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SMALE'S CONJECTURE ON MEAN VALUES OF POLYNOMIALS AND ELECTROSTATICS

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Dedicated to Academician Blagovest Sendov
on the occasion of his seventy-fifth anniversary

ABSTRACT. A challenging conjecture of Stephen Smale on geometry of polynomials is under discussion. We consider an interpretation which turns out to be an interesting problem on equilibrium of an electrostatic field that obeys the law of the logarithmic potential. This interplay allows us to study the quantities that appear in Smale's conjecture for polynomials whose zeros belong to certain specific regions. A conjecture concerning the electrostatic equilibrium related to polynomials with zeros in a ring domain is formulated and discussed.

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1. Introduction. There are two challenging problems which inspire the development of the Geometry of Polynomials. They are due to the celebrated Bulgarian mathematician Blagovest Sendov and to the Fields Medal winner Steve Smale. Sendov's conjecture states that for every nonconstant polynomial $p(z)$ of degree $n \geq 2$ with zeros in the closed unit disc, and for each of its zeros z_k, there is a critical point ξ_j of $p(z)$ (zeros of $p'(z)$) such that $|z_k - \xi_j| \leq 1$. As Academician Sendov uses to confess, he posed this conjecture in 1959 because he was asked to formulate a beautiful and difficult problem to himself, and eventually solve it, by his tutor Nikola Obrechkoff. It turned out the problem was so beautiful and so difficult that it is still open despite the nearly one hundred publications devoted to it.

In his 1981 paper [9] Steve Smale investigated thoroughly Newton's method for approximate calculation of zeros of a complex polynomial. He showed that the cost of solving this fundamental problem does not grow too fast, when the degree of the polynomial increases, in a certain statistical sense. Denote by π_n the set of complex algebraic polynomials of degree n. If $f \in \pi_n$, then $\xi \in \mathbb{C}$ is said to be a critical point of f if $f'(\xi) = 0$. The main tools employed by Smale in this investigation were results of the nature of the following theorem:

Theorem A. *For any $n \in \mathbb{N}$ and each $f \in \pi_n$ the inequality*

$$\frac{|f(\xi) - f(z)|}{|\xi - z||f'(z)|} \leq 4 \tag{1.1}$$

holds for at least one critical point ξ of $f(z)$ and all $z \in \mathbb{C}$ for which $f'(z) \neq 0$.

A straightforward transformation shows that, an equivalent statement of Theorem A, is to consider polynomials from

$$\pi_n^0 := \{f \in \pi_n \ : \ f(0) = 0, f'(0) \neq 0\}$$

and, for any such a polynomial, prove that

$$\min \left\{ \frac{|f(\xi)|}{|\xi||f'(0)|} \ : \ f'(\xi) = 0 \right\} \leq 4 \tag{1.2}$$

for at least one critical point ξ. The natural question of determining the smallest possible bound of the quantity which appears on the left-hand side of (1.2), when $f \in \pi_n^0$, arose. Thus, Smale stated the following conjecture (see Problem 1F in [9]):

Conjecture 1 (Smale). *Let $f \in \pi_n^0$. Then*

$$\min \left\{ \frac{|f(\xi)|}{|\xi||f'(0)|} \ : \ f'(\xi) = 0 \right\} \leq K, \tag{1.3}$$

where $K = 1$ or possibly $K = (n-1)/n$.

Observe that the statement of Theorem A, or equivalently the inequality (1.2), is nothing but (1.3) with $K = 4$. The polynomials $f(z) = a_1 z + a_n z^n$, $a_1 a_n \neq 0$, show that K cannot be less than $(n-1)/n$. There exist strong indications that these are the only "extremal polynomials" for Conjecture 1, which allows the following speculation which is sometimes called "strengthened form of Smale's conjecture":

Conjecture 2 (**Smale, strengthened form**). *Let $f \in \pi_n^0$. Then the inequality* (1.3) *holds with $K = (n-1)/n$ and equality is attained if and only if $f(z) = a_1 z + a_n z^n$, where a_1 and a_n are any complex numbers with $a_1 a_n \neq 0$.*

Nice recent surveys on Sendov's and Smale's conjectures and further open problems were given by Sendov himself [6, 7] and by Schmeisser [4]. Smale's conjecture was established in the special case when all the zeros of $f(z)$ lie on a circumference centered at the origin by Tischler [11] who proved that for such polynomials the extremal value $(n-1)/n$ as attained only for the polynomials from Conjecture 1. Beardon, Minda and Ng [1] proved that

$$\sup_{f \in \pi_n^0} \min \left\{ \frac{|f(\xi)|}{|\xi||f'(0)|} \ :\ f'(\xi) = 0 \right\} \leq 4^{(n-2)/(n-1)}.$$

Recently Dubinin [2] developed a method which allows an improvement of this results. More precisely, the right-hand side of the latter inequality can be reduced by introducing the factor $(n-1)/n$.

Conjecture 1 is known to be true for polynomials of degree up to five and for $n = 5$ it was proved by Sendov and Marinov [8].

In this paper we study the the quantities that appear in Smale's conjecture and formulate a natural problem on electrostatic equilibrium.

2. A problem on electrostatic equilibrium.

In this paper we discuss an electrostatic equilibrium problem related to the conjecture of Smale. Our approach is pretty simple. Instead of investigating the minimum of the $n-1$ quantities $|f(\xi_j)|/\left(|\xi_j||f'(0)|\right)$, where $\xi_j,\ j = 1, \dots, n-1$, are the critical points of $f \in \pi_n^0$, we consider their product.

Theorem 1. *If $f(z)$ is a polynomial from π_n^0 with zeros $z_0 = 0, z_1, \dots, z_{n-1}$ and critical points $\xi_1, \dots, \xi_{n-1}$, then*

$$\prod_{j=1}^{n-1} \frac{|f(\xi_j)|}{|\xi_j||f'(0)|} = \frac{1}{n^{n-1}} \frac{\prod_{1 \leq j < k \leq n-1} |z_k - z_j|^2}{\prod_{j=1}^{n-1} |z_j|^{n-2}}. \tag{2.1}$$

P r o o f. Let $f(z)$ be monic. Then

$$f(z) = (z - z_0)(z - z_1) \cdots (z - z_{n-1}) \tag{2.2}$$

and then

$$f'(z_j) = (z_j - z_0) \cdots (z_j - z_{j-1})(z_j - z_{j+1}) \cdots (z_j - z_{n-1}). \tag{2.3}$$

On the other hand, we have

$$f'(z) = n(z - \xi_1)(z - \xi_2) \cdots (z - \xi_{n-1}),$$

which yields

$$f'(z_j) = n(z_j - \xi_1)(z_j - \xi_2) \cdots (z_j - \xi_{n-1}), \quad j = 0, \ldots, n-1. \tag{2.4}$$

Multiplying the latter identities and having in mind (2.2), we obtain

$$\begin{aligned} f'(z_0) \cdots f'(z_{n-1}) &= n^n \prod_{j=0}^{n-1} (z_j - \xi_1)(z_j - \xi_2) \cdots (z_j - \xi_{n-1}) \\ &= n^n \prod_{k=1}^{n-1} (-1)^n (\xi_k - z_0)(\xi_k - z_1) \cdots (\xi_k - z_{n-1}) \\ &= n^n \prod_{k=1}^{n-1} f(\xi_k). \end{aligned}$$

This immediately implies

$$\frac{f(\xi_1) f(\xi_2) \cdots f(\xi_{n-1})}{\xi_1 \xi_2 \cdots \xi_{n-1} [f'(0)]^{n-1}} = \frac{1}{n^n} \frac{f'(z_0) f'(z_1) \cdots f'(z_{n-1})}{\xi_1 \xi_2 \cdots \xi_{n-1} [f'(0)]^{n-1}}. \tag{2.5}$$

Since, by (2.4), $f'(z_0) = f'(0) = (-1)^{n-1} n \xi_1 \xi_2 \cdots \xi_{n-1}$, then the right-hand side of (2.5) reduces to

$$\frac{(-1)^{n-1}}{n^{n-1}} \frac{f'(z_1) \cdots f'(z_{n-1})}{[f'(0)]^{n-1}}.$$

Now we substitute the values of the derivatives both in the numerator and the denominator of the latter by their values, given as in (2.4), to obtain

$$\begin{aligned}\left|\frac{f(\xi_1)f(\xi_2)\cdots f(\xi_{n-1})}{\xi_1\xi_2\cdots\xi_{n-1}[f'(0)]^{n-1}}\right| &= \frac{1}{n^{n-1}}\frac{|f'(z_1)|\cdots|f'(z_{n-1})|}{|f'(0)|^{n-1}}\\ &= \frac{1}{n^{n-1}}\frac{\prod_{j=1}^{n-1}\left(|z_j|\prod_{k=1,k\neq j}^{n-1}|z_j-z_k|\right)}{|z_1z_2\cdots z_{n-1}|^{n-1}}\\ &= \frac{1}{n^{n-1}}\frac{\prod_{1\le j<k\le n-1}|z_j-z_k|^2}{\prod_{j=1}^{n-1}|z_j|^{n-2}}.\end{aligned}$$

□

It is worth mentioning that a similar argument appeared already in a short note of Szegő [10] in connection with a famous problem of Erdös on lemniscates of polynomials. Szegő observed that for every monic polynomial $p(z)$ of degree n with zeros $z_1,\ldots,z_n$, and critical points $\xi_1,\ldots,\xi_{n-1}$, we have

$$\prod_{j=1}^{n-1}|p(\xi_j)| = \prod_{1\le j<k\le n-1}|z_k-z_j|^2.$$

In the same short note Szegő points out that Schur [5] had proven that the maximum of the latter Vandermonde determinant, when the zeros are restricted to be in the closed unit disc, is attained when they are, up to a rotation the roots of unity. Thus, Sezegő's and Schur's observations immediately yield the above mentioned Tischler's result.

In what follows we shall study the behaviour of the quantity

$$T(z_1,\ldots,z_{n-1}) = \frac{\prod_{1\le j<k\le n-1}|z_k-z_j|}{\prod_{j=1}^{n-1}|z_j|^{(n-2)/2}}. \tag{2.6}$$

Given a limited domain E in the complex plane whose complement contains a neighbourhood of the origin, we shall investigate

$$\max_{z_1,\ldots,z_{n-1}\in E} T(z_1,\ldots,z_{n-1}). \tag{2.7}$$

Consider the the following electrostatic field. Given a positive integer $n\ge 2$, a fixed negative charge with force of absolute value $n/2-1$ is located at the origin and $n-1$ free unit charges are located at $z_1,\ldots,z_{n-1}$. Suppose that the electrostatic field generated by these charges obey the law of the logarithmic potential which means that all the charges are uniformly distributed along infinite straight lines perpendicular to the complex plane. Thus, the total energy of this field is given by

$$L(z_1,\dots,z_{n-1}) = -\frac{n-2}{2}\sum_{k=1}^{n-1}\log\frac{1}{|z_k|} + \sum_{1\le i<k\le n-1}\log\frac{1}{|z_i-z_k|}.$$

Then obviously

$$\prod_{j=1}^{n-1}\frac{n|f(\xi_j)|}{|\xi_j||f'(0)|} = T^2(z_1,\dots,z_{n-1}) = \exp\left(-2\,L(z_1,\dots,z_{n-1})\right)$$

and the problem of minimizing the energy of the filed is equivalent to the problem of maximizing the generalized Vandermonde determinant T which itself is equivalent to maximizing the product of Smale's quantities $|f(\xi_j)|/(|\xi_j||f'(0)|)$. Since each $|f(\xi_j)|/(|\xi_j||f'(0)|)$, $j=1,\dots,n-1$, and thus the function $T(z_1,\dots,z_{n-1})$, are invariant upon a transformation $z\mapsto kz$, for any $k\in\mathbb{C}$, $k\neq 0$, without loss of generality we may assume that all zeros of f lie outside an open disc with radius $a>0$, i.e. that $|z_k|\ge a$. Moreover, if $z_1,\dots,z_{n-1}$ belong to a compact set $E\subset D_\infty(0,a)=\{z\ :\ |z|\ge a\}$, then the Weierstrass theorem guarantees the existence of points $z_1^*,\dots,z_{n-1}^*\in E$ that are solutions of the problem (2.7) and then these are points of an equilibrium of the described electrostatic field.

In what follows we discuss the particular case when E is an annulus in $\mathbb{C}$. Let $0<a<b<\infty$ and let us consider the closed ring domain

$$R(a,b)=\{z\ :\ a\le|z|\le b\}.$$

We shall be interested in the extremal problem

$$\max\left\{T(z_1,\dots,z_{n-1})\ :\ z_k\in R(a,b),\ k=1,\dots,n-1\right\}$$

By the above observation there are points $z_k^*\in R(a,b)$ for which this maximum is attained,

$$T(z_1^*,\dots,z_{n-1}^*) = \max\left\{T(z_1,\dots,z_{n-1})\ :\ z_k\in R(a,b),\ k=1,\dots,n-1\right\}. \tag{2.8}$$

In the next section we formulate a conjecture about the location of the extremal points of z_k^* when n is an odd integer as well as about the maximal value $T(z_1^*,\dots,z_{n-1}^*)$.

3. The new conjecture and further comments. We begin this section with our conjecture:

Conjecture 3. *Let n be an odd integer, $n-1=2m$, $m \in \mathbb{N}$. Then the unique, up to a rotation, extremal points $z_1^*, \ldots, z_{2m}^*$ for the problem (2.8) are the zeros of the polynomial $q(z)=(z^m-b^m)(z^m+a^m)$. Moreover, we have*

$$T(z_1^*, \ldots, z_{2m}^*) = \left(m\frac{a^m+b^m}{\sqrt{a^m b^m}}\right)^m .$$

First we comment some straightforward consequences of the conjecture. Obviously, for odd n it reads

$$T(z_1^*, \ldots, z_{n-1}^*) = (n-1)^{(n-1)/2}\left(\frac{a^{(n-1)/2}+b^{(n-1)/2}}{2a^{(n-1)/4}b^{(n-1)/4}}\right)^{(n-1)/2},$$

which immediately implies that there exist a critical point ξ_j such that

$$\frac{|f(\xi_j)|}{|\xi_j||f'(0)|} \le \frac{n-1}{n}\frac{a^{(n-1)/2}+b^{(n-1)/2}}{2a^{(n-1)/4}b^{(n-1)/4}}.$$

In general, the second quotient on the right-hand side of this inequality is greater than one and is equal to one if and only if $a=b$ which implies Tischler's result. On the other hand, for this location of the zeros z_k, always exists a critical point with

$$\frac{|f(\xi_j)|}{|\xi_j||f'(0)|} \le \frac{n-1}{n}.$$

Indeed, when $a<b$ the critical points ξ of the polynomial $p(z)=z(z^m-b^m)(z^m+a^m)$ satisfy

$$\xi^m = \frac{(m+1)(b^m-a^m) \pm \sqrt{(m+1)^2(b^m-a^m)^2+4(2m+1)a^m b^m}}{2(2m+1)},$$

so that they are located on two concentric circumferences centered at the origin, m critical points on each of them the radius of the smaller one being smaller than a. For those zeros ξ_k of $p'(z)$ with smaller modulus the strict inequality $|f(\xi_k)|/(|\xi_k||f'(0)|) < (n-1)/n$ holds. We omit these calculation in the general case because they are rather straightforward. We only show the situation on the figure below for the case $n=21$, $m=10$, when $a=1, b=2$. The larger points are on the figure the zeros of $p(z)$, i.e. the extremal charges, and the smaller one are the critical points of $p(z)$. Then the critical points are located on two circumferences with radii $r_1 = 0.786857$ and $r_2 = 1.87462$. For the critical points ξ_k with $|\xi_k| = r_1$ we have

$$\frac{|f(\xi_k)|}{|\xi_k||f'(0)|} = 0.909098 < 0.952381 = 20/21$$

and for those critical points ξ_j with $|\xi_j| = r_2$ we have $|f(\xi_j)|/(|\xi_j||f'(0)|) = 255.917$. Thus, Smale's conjecture is true for this configuration.

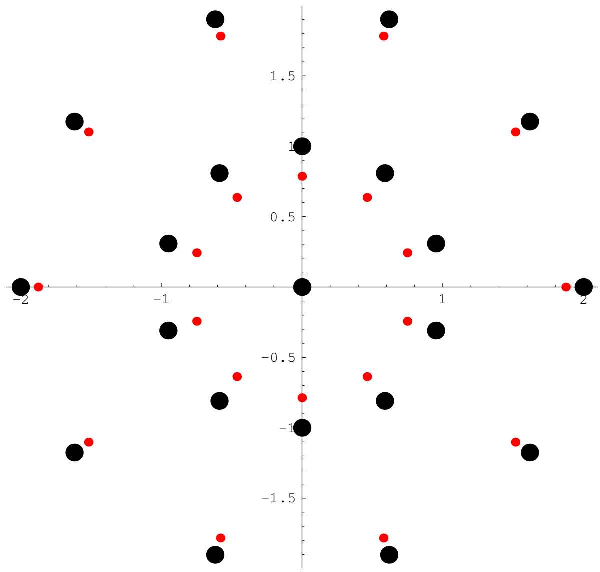

Fig. The conjectured extremal configuration for $n = 21$, $m = 10$, $a = 1$ and $b = 2$.

In what follows we justify our conjecture furnishing arguments that led us to it. In order to do this, first we provide brief information and some basic properties of the so-called weighted Fekete points. For details see Chapter III of Saff and Totik's monograph [3].

Definition 1. *Given a compact set E of the complex plane, a positive integer n, and a continuous weight function $w : E \to I\!R^+$, we define the n-th weighted Fekete set $\mathcal{F}_n := \{\zeta_1^*, \ldots, \zeta_n^*\} \subset E$ to be a set which maximizes the product*

$$\prod_{1 \le i < j \le n} |\zeta_i - \zeta_j| w(\zeta_i) w(\zeta_j). \tag{3.1}$$

The points $\{\zeta_i^*\}_1^n$ *are called weighted Fekete points.*

We immediately obtain the following:

Proposition 1. *Let* E *be a compact set and* $W(z)$ *be an analytic function in some open domain containing* E *with* $W(z) \neq 0$ *for every* $z \in E$. *Then for any* $n \in \mathbb{N}$ *the weighted Fekete points for the weight* $w(z) := |W(z)|$ *are on the boundary of* E.

Proof. Suppose that $n \in \mathbb{N}$ is fixed and let $\mathcal{F}_n = \{\zeta_1, \dots, \zeta_n\}$ be a weighted Fekete set for E. Fix $\zeta_k \in \mathcal{F}_n$ and consider the function $g_k(z) := W^{n-1}(z) \prod_{i \neq k}(z - \zeta_i)$. It is analytic in $\mathrm{int}(E)$ and $|g_k(z)|$ achieves its maximum at ζ_k. Therefore, by the maximum principle $\zeta_k \in \partial E$ (unless g_k is a constant, which is impossible because of the analyticity of W). Since ζ_k was arbitrary, we obtain $\mathcal{F}_n \subset \partial E$. $\Box$

The close relation of our problem to weighted Fekete points is revealed by the fact that (3.1) reduces to $T(\zeta_1, \dots, \zeta_n)$ when $w(z) = 1/\sqrt{|z|}$. Therefore, in fact, we need to determine the weighted Fekete points for the ring $R(a, b)$ associated with this weight. Our conjecture is that for even n they are equally distributed on the circumferences $|z| = a$ and $|z| = b$ is such a way that the arguments of the points on $|z| = a$ are rotated π/n in comparison with those of the points on $|z| = b$.

Related to the weighted Fekete points is the minimal energy problem with weight w. For a compact set E, denote by $\mathcal{M}_E$ the set of all unit Borel measures. Given a continuous weight function w, an equilibrium measure of E associated with w is defined to be a measure $\mu_w := \mu_{w,E}$ such that

$$I_w(\mu_w) = \min\{I_w(\mu) : \mu \in \mathcal{M}_E\},$$

where

$$I_w(\mu) := \int \log \frac{1}{|x - z| w(x) w(z)} \, d\mu(x) d\mu(z)$$

is the weighted energy of a measure $\mu \in \mathcal{M}_E$. The equilibrium measure is unique and is characterized by the Euler-Lagrange variational conditions

$$U^\mu(x) + Q(x) = F \qquad \text{for } x \in \mathrm{supp}(\mu), \tag{3.2}$$

$$U^\mu(x) + Q(x) \geq F \qquad \text{for } x \in E, \tag{3.3}$$

where $Q(z) = -\ln|z|$ is called *external field* and $U^\mu(x) = \int \log(1/|x|) \, d\mu(x)$ is the *logarithmic potential* of μ.

Let us denote the support of the weighted equilibrium measure μ_w with S_w. It is known that the Fekete sets $\mathcal{F}_n$ are subsets of S_w and the discrete counting measures

$$\nu(\mathcal{F}_n) := \frac{1}{n} \sum_{z \in \mathcal{F}_n} \delta_z$$

have a weak* limit μ_w (see [3, Theorems III.1.2 and III.1.3]).

Next we find the weighted equilibrium measure for $R(a, b)$ with $w(z) = 1/\sqrt{|z|}$.

Proposition 2. *Let $0 < \alpha < 1$ and $w_\alpha(z) = |z|^{-\alpha}$. Then the weighted equilibrium measure μ_{w_α} on $D(a, b)$ associated with the weight w_α is*

$$\mu_{w_\alpha} = \alpha \mu_a + (1 - \alpha) \mu_b, \tag{3.4}$$

where μ_a and μ_b are the unit one-dimensional Lebesgue measures on the circles $|z| = a$ and $|z| = b$, respectively.

Proof. Let $\mu := \alpha\mu_a + (1 - \alpha)\mu_b$. Certainly $\operatorname{supp}(\mu) = \{|z| = a\} \cup \{|z| = b\} = \partial D(a, b)$. Recall that for each $r > 0$

$$U^{\mu_r}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z - re^{i\phi}|} \, d\phi = \begin{cases} \log 1/r & \text{if } |z| \le r \\ \log 1/|z| & \text{if } |z| > r \end{cases}.$$

Thus, for the logarithmic potential of $\mu(z)$ we have

$$U^{\mu}(z) = \alpha U^{\mu_a}(z) + U^{\mu_b}(z) = \begin{cases} \alpha \log 1/a + (1 - \alpha) \log 1/b & \text{if } |z| \le a \\ \alpha \log 1/|z| - (1 - \alpha) \log b & \text{if } a < |z| \le b \\ \log 1/|z| & \text{if } |z| > b \end{cases}. \tag{3.5}$$

Since the external field is $Q(z) = \alpha \log |z|$, we can easily verify that conditions (3.2) and (3.3) hold for μ. In fact equality holds on all of $D(a, b)$ and not only on $\partial D(a, b)$. The uniqueness of the equilibrium measure now implies that $\mu_{w_\alpha} = \mu$ and (3.4) follows. $\square$

Since in our case $\alpha = 1/2$, the weighted equilibrium measure associated with the electrostatic problem is $(\mu_a + \mu_b)/2$ which shows that when n goes to infinity, the charges that minimize the energy must be distributed in such a way that approximately half of them should lie on $|z| = a$ and the other half on

$|z| = b$. It is natural to try to guess the situation when the number of the free charges is odd, i.e. when $n - 1 = 2m - 1$. Despite that we know the approximate asymptotic behaviour, it is not clear if there will be $m - 1$ extremal points z_k^* on $|z| = a$ and m on $|z| = b$ or vice versa: m Fekete points with $|z| = a$ and $m - 1$ with $|z| = b$. The exact location of the extremal point in this case is still a mystery for me despite some numerical tests.

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