^{\sigma|\Im(z)|}), \quad |z| \to \infty \ \ (z \in \mathbb{C}).$$

Then obviously,

$$\mathrm{PWS} = \cup_{\sigma \geq 0}\, \mathrm{PWS}_\sigma\ .$$

According to a result of Hörmander [10], [12], [17] an entire function is from PWS_σ if and only if its Fourier transform is a tempered distribution with a compact support in $[-\sigma, \sigma]$. If $\sigma = 0$, i.e., $f(z) \in \mathrm{PWS}_0$, then $f(z)$ is a polynomial of degree at most $m-1$ and its Fourier transform has a support at 0. Hence, the non-trivial considerations are for $\sigma > 0$.

Let $\mathbb{N}_0$ denote the set of all non-negative integers.

Definition. *Let $m \in \mathbb{N}_0$ and $\sigma \geq 0$ be fixed. An entire function $f(z)$ belongs to* $\mathrm{PWS}_{\sigma,m}$ *(a sub-space of* PWS_σ*) with parameters σ and m if*

$$|f(z)| = o(|z|^m\, e^{\sigma|\Im(z)|}), \quad |z| \to \infty \ \ (z \in \mathbb{C}).$$

Obviously,

$$\mathrm{PWS}_\sigma = \cup_{m\in\mathbb{N}_0} \mathrm{PWS}_{\sigma,m} \quad \text{and} \quad \mathrm{PWS} = \cup_{\sigma\ge 0} \mathrm{PWS}_\sigma = \cup_{\sigma\ge 0, m\in\mathbb{N}_0} \mathrm{PWS}_{\sigma,m}\,.$$

We prove that the set $\mathrm{PWS}_{\sigma,m}$ is in fact the linear space of all entire functions of exponential type σ having $o(|x|^m), |x| \to \infty$ polynomial asymptotic on the real line. If $f \in \mathrm{PWS}_{\sigma,m}$ then trivially, it is of exponential type σ with a polynomial asymptotic $o(|x|^m), |x| \to \infty$ on $\mathbb{R}$ but the converse statement is not obvious. In order to establish the converse statement we need auxiliary results, some of them well known.

Definition of a harmonic measure of an arc. *Let $\mathbb{D}$ be a domain bounded by a finite number of rectifiable Jordan curves Γ. Let $\Gamma = \alpha_1 \cup \alpha_2$ and* $\mathrm{int}(\alpha_1)\cap\mathrm{int}(\alpha_2) = \emptyset$, *where α_1 and α_2 are finite sets of Jordan arcs. The function $\omega(z, \alpha_1; \mathbb{D})$ which is harmonic in $\mathbb{D}$ and takes value 1 on α_1 and value 0 on α_2 is called* a harmonic measure of α_1. *Then $\omega(z, \alpha_1; \mathbb{D}) + \omega(z, \alpha_2; \mathbb{D}) = 1 (z \in \mathbb{D})$ and this is called a harmonic unit decomposition.*

Here is an important theorem that plays an auxiliary role in our considerations.

Theorem A (brothers Nevanlinna and Ostrowskii n-constants theorem) [11]**.** *Suppose f is harmonic in a domain $\mathbb{D}$ with a boundary, the union of n distinct rectifiable arcs $\alpha_1, \dots, \alpha_n$ and for each j there is a constant M_j such that if z approaches any point from α_j, the limits of $|f(z)|$ do not exceed M_j (in absolute value). Then,*

$$\log|f(z)| \le \sum_{j=1}^{n} \omega(z, \alpha_j; \mathbb{D}) \log M_j\,.$$

Next, we present a Lindelöf type result [11] on uniform limiting behavior of a holomorphic function.

Lemma A. *Let $f(z)$ be holomorphic and bounded in the upper half plane. If $f(z)$ is continuous at all finite points of the real axis, and $f(x) \to l$, $x \to \infty$, then*

$$\lim_{z\to\infty} f(z) = l$$

uniformly in any sector $0 \le \arg(z) \le \pi - \delta,\ \ \delta > 0$.

P r o o f o f L e m m a A. Without any restriction we suppose that

$$|f(z)| \leq 1 \quad (\Im(z) > 0) \quad \text{and} \quad l = 0.$$

Given $\varepsilon > 0$ we can find x_ε large enough $x_\varepsilon \in \mathbb{R}$ such that

$$|f(x)| \leq \varepsilon \qquad (x_\varepsilon \leq x < \infty).$$

We apply the n-constants theorem of brothers Nevanlinna and Ostrowskii for $n = 2$ in the upper half plane $\mathbb{D}$ with arcs $\alpha_2 = (-\infty, x_\varepsilon)$, $M_2 = 1$ and $\alpha_1 = (x_\varepsilon, \infty)$, $M_1 = \varepsilon$ to obtain

$$\log |f(z)| \leq \omega_1(z, \alpha; \mathbb{D}) \log \varepsilon \quad (0 < \varepsilon < 1),$$

where $\omega_1(z, \alpha; \mathbb{D})$ is the harmonic measure of (x_ε, ∞) with respect to $\mathbb{D}$ and evaluated at the point z. Observe that

$$\omega_1(z, \alpha; \mathbb{D}) = \frac{\varphi}{\pi} = 1 - \frac{1}{\pi} \arg(z - x_\varepsilon),$$

where φ is the angle under which (x_ε, ∞) is seen from the point z. In the sector $0 \leq \arg(z - x_\varepsilon) \leq \pi(1 - \gamma)$ we have

$$\gamma \leq \omega_1(z, \alpha; \mathbb{D}) \leq 1.$$

Hence,

$$\log |f(z)| \leq \gamma \log \varepsilon, \qquad |f(z)| \leq \varepsilon^\gamma$$

and we complete the proof. □

In Lemma 1 we establish a result on asymptotic global behavior of entire functions of exponential type on the complex plane $\mathbb{C}$ with isolated point at ∞. It shows that a function is from $\mathrm{PWS}_{\sigma,m}$ if and only if it is entire of exponential type σ and has asymptotic $o(|x|^m), |x| \to \infty$ on the real line.

Lemma 1. *Let f be entire function of exponential type σ. Then*

$$f(x) = o(x^m), \quad |x| \to \infty \ (x \in \mathbb{R})$$

if and only if

$$f(z) = o(z^m e^{\sigma |y|}), \ |z| \to \infty \ (z \in \mathbb{C}, z = x + iy, i^2 = -1)$$

uniformly on the complex plane $\mathbb{C}$*.*

Proof. First note that if $f(x) = O\,(\,x^m\,)$, $|x| \to \infty$ $(x \in \mathbb{R})$, then the auxiliary entire function

$$r(z) \;:=\; \left(f(z) \,-\, \sum_{j=0}^{m-1} (\, f^{(j)}(0)/j!\,)\, z^j \right) \Big/ \; z^m$$

is $O(1)$, $|x| \to \infty$. Applying Bernstein's inequality for entire functions of exponential type [1] we obtain $(z = x + iy,\; i^2 = -1)$

$$|r(z)| \le \sum_{k=0}^{\infty} |r^{(k)}(x)||y|^k/k! \le |r|_{C(\mathbb{R})} \sum_{k=0}^{\infty} (\sigma|y|)^k/k! = O(e^{\sigma\,|y|}),$$

uniformly on the complex plane $\mathbb{C}$. Hence,

$$f(z) = O\left(z^m e^{\sigma\,|y|}\right)$$

and this is a well known result [1].

However, our goal is to prove not $O(\circ)$ but $o(\circ)$ estimate. Analogously, consider the auxiliary entire function of exponential type σ

$$r(z) := \left(f(z) \,-\, \sum_{j=0}^{m-1} (f^{(j)}(0)/j!)\, z^j \right) \Big/ \; z^m.$$

Then, the entire function $R(z) \;:=\; r(z)\, e^{i\sigma z}$ is of exponential type with a Phragmén-Lindelöf logarithmic indicator [4] at $\pi/2$

$$h_R\,(\pi/2) \;:=\; \limsup_{y\to\infty} \frac{\log |R(iy)|}{y} \;\le\; 0.$$

In view of

$$|R(x)| \;=\; |r(x)| \;=\; o(1), \quad |x| \to \infty,$$

the function R is bounded on $\mathbb{R}$. Hence [4, Theorem 6.2.4], the function R is bounded on the upper half plane. Since $R(x) \to 0$, $x \to \infty$, Lemma A implies that $R(z) \to 0$, $|z| \to \infty$ and uniformly in $0 \le \arg(z) \le \pi/2$. In view of this we obtain

$$r(z)\, e^{-\sigma\,|y|} \;\to\; 0, \quad |z| \;\to\; \infty,$$

uniformly in $0 \leq \arg(z) \leq \pi/2$. Similar arguments show that in any sector of opening $\pi/2$ the above limit holds hence, we can conclude that the above limit holds uniformly in $0 \leq \arg(z) \leq 2\pi$,i.e., uniformly on $\mathbb{C}$. This completes the proof. $\square$

Remark. Lemma 1 shows that the Paley-Wiener-Schwartz space can be characterized as the linear space of all entire functions of exponential type having a polynomial asymptotic on the real line.

3. Shannon Sampling Theorem for Bandlimited Signals with Polynomial Time Asymptotic. Let us denote by $B_{\sigma,p}$ the normed linear space of all entire functions of exponential type that belong to $L_p(\mathbb{R})$ on the real line, $1 \leq p < \infty$. A σ band-limited signal is a function of continuous time (the real line) such that its Fourier transform, i.e., its representation as a function of the frequency variable has a finite support in $[-\sigma, \sigma]$. A signal is said to have a bounded energy if it is $L_2(\mathbb{R})$-summable. Hence, the set of all σ band-limited signals with bounded energy is in fact the normed linear space $B_{\sigma,2}$ equipped with $L_2(\mathbb{R})$ norm. According to a result of Paley-Wiener [1]: $f \in B_{\sigma,2}$ if and only if $\operatorname{supp}(\hat{f}) \subset [-\sigma, \sigma]$ and $\hat{f} \in L_2[-\sigma, \sigma]$. Sampling is a process of converting a signal (function of continuous time) into a numeric sequence (function of discrete time). The process is based on an appropriate interpolating formula given below and it is called also analog-to-digital conversion, or digitizing.

Shannon Sampling Theorem [8], [13], [18], [20]**.** *Let $f \in B_{\sigma,2}$. Then*

$$f(z) = \sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi},$$

the sampling series being absolutely and uniformly convergent on $\mathbb{R}$ and l.u.c. on $\mathbb{C}$.

Remark. The meaning of the Shannon sampling formula (see above) is that the signal $f(x)$ of continuous time x is completely determined by the digit sequence $\{f(k\pi/\sigma),\ k \in \mathbb{Z}\}$ that is a function of discrete time. The Shannon sampling theorem holds also for the normed linear space $B_{\sigma,p}$, endowed with $L_p(\mathbb{R})$ norm, of all σ-band limited signals that are $L_p(\mathbb{R})$-summable on the real line $(1 \leq p < \infty)$ [9]. It is known [1] that $B_{\sigma,p_1} \subset B_{\sigma,p_2}$ for $1 \leq p_1 < p_2 < \infty$. Note that for $p > 2$ the Fourier transform of $f \in B_{\sigma,p}$ is a generalized function so, some additional considerations are needed in the proof of the Shannon formula for $B_{\sigma,p}$ signals, $p > 2$. The Shannon sampling theorem states conditions under which a sampling of σ band-limited signal at a rate π/σ represents no loss of information and can therefore be used to reconstruct (recover) the original signal with arbitrary at the very beginning chosen accuracy. The sampling theorem states that the signal must be band-limited which is a natural restriction assuming that there are no waves with arbitrary small length, and that the rate of sampling must be at least twice the reciprocal value of the signal bandwidth (multiplied by π). *The sampling rate π/σ is called Nyquist rate and it is optimal in a sense that no information is lost if a signal is sampled at the Nyquist rate, and no additional information is gained by sampling faster than this rate.*

Remark. We obtain exact sampling recovery by digitizing of band-limited signals having a polynomial time asymptotic at the optimal Nyquist rate (see Theorem 1 and Theorem 2). Even in the classical case of $B_{\sigma,2}$ signals the Shannon sampling series can be very slowly convergent. By using the sampling formulas obtained we can accelerate the convergence of the sampling series at the cost of a finite number of additional digits (functional values and values of the derivatives) to be added to the sampling sequence.

The proof of the sampling formula for PWS functions (see Theorem 1) uses the following *Hermite weighted polynomial interpolation.*

Lemma 2. *Let f be enough smooth function, let $S_1 := \{m_\mu \pi/\sigma,\ \mu = 1,\dots,r_1\}$ be a set of zeros of $\sin\sigma z$, and let $S_2 := \{z_\nu,\ \nu = 1,\dots,r_2\}$ be a set of points non of them zero of $\sin\sigma z$. Let $\{\lambda_1,\dots,\lambda_{r_1}\}$ and $\{\beta_1,\dots,\beta_{r_2}\}$ be two sets of positive integers such that*

$$\sum_{\mu=1}^{r_1} \lambda_\mu + \sum_{\nu=1}^{r_2} \beta_\nu = m\,.$$

Then, there exists a unique algebraic polynomial q_{m-1} of degree $\le m-1$ which is a solution of the following weighted interpolating problem:

$$(\sin\sigma z\ q_{m-1}(z))^{(j)}_{z=m_\mu\pi/\sigma} = \left(f(z) - \sum_{\kappa=1}^{r_1} f\left(\frac{m_\kappa \pi}{\sigma}\right) \frac{\sin(\sigma z - m_\kappa\pi)}{\sigma z - m_\kappa \pi} \right)^{(j)}_{z=\frac{m_\mu\pi}{\sigma}}$$

for $\quad j = 1,\dots,\lambda_\mu \quad$ and $\quad \mu = 1,\dots,r_1\,,$

and

$$(\sin\sigma z\; q_{m-1}(z))^{(j)}_{z=z_\nu} = \left(f(z) - \sum_{\kappa=1}^{r_1} f\left(\frac{m_\kappa\pi}{\sigma}\right)\frac{\sin(\sigma z - m_\kappa\pi)}{\sigma z - m_\kappa\pi}\right)^{(j)}_{z=z_\nu}$$

$$\text{for} \quad j = 0,\dots,\beta_\nu - 1 \quad \text{and} \quad \nu = 1,\dots,r_2\,.$$

Remark. The coefficients of q_{m-1} are uniquely determined by the interpolation data $f^{(j)}(m_\mu\pi/\sigma)$ $(j = 0,1,\dots,\lambda_\mu;\ \mu = 1,\dots,r_1)$ and $f^{(j)}(z_\nu)$ $(j = 0,1,\dots,\beta_\nu - 1;\ \nu = 1,\dots,r_2)$.

Proof of Lemma 2. Applying Newton-Leibnitz differentiation rule we obtain

$$(\sin\sigma z\; q_{m-1}(z))^{(j)}_{x=m_\mu\pi/\sigma} = j\,\sigma\,(-1)^{m_\mu}\, q^{(j-1)}_{m-1}(m_\mu\pi/\sigma) + \sum_{s=0}^{j-2}\alpha_{s,j}\, q^{(s)}_{m-1}(m_\mu\pi/\sigma)$$

to conclude that the interpolating conditions for the function $\sin\sigma z\; q_{m-1}(z)$ at the point $m_\mu\pi/\sigma \in S_1$, are in fact Taylor's type interpolating conditions up to $(\lambda_\mu - 1)$-derivative for $q_{m-1}(z)$ at the same point $m_\mu\pi/\sigma$. Summing up, we have Hermite interpolation problem for q_{m-1} based on m interpolation conditions and this ends the proof. □

Let us denote by E_σ the complex vector space of all entire functions of exponential type $\sigma > 0$ and let us define for fixed $\sigma > 0$, $m \in \mathbb{N}_0$, and $p \geq 1$ the following subspace of E_σ:

$$A_{\sigma,p,m} := \left\{ f : f \in E_\sigma;\ f(x) = o(x^m),\ |x| \to \infty;\ \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|f(k\pi/\sigma)|^p}{|k|^{mp}} < \infty \right\}.$$

Then by Lemma 1

$$\mathrm{PWS} = \cup_{\sigma\geq 0,\, m\in\mathbb{N}_0,\, p\geq 1} A_{\sigma,p,m}$$

and

$$\mathrm{PWS}_\sigma = \cup_{m_1\in\mathbb{N}_0}\mathrm{PWS}_{\sigma,m_1} = \cup_{m_2\in\mathbb{N}_0,\ p\geq 1} A_{\sigma,p,m_2}$$

and for $1 \leq p_1 < p_2 < \infty$: $A_{\sigma,p_1,m} \subset A_{\sigma,p_2,m}$. Note that if $f \in \mathrm{PWS}_{\sigma,m}$, then for sure $f \in A_{\sigma,p,m+1}$, $p > 1$. Later, we shall prove that in the case $1 < p < \infty$, the linear space $A_{\sigma,p,m}$ coincides with the linear normed space of all functions f

from E_σ, satisfying the following Sobolev type of weighted integrable condition (weighted L_p-norm):

$$\int_{\mathbb{R}} \frac{|f(x)|^p}{(1+x^2)^{mp/2}} dx < \infty .$$

Remark. The example $z^{m-1} \sin(\sigma z)$ $(\sigma > 0, p = 1)$ shows that both functional classes are not equivalent when $p = 1$. Obviously, $z^{m-1} \sin(\sigma z) \in A_{\sigma,1,m}$ but does not satisfy the above Sobolev type integrable condition ($p = 1$).

Theorem 1. *Let $f \in A_{\sigma,p,m}$, where $p \geq 1$ and $m \in \mathbb{N}_0$. Given two sets $S_1 = \{m_\mu \pi/\sigma,\ \mu = 1, \dots, r_1\}$ and $S_2 := \{z_\nu,\ \nu = 1, \dots, r_2\}$ together with positive integer multiplicities $\{\lambda_\mu,\ \mu = 1, \dots, r_1\}$ $\{\beta_\nu,\ \nu = 1, \dots, r_2\}$, let q_{m-1} be the unique interpolating solution from Lemma 2. Let us define*

$$\Omega_m(z) := \prod_{\mu=1}^{r_1} (z - m_\mu \pi/\sigma)^{\lambda_\mu} \prod_{\nu=1}^{r_2} (z - z_\nu)^{\beta_\nu} \quad \text{and} \quad \omega_1 := \{m_1, \dots, m_{r_1}\} .$$

Then, the following Shannon type sampling formula holds

$$\begin{aligned} f(z) \;=\; & \sum_{\mu=1}^{r_1} f\left(\frac{m_\mu \pi}{\sigma}\right) \frac{\sin(\sigma z - m_\mu \pi)}{\sigma z - m_\mu \pi} \\ & + \Omega_m(z) \sum_{k \in \mathbb{Z} \setminus \omega_1} \frac{f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)} \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi} \\ & + \sin \sigma z \, q_{m-1}(z) . \end{aligned} \tag{1}$$

The infinite series in the sampling formula (1) is absolutely and uniformly convergent on $\mathbb{R}$ and l.u.c on $\mathbb{C}$. The sampling representation (1) *is l.u.c. on $\mathbb{C}$.*

Remark. The sampling formula (1) can be considered as a blending interpolant based on two processes: One of them is the usual Shannon sampling process at the Nyquist rate π/σ and the other one is weighted Hermite interpolation process by using additional finite m bits of information, functional and derivative values. Blending these two interpolating procedures results in a new sampling formula (1), where the digital sampling sequence consists of the functional values multiplied by appropriate sampling multipliers

$$\left\{ \frac{f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)}, k \in \mathbb{Z} \setminus \omega_1; \quad f(l\pi/\sigma), l \in \omega_1 \right\}$$

plus m functional and derivative values of f. The sampling sequence has asymptotic $o(1)$, $|k| \to \infty$. Hence, at the cost of additional m bits of information we digitize a band-limited signal with polynomial time asymptotic without any loss of information by using the Nyquist optimal rate. *This gives a new sampling characterization of the PWS space in terms of classical functions, not distributions.*

Proof of Theorem 1. Let us consider the series

$$\sum_{k\in\mathbb{Z}\setminus\omega_1} \frac{f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)} \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi}.$$

In view of $\sum_{k\in\mathbb{Z}\setminus\omega_1} |f(k\pi/\sigma)/\Omega_m(k\pi/\sigma)|^p < \infty$, where $p \geq 1$ we conclude (see [14]) that the above series, being locally uniformly convergent on $\mathbb{C}$, represents an entire function of exponential type σ which is $L_p(\mathbb{R})$ integrable on the real line [14]. By using Bernstein's inequality for entire functions of exponential type [1] it is seen that the function is $o(1)$, $|x| \to \infty$ on the real axis and by Lemma 1, it is $o\left(e^{\sigma|y|}\right)$, $|z| \to \infty$, $z = x + iy$, $i^2 = -1$ uniformly on the complex plane.

Let us denote by $h(z)$ the right hand side of the formula (1). Then, assuming that $f(x) = O(x^m)$, $|x| \to \infty$ and by using Lemma 1 and the inequality

$$|\sin\sigma z| \geq \frac{1}{3} e^{\sigma|y|} \quad (z \in \Gamma_n),$$

where Γ_n is the square contour with corners $(n+1/2)\frac{\pi}{\sigma}(\pm 1 \pm \mathrm{i})$ $(n \in \mathbb{N}_0)$, we conclude that the auxiliary entire function

$$r(z) := (f(z) - h(z)) / (\Omega_m(z) \sin\sigma z)$$

is bounded on Γ_n $(n \to \infty)$ and from the maximum modulus principle, the function $r(z)$ is bounded on the complex plane. By Liouville's theorem $r(z)$ must be a constant on $\mathbb{C}$, i.e.,

$$\begin{aligned} f(z) &= \sum_{\mu=1}^{r_1} f\left(\frac{m_\mu \pi}{\sigma}\right) \frac{\sin(\sigma z - m_\mu \pi)}{\sigma z - m_\mu \pi} \\ &\quad + \Omega_m(z) \sin\sigma z \sum_{k\in\mathbb{Z}\setminus\omega_1} \frac{(-1)^k f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)\,(\sigma z - k\pi)} \\ &\quad + \sin\sigma z\, q_{m-1}(z) + \mathbf{A}\,\Omega_m(z) \sin\sigma z. \end{aligned}$$

The restriction $f(x) = o(x^m)$, $|x| \to \infty$ ($x \in \mathbb{R}$) implies that the constant $\mathbf{A}$ must be zero.

The sharpness of our result. Estimating the sharpness of Theorem 1 we may ask the following question: *Can we recover exactly σ-bandlimited signals with $O(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$ time asymptotic by sampling at the optimal Nyquist rate π/σ with an additional m – bits Hermite type of information to the sampling sequence, as it is in Theorem 1?*

The simple example $\Omega_m(z)\sin\sigma z$ shows that this is not possible. In other words, the asymptotic

$$f(x) = o(x^m),\ |x| \to \infty\ \ (x \in \mathbb{R})$$

is *the best possible* for exact sampling recovery of $f \in \mathrm{PWS}_\sigma$ based on interpolation data at $\{k\pi/\sigma + \alpha,\ k \in \mathbb{Z},\ \alpha \in \mathbb{R}\}$ and an additional, m - bits Hermite type information - functional values and derivatives of $f(z)$. Similar example shows that σ-bandlimited signals with $o(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$ time asymptotic can not be recovered by sampling at rate π/σ plus less than m additional bits of information.

Remark. Let us clarify that *the sampling formula (1) is interpolating at $k\pi/\sigma$, $k \in \mathbb{Z}$, satisfies additional m Hermite interpolating condition, and has been constructed by the following interpolation data*:

$$\begin{aligned}
&f(k\pi/\sigma),\ k \in \mathbb{Z};\\
&f^{(j)}(m_\mu\pi/\sigma),\ j = 1,\dots,\lambda_\mu,\ \mu = 1,\dots,r_1;\\
&f^{(j)}(z_\nu),\ j = 0,\dots,\beta_\nu - 1,\ \nu = 1,\dots,r_2\ \ (z_\nu \neq k\pi/\sigma);\\
&\sum_{\mu=1}^{r_1} \lambda_\mu + \sum_{\nu=1}^{r_2} \beta_\nu = m\,.
\end{aligned}$$

Remark. The case $m = 0$ covers the well known Shannon sampling theorem (see p. 417). In the particular case $m = 1$, $S_1 = \{0\}$, and $S_2 = \emptyset$ the corresponding interpolating formula has been known since 1934 [19], [6, Theorem 4, and the footnotes on p. 47] but under more stringent conditions.

Example 1. The case $\omega_1 = \{0\}$ $(r_1 = 1,\ m_1 = 0,\ \lambda_1 = m)$ and $S_2 = \emptyset$ has been considered by Bernstein [3] under the condition

$$\text{(2)}\qquad f(k\pi/\sigma) = O(|k|^\alpha),\ |k| \to \infty\quad (k \in \mathbb{Z},\ \alpha < m),$$

which is a bit more restrictive than $f \in A_{\sigma,p,m}$. In this particular case of Theorem 1 we have $\Omega_m(z) = z^m$ and the corresponding sampling formula reads as follows

$$f(z) \;=\; f(0)\frac{\sin\sigma z}{\sigma z} + (\sigma z)^m \sin\sigma z \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^k f(k\pi/\sigma)}{(k\pi)^m(\sigma z - k\pi)} + \sin\sigma\, z\, q_{m-1}(z)\,,$$

The coefficients of q_{m-1} are linear combinations of the data

$$\left\{f(0),\, f'(0), \ldots,\, f^{(m)}(0)\right\}\,.$$

and they are uniquely determined by the Hermite interpolating conditions (see Lemma 2)

$$\left[f(x) - f(0)\frac{\sin\sigma\, x}{\sigma\, x}\right]^{(j)}_{x=0} = [\sin\sigma\, x\, q_{m-1}(x)]^{(j)}_{x=0}\,,\ j = 1, 2, \ldots, m$$

and in view of this the coefficients of q_{m-1} can be computed by using the recursion formulas

$$q^{(2l)}_{m-1}(0) = \frac{1}{2l+1}\sum_{k=1}^{l}\sigma^{2k}\binom{2l+1}{2k+1}(-1)^{k+1}\, q^{(2l-2k)}_{m-1}(0) + \frac{1}{(2l+1)\sigma}\, f^{(2l+1)}(0)$$

$(l = 0, 1, \ldots;\ 2l \le m-1)$

$$q^{(2l+1)}_{m-1}(0) = \frac{1}{2l+2}\sum_{k=1}^{l}\sigma^{2k}\binom{2l+2}{2k+1}(-1)^{k+1}\, q^{(2l-2k+1)}_{m-1}(0) + \frac{1}{(2l+2)\sigma}\left(f^{(2l+2)}(0) + f(0)\frac{(-1)^l}{2l+3}\sigma^{2l+2}\right)\ (l = 0, 1, \ldots;\, 2l+1 \le m-1).$$

In particular, $q_{m-1}(0) = f'(0)/\sigma$, $q'_{m-1}(0) = f''(0)/(2\sigma) + \sigma\, f(0)/6$, $q''_{m-1}(0) = f'''(0)/(3\sigma) + \sigma\, f'(0)/3$, $q'''_{m-1}(0) = f^{(4)}(0)/(4\sigma) + \sigma\, f''(0)/2 + 7\,\sigma^3\, f(0)/60$, etc.

Concerning Bernstein condition (2), we take as an example

$$f(z) \;:=\; (\sigma\, z)^m \sum_{k\,\in\,\mathbb{Z}\setminus\{0\}} \frac{y_k}{(k\pi)^m}\, \frac{\sin(\sigma\, z - k\pi)}{\sigma\, z - k\pi},$$

where $y_{2^s} := 2^{ms-\sqrt{s}}$ $(s \in \mathbb{N}_0)$ and $y_k := 0$ $(k \ne 2^s$ $(s \in \mathbb{N}_o,\ k \in \mathbb{Z})$. Then, $f \in A_{\sigma,m,p}$ $(p \ge 1)$ but it does not satisfy the condition (2). Conversely, each function from E_σ which is $o(x^m)\,(|x| \to \infty)$ and satisfying (2), belongs to $A_{\sigma,m,p}$ for some $p \geq 1$. *Hence, our condition of weighted l_p-summability (see the definition of the linear space $A_{\sigma,m,p}$) of the sampling sequence is weaker than that proposed by Bernstein* (2) *and gives a complete characterization of the PWS space. Moreover, we shall see that the class $A_{\sigma,m,p}$ can be completely characterized by an $L_p(\mathbb{R})$ weighted integral norm of Sobolev type.*

Example 2. Let $z_1, z_2, \ldots, z_m$ be m numbers on the complex plane such that $\sin\sigma z_l \neq 0,\ l = 1, \ldots, m,$. So, we have $S_1 = \emptyset$ and $S_2 = \{z_1, \ldots, z_m\}$ ($r_2 = m,\ \beta_1 = \cdots = \beta_m = 1$). The unique solution $q_{m-1}(z)$ of the interpolating problem (see Lemma 2)

$$q_{m-1}(z_l) = \frac{f(z_l)}{\sin\sigma z_l} \qquad (l = 1, \ldots, m)$$

is easily constructed by Lagrange interpolating formula. Thus, in this particular case of Theorem 1, we have $\Omega_m(z) = \prod_{\nu=1}^{m}(z - z_\nu)$ and the corresponding sampling formula (see Theorem 1) has the form

$$f(z) = \Omega_m(z)\, \sin\sigma z \sum_{k\in\mathbf{Z}} \frac{(-1)^k f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)\,(\sigma z - k\pi)} + \sin\sigma z\, q_{m-1}(z),$$

where

$$q_{m-1}(z) = \sum_{l=1}^{m} \frac{f(z_l)}{\sin\sigma z_l} \frac{\Omega_m(z)}{(z - z_l)\Omega_m'(z_l)}.$$

Remark. It is known [14] that the set $A_{\sigma,p,0},\ 1 < p < \infty$ coincides with the set of functions from E_σ which are $L^p(\mathbb{R})$ integrable on the real line. Let $f \in A_{\sigma,p,0},\ p \geq 1$. Then, f is $L_p(\mathbb{R})$ integrable and according to a result in [14]:

$$f(z) = \sum_{k\in\mathbb{Z}} f\,(k\pi/\sigma)\, \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi},$$

where the Shannon sampling series is absolutely and uniformly convergent on the real line an locally uniformly convergent on the complex plane.

For $\sigma > 0,\ m \in \mathbb{N}_0$, and $p \geq 1$ we define the following *Sobolev type normed linear subspace of E_σ*:

$$B_{\sigma,p,m} := \left\{ f\, :\, f \in E_\sigma,\ |f|_{\sigma,p,m} = \left(\int_{\mathbb{R}} \frac{|\,f(x)\,|^p}{(1 + x^2)^{mp/2}}\, dx \right)^{1/p} < \infty \right\}.$$

Theorem 2. *Let* $\sigma > 0,\ m \in \mathbb{N}_0,\ 1 < p < \infty$. *Consider*

$$A_{\sigma,p,m} = \{\, f : f \in E_\sigma;\ f(x) = o(x^m),\ |x| \to \infty;\ \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|f(k\pi/\sigma)|^p}{|k|^{mp}} < \infty\}$$

and

$$B_{\sigma,p,m} = \left\{ f \,:\, f \,\in\, E_\sigma,\ |f|_{\sigma,p,m} = \left(\int_{\mathbb{R}} \frac{|\, f(x)\,|^p}{(1\,+\,x^2)^{mp/2}}\, d\,x \right)^{1/p} < \infty \right\}.$$

Then

$$B_{\sigma,p,m} \,\equiv\, A_{\sigma,p,m}$$

and following the notations of Theorem 1, *each* $f \in B_{\sigma,p,m}$ $(A_{\sigma,p,m})$ *is exactly recovered by the sampling formula*

$$\begin{aligned} f(z) \quad = \quad & \sum_{\mu=1}^{r_1} f\left(\frac{m_\mu \pi}{\sigma}\right) \frac{\sin(\sigma z \,-\, m_\mu\, \pi)}{\sigma\, z \,-\, m_\mu\, \pi} \\ & + \Omega_m(z) \sum_{k\in\mathbb{Z}\setminus\omega_1} \frac{f(k\pi/\sigma)}{\Omega_m\,(k\pi/\sigma)} \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi} \\ & + \,\sin \sigma\, z\, q_{m-1}(z) \end{aligned}$$

that is l.u.c. on $\mathbb{C}$.

Corollary 1. *If* $1 \le p_1 < p_2$, *then* $B_{\sigma,p_1,m} \subset B_{\sigma,p_2,m}$.

Decomposition of the PWS space in terms of $\boldsymbol{B_{\sigma,p,m}}$**.** Consider $\mathrm{PWS}_{\sigma,m}$, the linear subspace of PWS functions with parameters σ and m (see p. 3). According to Lemma 1 and Theorem 2

$$\cup_{m_1\in\mathbb{N}_0}\, \mathrm{PWS}_{\sigma,m_1} = \cup_{m_2\in\mathbb{N}_0,\ 1\le p<\infty} B_{\sigma,p,m_2}.$$

Obviously, if $f \in \mathrm{PWS}_{\sigma,m}$, then $f \in B_{\sigma,p,m+1}$ for each $p > 1$. *The practical meaning of the above decomposition is that each* $\mathrm{PWS}_{\sigma,m}$ *function can be exactly recovered by making use of the sampling formula of Theorem 2, i.e., with an optimal rate* π/σ *of sampling at the cost of a finite number bits of information to be added to the sampling data.*

Interpolation theory in $B_{\sigma,p,m}$ ($\sigma > 0$, $m \in \mathbb{N}_0$, $1 < p < \infty$).

Corollary 2. *Consider the following linear space of infinite sequences of complex numbers:*

$$l_{p,m} := \left\{ \mathbf{y} := \{\ldots, y_{-1}, y_0, y_1, \ldots\} \,:\, \sum_{k\in\mathbb{Z}} \frac{|y_k|^p}{(1+|k|)^{mp}} < \infty,\ y_k \in \mathbb{C} \right\}.$$

Then, following the notations of Theorem 1, for each $\mathbf{y} \in l_{p,m}$ there exists a unique within m Hermite type interpolation conditions function $f_{\mathbf{y}} \in B_{\sigma,p,m}$ with an explicit construction by the l.u.c. Shannon type sampling series

$$\begin{aligned} f_{\mathbf{y}}(z) \; &= \; \sum_{\mu=1}^{r_1} y_{m_\mu} \frac{\sin(\sigma z \,-\, m_\mu \pi)}{\sigma\, z \,-\, m_\mu\, \pi} \\ &\quad + \Omega_m(z) \sum_{k\in\mathbb{Z}\setminus\omega_1} \frac{y_k}{\Omega_m\,(k\pi/\sigma)} \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi} \\ &\quad + \,\sin\sigma\, z\, q_{m-1}(z) \end{aligned}$$

satisfying the interpolation conditions $f_{\mathbf{y}}(k\pi/\sigma) = y_k$ $(k \in \mathbb{Z})$. Conversely, by Theorem 2, for each $f \in B_{\sigma,p,m}$, the sequence $\{\, f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$ belongs to $l_{p,m}$ and f can be recovered by the sampling formula (1).

Proof of Theorem 2. Suppose that $f \in B_{\sigma,p,m}$, $1 \le p < \infty$. Then the auxiliary function $F(z) := \left(f(z) - \sum_{k=0}^{m-1} (f^{(k)}(0)/k!)\, z^k \right) / z^m$ is from E_σ and it is $L_p(\mathbb{R})$ integrable on the real line. Bernstein's L_p-inequality for entire functions of exponential type [1] implies that

$$\sum_{k\,\in\,\mathbb{Z}} |\, F(\xi_k)\,|^p \,\le\, 2^{p-1} \left(\pi^{p-1} \,+\, \pi^{-1}\right) \sigma \,\int_{\mathbb{R}} |\, F(t)\,|^p \; d\,t,$$

where $\xi_k \in [\, k\pi/\sigma\,, (k+1)\pi/\sigma\,]$, $k \in \mathbb{Z}$. First, substituting $\xi_k := k\pi/\sigma$ we conclude that

$$\sum_{k\,\in\,\mathbb{Z}} \frac{|\, f(k\pi/\sigma)\,|^p}{(1+|k|)^{m\,p}} \,\le\, c(p,\sigma,m) \left[\max_{0\le k\le m-1} |\, f^{(k)}(0)\,| + \int_{\mathbb{R}} |\, F(t)\,|^p \; d\,t \right].$$

On the other hand, substituting $\xi_k := \xi_k^*$, where

$$|F\,(\xi_k^*)\,| := \max_{x\,\in\,[\,k\pi/\sigma\,,\,(k+1)\pi/\sigma\,]} |\, F(x)\,|$$

we obtain that $F(x) = o(1)\,(|x| \to \infty)$ so, $f(x) = o(x^m)\,(|x| \to \infty)$. Hence, $f \in A_{\sigma,p,m}$.

Conversely, let $f \in A_{\sigma,p,m}$, where $1 < p < \infty$. Then, applying the sampling formula of Theorem 1 we obtain:

$$\begin{aligned}\left(\int_{\mathbb{R}} \frac{|f(x)|^p}{(1+x^2)^{mp/2}}\,dx\right)^{1/p} &\leq c_1(p,m)\left(\sum_{k\in\mathbb{Z}\setminus\omega_1} \frac{\pi}{\sigma}\left|\frac{f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)}\right|^p\right)^{1/p} \\ &+ c_2(p,m,\sigma)\sum_{\mu=1}^{r_1}\sum_{j=0}^{\lambda_\mu}\left|f^{(j)}\left(\frac{m_\mu\pi}{\sigma}\right)\right| \\ &+ c_3(p,m,\sigma)\sum_{\mu=1}^{r_2}\sum_{j=0}^{\beta_\nu-1}\left|f^{(j)}(z_\nu)\right|\end{aligned}$$

and in view of this $f \in B_{\sigma,p,m}$. Note that the method of proof fails if $p = 1$.

The case $\mathbf{p = 1}$. Let $f \in B_{\sigma,1,m}$. Then, let us consider the auxiliary function

$$F(z) := \left(f(z) - \sum_{k=0}^{m-1} f(k\pi/\sigma)\,\frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi}\right) \Big/ \left(\prod_{k=0}^{m-1}\left(z - \frac{k\pi}{\sigma}\right)\right).$$

Obviously, $F \in B_{\sigma,1,0}$ and from here we conclude that $f \in A_{\sigma,1,m}$. However, the function $g(z) = z^{m-1}\sin\sigma z$ belongs to $A_{\sigma,1,m}$ but $g(x)/(1+x^2)^{m/2} = \left(x^{m-1}\sin\sigma x\right)/(1+x^2)^{m/2}$ is not $L^1(\mathbb{R})$-integrable. Hence, $g \notin B_{\sigma,1,m}$ and in view of this $B_{\sigma,1,m} \subset A_{\sigma,1,m} \subset A_{\sigma,p,m} = B_{\sigma,p,m} \quad (1 < p < \infty)$.

Acceleration of the convergence of sampling series, including the sampling of classical Shannon type $\mathbf{B_{\sigma,2}}$ signals. As we mentioned even for $\mathbf{B_{\sigma,2}}$ signals the corresponding Shannon sampling series (see Shannon Sampling Theorem [8], [13], [18], [20]) could be slowly convergent. Here we show how the convergence can be accelerated by adding a finite Hermite type of information to the sampling representation, following the representation sampling formula (1) (see Theorem 1 and Theorem 2, Example 1, and Example 2). Let $f(x) \in B_{\sigma,p,m}$. Then, $f \in B_{\sigma,p,N}$ for $N \geq m$ $(N \in \mathbb{N})$ and $p > 1$. Following Example 1, for a fixed $S \in \mathbb{N}$ consider the truncated sampling approximant to $f(z)$ in

$D := \{z : |z| \le R\}$, $R > 1$ of the form

$$f_{S,N}(z) = (\sigma z)^N \sum_{|k| \le S-1,\ k \ne 0} \frac{f(k\pi/\sigma)}{(k\pi)^N} \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi} + f(0)\frac{\sin\sigma z}{\sigma z} + \sin\sigma z\, q_{N-1}(z)$$

Then, by using Theorem 2 and well known technique given in [1], [14] we obtain

$$\max_{z\in D}|f(z) - f_{S,N}(z)| \le c(p,\sigma)\frac{R^N e^{\sigma R}}{S^{N-m}}.$$

Let $S \ge R^{(2N)/(N-m)}$. Then by making use of the above estimate we obtain

$$\max_{z\in D}|f(z) - f_{S,N}(z)| \le c(p,\sigma)\frac{e^{\sigma R}}{R^N}$$

to conclude that by choosing N larger we accelerate the l.u.c. of the truncated sampling approximant. Similar approximation estimates can be obtained by using the same technique and Example 2 based on Lagrange interpolating formula. For $B_{\sigma,p}$-signals, obviously, $m = 0$ hence, $S \ge R^2$ and if we take $S \ge R^k$, $k \ge 2$ then, the estimate will be

$$\max_{z\in D}|f(z) - f_{S,N}(z)| \le c(p,\sigma)\frac{e^{\sigma R}}{R^{(k-1)N}}.$$

Next, we compare Theorem 1 and Theorem 2 with some similar statements in the sampling theory [6], [5], [16] in order to see the usefulness of the results obtained in the present article.

Campbell's method [5] for exact recovery of bandlimited signals with polynomial asymptotic on the time axis. The following method can be applied also for exact recovery of PWS functions. Let $f \in E_\sigma$ and $f(x) = o(x^m)$ $(|x| \to \infty)$. We take a Sobolev function

$$\psi(x) := c\exp\left(-\frac{1}{1-x^2}\right) \quad \text{for} \quad x \in (-1,1),$$

$\psi(x) = 0$ for $|x| \ge 1$, and $\int_{\mathbb{R}} \psi(x)\,dx = 1$. The function ψ is infinitely differentiable on the real line and with a compact support in $[-1,1]$ so, it is fast decreasing. The Fourier transform $\varphi_\varepsilon(z)$ of $(1/\varepsilon)\psi(x/\varepsilon)$ will be also a fast decreasing function which is entire of exponential type $\varepsilon > 0$ and such that $\varphi_\varepsilon(0) = 1$.

Then, for a fixed x, the auxiliary function $F_\varepsilon(z) := f(z)\,\varphi_\varepsilon(x-z)$ is an entire function of exponential type $\sigma+\varepsilon$ which is $L^2(\mathbb{R})$ integrable. Applying Shannon sampling theorem to $F_\varepsilon(z)$ for a fixed x and substituting $z=x$ we obtain the sampling formula:

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma+\varepsilon}\right)\frac{\sin((\sigma+\varepsilon)x-k\pi)}{(\sigma+\varepsilon)x-k\pi}\,\varphi_\varepsilon\left(x-\frac{k\pi}{\sigma+\varepsilon}\right).$$

First, when $\varepsilon\to 0$, then $\varphi_\varepsilon(z)$ will tend uniformly to 1 in each time interval $[-T,T]$ which makes the above sampling series slowly convergent. Second, if we take $\pi/(\sigma+\varepsilon)$ rate of sampling then the sampling sequence to recover a function from $B_{\sigma,p,m}$ in the time interval $[-T,T]$, $T>0$ consists of $2T(\sigma+\varepsilon)/\pi$ digits that is with $2T\varepsilon/\pi$ digits more than in the case of sampling at the optimal Nyquist π/σ rate and as we observed before the parameter ε can not be taken too small. Now, what gives the sampling formula from Theorem 1? By Theorem 1, for exact sampling recovery of functions from $B_{\sigma,p,m}$, i.e., for digitizing of a time polynomial signal without any loss of information, we need the functional values at $k\pi/\sigma$, $k\in\mathbb{Z}$, i.e., *digitizing at the optimal Nyquist rate π/σ and the digit sequence contains m more bits of information (finite number of data, functional values and derivatives) and this is in fact the Hermite type of information needed to construct the auxiliary polynomial q_{m-1} (see Lemma 1).*

The sampling observation of Cartwright [6]. Sampling of bandlimited and time-bounded (bounded as functions of the time) signals at the optimal Nyquist rate by using the sampling formula (1). Let $B_{\sigma,\infty}$ denote the linear space of all functions $f\in E_\sigma$ which are $O(1)$, $|x|\to\infty$ $(x\in\mathbb{R})$, i.e., we consider the class of all σ band-limited, time-bounded signals. Cartwright showed [4, Theorem 10.2., Theorem 10.2.3], [6], [7] that for each $\varepsilon>0$ the sequence $\{k\pi/(\sigma+\varepsilon)\}$, $k\in\mathbb{Z}$ is sampling for $B_{\sigma,\infty}$, i.e., for σ band-limited signal that are time-bounded. However, if we take $\pi/(\sigma+\varepsilon)$ rate of sampling data to recover exactly a $B_{\sigma,\infty}$ function, then in the time interval $[-T,T]$, $T>0$ we need $2T(\sigma+\varepsilon)/\pi$ functional values which is $2T\varepsilon/\pi$ functional values more than in the case of recovery by using the optimal Nyquist rate π/σ of sampling. On the other hand, obviously, $B_{\sigma,\infty}\subset B_{\sigma,p,1}$ $(p>1)$. In view of this, by using Theorem 1 and Theorem 2 each $B_{\sigma,\infty}$ signal can be recovered by sampling at the optimal Nyquist rate π/σ *plus only one bit of information in addition:* A functional value or a value of the derivative at a point x_*: $f(x_*)$ if $\sin\sigma x_*\neq 0$ or $f'(k_*\pi/\sigma)$ if $x_*=k_*\pi/\sigma$, $k_*\in\mathbb{Z}$.

The sampling observation of Lyubarskii and Madych [16]. The results in the present article are closely related to those obtained in [16] obtained by a different approach. The authors consider a generalized Paley-Wiener space $W^m_{\sigma,p}$ of entire functions $f(z)$ of exponential type σ such that $f^{(m)}(z) \in B_{\sigma,p}$, $1 < p < \infty$ endowed with the semi-norm $|f^{(m)}|_{L_p(\mathbb{R})}$. They prove that each interpolating sequence in $B_{\sigma,p}$ is an interpolating sequence in $W^m_{\sigma,p}$, in particular that, obtained by sampling at the Nyquist rate. However, the Sobolev type normed linear space $B_{\sigma,p,m}$ considered in the present paper includes functions like $z^{m-1}\sin\sigma z$, $z^{m-1}[\sin(\sigma z/k)]^k$ which are not in $W^m_{\sigma,p}$ for any $m \in \mathbb{N}$ and can not be recovered by using Nyquist rate of sampling without additional sampling information. Also, if $f \in W^m_{\sigma,p}$, then easy calculations show that $f \in B_{\sigma,p_1,m}$ for each $p_1 > p$ hence, the sampling representation (1) holds for such functions. The sampling formula (1) is using m bits more information than the sampling representation given in [16] but on the other side taking m bigger makes the infinite part of the sampling representation faster convergent. From practical, signal point of view, we have a signal developing during the time so, only sampling at real (time) moments is accessible and that is because the Nyquist (real) time rate is important, being optimal. Also, it is not so straight to be seen that $f^{(m)}$ is $L_p(\mathbb{R})$-summable because the signal practically is accessible only at discrete time (real) moments and the numerical differentiation is not a stable process with respect to the initial data. Similar problem we have with computing the Hermite polynomial q_{m-1} in (1) but the advantage here is that we are to compute a finite number of data and it is a simple procedure based only on functional values (samples) in the Lagrangian case (see Example 2). In addition, the time asymptotic of a signal can be easily estimated by using the sampling data.

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