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DOES ATKINSON-WILCOX EXPANSION CONVERGES FOR ANY CONVEX DOMAIN?

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Abstract. The Atkinson-Wilcox theorem claims that any scattered field in the exterior of a sphere can be expanded into a uniformly and absolutely convergent series in inverse powers of the radial variable and that once the leading coefficient of the expansion is known the full series can be recovered uniquely through a recurrence relation. The leading coefficient of the series is known as the scattering amplitude or the far field pattern of the radiating field. In this work we give a simple characterization of the strictly convex domains, such that a reasonable generalization of the Atkinson-Wilcox expansion converges uniformly in the corresponding exterior domain. All these strictly convex domains are spheres.

1. Introduction. Given any compact set $B \subset \mathrm{R}^3$ with smooth boundary ∂B we consider the Helmholtz equation

$$\Delta u\,(\mathrm{r}) + k^2 u\,(\mathrm{r}) = 0, \qquad \mathrm{r} \in V, \tag{1.1}$$

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where $V = \mathrm{R}^3 \setminus B$. The solution can be represented as a sum of an incident field u^i and scattered field u^s as follows

$$u(\mathrm{r}) = u^i(\mathrm{r}) + u^s(\mathrm{r}), \qquad \mathrm{r} \in V. \tag{1.2}$$

The incoming wave u^i is supposed to be a plane wave or a point source generated spherical wave or a superposition of plane waves and/or spherical waves. The scattered field u^s is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{|\mathrm{r}|\to\infty} r\left(\frac{\partial u^s(\mathrm{r})}{\partial r} - iku^s(\mathrm{r})\right) = 0 \tag{1.3}$$

at infinity [17].

The far field (radiation) pattern or the scattering amplitude u_∞ plays a central role in the direct and inverse scattering theory. It was Atkinson [3] in 1949 who showed that any solution of the Helmholz's equation (1.1), which satisfies the Sommerfeld radiation condition (1.3), has an absolutely and uniformly convergent series representation in inverse powers of the radial distance in all space, exterior to the above sphere.

In fact, Atkinson presented a wave analogue of Maxwell's multipole expansion in potential theory [13], i.e. he proved that this expansion separates the radial from the angular dependence of the solution, where the radial dependence characterizes the radiative nature of the scattered field, while the angular dependence incorporates the geometrical and the physical characteristics of each particular target.

The one-to-one correspondence between the the scattered fields and their radiation patterns for the exterior Helmholtz problem was established by Rellich in [16]. This correspondence was to make explicit by Wilcox, who made a generalization of the Atkinson's expansion theorem, both for the acoustic [19], as well as the electromagnetic case [20]. His works contain the important fact that all the angular dependent coefficients $\{F_n : n = 1, 2, \ldots\}$ in the Atkinson's expansion can be recovered through a recurrence relation via the leading coefficient F_0, known as the far field pattern u_∞ [8]. The above results were extended to the case of electromagnetic scattering by Wilcox [20] and Athanasiadis and Giotopoulos [2], to the scattering in $\mathbf{R}^2$ by Karp [11], to elasticity by Dassios [9], Cakoni and Dassios [6], and for inhomogeneous waves by Caviglia and Morro [7].

The expansion theorem brought up the questions of recovering the radiating solution of the Helmholtz equation from a knowledge of their far field patterns in the direct scattering problem, as well as the determination of the shape and the nature of the scatterer from a given scattering amplitude in the inverse obstacle problem.

During the 1990's there were many attempts on the so-called "expansion theorem" for arbitrary convex domain [1], for obstacles such as the prolate and the oblate spheroid [4] and the ellipsoid [5, 10], since it provides a very efficient way to construct methods (the infinite element method, the method of mirror images, etc.) for solving scattering problems.

The present work studies bounded domains in three-dimensional space, represented by orthogonal curvilinear coordinates that allow Atkinson-Wilcox series representation in the so-called far-field zone $r = |x| \gg 1$ and k varying in compact interval I in $(0, +\infty)$.

To be more precise, first we recall the classical Atkinson-Wilcox theorem.

Every solution u to the Helmholtz equation (1.1), satisfying the Sommerfeld radiating condition (1.3) has the asymptotic behavior of an outgoing spherical wave

$$u(\mathrm{r}) = \frac{e^{ikr}}{r}\left\{u_\infty(k, \hat{\mathrm{r}}) + O\left(\frac{1}{r}\right)\right\}, \quad r \to \infty, k \in I \subset (0, +\infty), \tag{1.4}$$

uniformly in all directions $\hat{\mathrm{r}} = \mathrm{r}/r$ and in all $k \in I$, where I is any compact interval in $(0, +\infty)$. The function u_∞, defined on $I \times S^2$ is known as the far field pattern of u and enjoys the following integral representation

$$u_\infty(k, \hat{\mathrm{r}}) = \frac{1}{4\pi}\int_{\partial B}\left\{u(\mathrm{r}')\frac{\partial e^{-ik\hat{\mathrm{r}}\cdot\mathrm{r}'}}{\partial \nu(\mathrm{r}')} - \frac{\partial u(\mathrm{r}')}{\partial \nu(\mathrm{r}')}e^{-ik\hat{\mathrm{r}}\cdot\mathrm{r}'}\right\}ds(\mathrm{r}'), \tag{1.5}$$

where ν denotes the outward unit normal to the scattering surface ∂B.

From the representation (1.5), it is obvious that the far field pattern is an analytic function with respect to $\hat{\mathrm{r}} \in S^2$ and $k \in I$, since the kernels in its representation are analytic on $I \times S^2$. We have the following theorem, due to Atkinson [3] and Wilcox [19].

Theorem 1.1. *Let u be a radiating solution to the Helmholtz equation* (1.1) *in the exterior $r > R > 0$ of a sphere and let r, θ and φ are spherical polar coordinates. Then u has an expansion of the form*

$$u(\mathrm{r}) = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{F_n(k, \theta, \varphi)}{r^n}, \tag{1.6}$$

that converges absolutely and uniformly for k in a compact interval $I \subset (0, \infty)$ and $r \in (R + \varepsilon, +\infty)$, where ε is any positive number. The series may be *differentiated any number of times and the resulting series all converges absolutely and uniformly in the same regions of k and r.*

Here, the function $F_0(k,\theta,\varphi)$ defined on $I \times S^2$, coincides with the far field pattern u_∞ of the radiating solution u (see relation (1.4)). Moreover, the coefficients F_n are independent of r and are uniquely determined from u_∞ by the following reciprocity relation

$$2iknF_n = n(n-1)F_{n-1} + \mathbf{B}F_{n-1}, \quad n = 1,2,\dots, \quad F_0 = u_\infty, \tag{1.7}$$

where $\mathbf{B} \equiv \mathbf{B}(\theta,\varphi,\partial_\theta,\partial_\varphi)$ is the Laplace-Beltrami operator on the unit sphere.

We shall made the following comment. The proof of Theorem 1.1 relies on the analytic properties of the integral kernel in the integral version of the problem (1.1)–(1.3). The relation (1.6) can be rewritten in the form

$$u(\mathrm{r}) = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{\widetilde{F}_n(k,\theta,\varphi)}{(kr)^n}, \tag{1.8}$$

where $\widetilde{F}_n$ satisfy the following relation (that follows from (1.7))

$$2in\widetilde{F}_n = n(n-1)\widetilde{F}_{n-1} + \mathbf{B}\widetilde{F}_{n-1}, \quad n = 1,2,\dots, \quad \widetilde{F}_0 = u_\infty. \tag{1.9}$$

This relation shows that in the particular case, when $\widetilde{F}_0(k,\theta,\varphi)$ is independent of k, the same is true for all $\widetilde{F}_n$. Generally, $\widetilde{F}_0(k,\theta,\varphi)$ depends on k and the expansions (1.6), (1.8) shall be considered for k varying in a fixed bounded interval I in $(0,+\infty)$.

Let us suppose that u is a radiating solution of the Helmholtz equation (1.1) in the exterior V of a convex domain $B = \mathbf{R}^3 \setminus V$ with smooth boundary ∂B. We assume that V can be represented as

$$V = \bigcup_{j=1}^{N} V_j, \tag{1.10}$$

such that for any $j = 1,\cdots,N$, the set V_j is an open connected domain in $\mathbf{R}^3$ and there exists an open domain $D \subseteq \mathbf{R}^2$ and a diffeomorphism

$$r,\xi,\eta \in (r_0,\infty) \times D \longrightarrow \mathrm{x} \in V_j. \tag{1.11}$$

The sets V_j can be considered as local charts for V and thus the diffeomorphism (1.11) and D formally depend on j. This dependence shall be assumed tacidly and we shall omit to write the index j.

We shall assume further, that the diffeomorphism (1.11) satisfies Assumptions 2.1–2.3 given below.

We shall give a relatively surprising characterization of all convex domains B, such that the solution of the Helmholtz equation (1.1) in the domain V_j satisfy the following natural generalization of the Atkinson-Wilcox expansion

$$u\left(x\right)=\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{F_n\left(k,\xi,\eta\right)}{r^n} \tag{1.12}$$

and the coefficients F_n, $n=1,2,\ldots$ are determined uniquely from the knowledge of the far field pattern F_0 by the second-order recurrence relation

$$kF_n(k,\xi,\eta)=C_1(\xi,\eta,n)F_{n-1}(k,\xi,\eta)+C_2(\xi,\eta,n)\mathbf{B}_SF_{n-1}(k,\xi,\eta). \tag{1.13}$$

Here, the functions C_1, C_2 are independent of r and the operator $\mathbf{B}_S$ is a second order differential operator with respect to the angular variables (i.e. $\mathbf{B}_S$ has the form

$$\mathbf{B}_S=\mathbf{B}_S(\xi,\eta,\partial_\xi,\partial_\eta)$$

in the coordinates ξ,η).

Our main result is the following.

Theorem 1.2. *Let ∂B be a smooth boundary surface of an arbitrary, strictly convex compact domain B in R^3, let $V=\mathrm{R}^3\setminus B$ can be represented as*

$$V=\bigcup_{j=1}^{N}V_j,$$

where V_j can be parametrized by (1.11)*, so that the Assumptions* 2.1–2.3 *are satisfied. If the Atkinson–Wilcox expansion* (1.12) *with respect to the radial variable is fulfilled and the second–order recurrence relation* (1.13) *holds in V_j for any $j=1,\cdots,N$, then ∂B is a sphere.*

The present result is strongly connected with the assumption that all the coefficients F_n in the expansion are independent of r and can be uniquely determined from the radiation pattern $u_\infty=F_0$ through a second-order recurrence formula of type (1.13), similar to the classical Atkinson-Wilcox situation. The above assumption is stronger compared, for example, with the one imposed by Dassios in [10], where the angular-dependent functions F_n depend on the wave number k in a more sophisticated way. For the case of an ellipsoidal boundary, Dassios derived an Atkinson-Wilcox-like series expansion, equipped with a sixth-order recurrence relation (see (15) and (57) in [10]). Note that our recurrence relation is of second-order as the classical one in Atkinson-Wilcox theorem (see (1.7) and (2.13) below).

The main objective that one can have is the following: why this particular choice of a candidate for a generalization of Atkinson-Wilcox expansion as the one in Theorem 1.2? For the ellipsoid, the result in [10] need a sixth-order recurrence relation. In this sense our result explains why the order of the recurrence relation can not be diminished essentially and the results of type [10] need higher order recurrence relations.

The next Section begins with an analogue of the classical Atkinson-Wilcox expansion, stated by Corollary 2.1. Then we introduce curvilinear coordinates r, ξ and η, so that the surface $r =$ constant, describes the boundary of an unknown scatterer. Rewriting the Laplace operator in terms of the new coordinate variables, substituting the series expansion of the radiation solution into the Helmholtz equation and equating like powers of r, we arrive at the recurrence relation for the orientation dependent coefficients. The condition for a second-order recurrence formula, independent of the wave number k, leads to the sequence of relations for the corresponding metric tensors $g_{\alpha,\beta}$ and scale factors h_α. In this way we prove that the only surface for which the Atkinson-Wilcox expansion theorem is true is the sphere.

2. Geometrical interpretation of the Expansion Theorem.

We have the following corollary of Theorem 1.1.

Corollary 2.1. *Let u be the radiating solution defined in Theorem* 1.1. *Then the expansion* (1.6) *is equivalent to the series*

$$u\,(\mathrm{r}) = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{F_n\,(k,\theta,\varphi)}{(kr)^n}. \tag{2.1}$$

Moreover, the coefficients F_n are uniquely determined from the leading coefficient F_0 by the reciprocity relation

$$2inF_n = n(n-1)F_{n-1} + \mathbf{B}F_{n-1}, \quad n = 1, 2, ... \tag{2.2}$$

The proof follows immediately from the relation (1.7).

Let V be an open domain in R^3, so that $B = \mathrm{R}^3 \backslash V$ is a strictly convex compact set in R^3, with smooth boundary $S = \partial B$. Our first assumption is the following one.

Assumption 2.1. There exists a smooth foliation of strictly convex surfaces

$$S_r, \quad r \geq r_0 > 0, \tag{2.3}$$

such that

$$V = \bigcup_{r > r_0} S_r \tag{2.4}$$

and $S_{r_0} = S$.

The existence of this foliation is equivalent to the existence of a smooth lapse function

$$r = r(\mathrm{x}),\ \mathrm{x} = (x_1, x_2, x_3) \in V \cup \partial B \tag{2.5}$$

defined on the closure of V such that the level surfaces $r(\mathrm{x}) = \text{const} > r_0$ are strictly convex and $r(\mathrm{x}) = r_0$ coincides with S (see [18]). One can represent V as

$$V = \bigcup_{j=1}^{N} V_j, \tag{2.6}$$

such that for any $j = 1, \cdots, N$, the set V_j is an open connected domain in $\mathbf{R}^3$ and there exists an open domain $D \subseteq \mathbf{R}^2$ and a diffeomorphism

$$r, \xi, \eta \in (r_0, \infty) \times D \longrightarrow \mathrm{x} \in V_j \tag{2.7}$$

so that

$$\mathrm{x} = \mathrm{x}\,(r, \xi, \eta)\,, \tag{2.8}$$

specifies the Cartesian coordinates in terms of r, ξ and η. Our next assumption is

Assumption 2.2. The parametrization (2.8) satisfies the property

$$x_j\,(r, \xi, \eta) = r A_j\left(r^{-1}, \xi, \eta\right), j = 1, 2, 3, \tag{2.9}$$

where A_j are analytic functions in r^{-1} for $r > r_0$ and smooth in $(\xi, \eta) \in D$.

The metric tensor associated with the parametrization (2.8) is

$$g_{\alpha,\beta} = \mathrm{x}_\alpha \cdot \mathrm{x}_\beta \qquad \alpha, \beta = r, \xi, \eta, \tag{2.10}$$

where $\mathrm{x}_\alpha = \partial_\alpha \mathrm{x}$, $\alpha = r, \xi, \eta$. The third assumption is

Assumption 2.3. *The parametrization* (2.8) *is orthogonal, i.e.*

$$g_{\alpha,\beta} = 0, \qquad \alpha \neq \beta. \tag{2.11}$$

As in (2.9) *we shall assume that the scale factors* $h_\alpha = \sqrt{g_{\alpha,\alpha}}$ *and the functions* $1/h_\alpha$, $\alpha = r, \xi, \eta$ *are analytic in* r^{-1} *for* $r > r_0$ *and smooth in* $(\xi, \eta) \in D$.

The parametrization defined by Assumptions 2.1–2.3 generalizes all the orthogonal coordinate systems that describe convex surfaces in $\mathbf{R^3}$ (see for instance Miller [14] and Morse and Feshbach [15]).

Following the statement of Atkinson-Wilcox theorem and Corollary 2.1, we shall suppose that the radiating solution $u = u(r, \xi, \eta)$ has an absolutely and uniformly convergent series expansion with respect to the inverse powers of kr of the form

$$u(r, \xi, \eta) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{F_n(k, \xi, \eta)}{(kr)^n} \tag{2.12}$$

and the coefficients $F_n, n = 1, 2, \dots$ are determined uniquely from the knowledge of the far field pattern F_0 by the second-order recurrence relation

$$F_n(k, \xi, \eta) = C_1(\xi, \eta, n) F_{n-1}(k, \xi, \eta) + C_2(\xi, \eta, n) \mathbf{B}_S F_{n-1}(k, \xi, \eta), \tag{2.13}$$

where the functions C_1, C_2 are independent of k, r and the operator $\mathbf{B}_S$ is a second order differential operator with respect to ξ and η.

P r o o f   o f   T h e o r e m 1.2. We shall take as our starting point the Helmholtz equation, rewriting it in the above curvilinear coordinates. Using the scale factors h_α and the general form of the Laplacian, we obtain

$$\begin{aligned} \Delta u + k^2 u &= \Big[\frac{1}{h_r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{h_r h_\xi h_\eta} \frac{\partial}{\partial r} \left(\frac{h_\xi h_\eta}{h_r} \right) \frac{\partial}{\partial r} \\ &+ \mathbf{P}_S(r, \xi, \eta, \partial_\xi, \partial_\eta) \Big] u + k^2 u = 0, \end{aligned} \tag{2.14}$$

with a second-order differential operator $\mathbf{P}_S$, defined by

$$\begin{aligned} \mathbf{P}_S(r, \xi, \eta, \partial_\xi, \partial_\eta) &= \frac{1}{h_\xi^2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{h_r h_\xi h_\eta} \frac{\partial}{\partial \xi} \left(\frac{h_r h_\eta}{h_\xi} \right) \frac{\partial}{\partial \xi} \\ &+ \frac{1}{h_\eta^2} \frac{\partial^2}{\partial \eta^2} + \frac{1}{h_r h_\xi h_\eta} \frac{\partial}{\partial \eta} \left(\frac{h_r h_\xi}{h_\eta} \right) \frac{\partial}{\partial \eta} \end{aligned} \tag{2.15}$$

From the Assumptions 2.1–2.3 we conclude that the coefficients of the Laplace operator (2.14) are analytic functions with respect to r^{-1}. Let us expand them in series in terms of the negative power of r with the form

$$\begin{aligned}
\frac{1}{h_r^2} &= \sum_{n=0}^{\infty} \frac{K_n(\xi,\eta)}{r^n}, \\
\frac{1}{h_r h_\xi h_\eta}\frac{\partial}{\partial r}\left(\frac{h_\xi h_\eta}{h_r}\right) &= \sum_{n=0}^{\infty} \frac{M_n(\xi,\eta)}{r^n}, \\
\mathbf{P}_S(r,\xi,\eta,\partial_\xi,\partial_\eta) &= \sum_{n=0}^{\infty} \frac{\mathbf{P}_{S,n}(\xi,\eta,\partial_\xi,\partial_\eta)}{r^n},
\end{aligned} \tag{2.16}$$

where we denote by $\mathbf{P}_{S,n}(\xi,\eta,\partial_\xi,\partial_\eta)$, $\deg(\mathbf{P}_{S,n}) \leq 2$ the differential operators with respect to the "angular variables" of our parametrization.

Substituting the above series representations and the solution (2.12) into the general form of the Laplacian (2.14) and omitting the e^{ikr}/r term, we obtain the relation

$$\begin{aligned}
-\sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{K_l F_{n-l}}{k^{n-l-1} r^{n+1}} - 2i\sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{(n-l+1)\,K_l F_{n-l}}{k^{n-l} r^{n+2}} \\
+\sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{(n-l+1)\,(n-l+2)\,K_l F_{n-l}}{k^{n-l+1} r^{n+3}} \\
+i\sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{M_l F_{n-l}}{k^{n-l} r^{n+1}} - \sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{(n-l+1)\,M_l F_{n-l}}{k^{n-l+1} r^{n+2}} \\
+\sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{\mathbf{P}_{S,l}\,(F_{n-l})}{k^{n-l+1} r^{n+1}} + \sum_{n=0}^{\infty} \frac{F_n}{k^{n-1} r^{n+1}} = 0.
\end{aligned} \tag{2.17}$$

By comparing the powers of r and after rearranging the summation index l, we arrive at

$$\begin{aligned}
\left\{k^2\,(K_0-1) - ikM_0 - \mathbf{P}_{S,0}\right\}\frac{F_n}{k^n} \\
+\left\{k^2 K_1 + ik\,(2nK_0 - M_1) + nM_0 - \mathbf{P}_{S,1}\right\}\frac{F_{n-1}}{k^{n-1}} \\
+\sum_{l=2}^{n}\Big\{k^2 K_l + ik[2\,(n-l+1)\,K_{l-1} - M_l] \\
-(n-l+1)\,[(n-l+2)\,K_{l-2} - M_{l-1}] - \mathbf{P}_{S,l}\Big\}\frac{F_{n-l}}{k^{n-l}} = 0,
\end{aligned} \tag{2.18}$$

for $n = 0, 1, 2, \ldots$.

The analysis of this expression, required for obtaining a recurrence formula of the form (2.13) leads to the following conclusions for the coefficients K_l, M_l and for the operators $\mathbf{P}_{S,l}$. Substituting $n = 0$ in (2.18) we obtain the equation

$$\left\{k^2 (K_0 - 1) - ikM_0\right\} F_0 = \mathbf{P}_{S,0}(F_0), \tag{2.19}$$

where $F_0 = u_\infty$ is the far field pattern, corresponding to the radiation solution of the Helmholtz equation.

Note that the operators $\mathbf{P}_{S,l}$ as well as the coefficients K_l, M_l are independent of k. Then, taking into consideration the analyticity of the far field pattern F_0 in k, we conclude that $K_0 = 1, M_0 = 0$ and the operator $\mathbf{P}_{S,0} \equiv 0$. For $n = 1$ the relation (2.18) takes a similar form

$$\left\{k^2 K_1 + ik(2K_0 - M_1) + M_0\right\} F_0 = \mathbf{P}_{S,1}(F_0) \tag{2.20}$$

or

$$\left\{k^2 K_1 + ik(2 - M_1)\right\} F_0 = \mathbf{P}_{S,1}(F_0). \tag{2.21}$$

From the arguments mentioned above we get $K_1 = 0, M_1 = 2$ and $\mathbf{P}_{S,1} \equiv 0$.

Having the zeroth and the first terms of the expansions (2.16), we can rewrite (2.18) in the form

$$\begin{aligned} 2inF_n = -\sum_{l=1}^{n} k^{l-1} \Big\{ & k^2 K_{l+1} + ik[2(n-l+1)K_l - M_{l+1}] \\ & -(n-l+1)[(n-l+2)K_{l-1} - M_l] - \mathbf{P}_{S,l+1} \Big\} F_{n-l}, \end{aligned} \tag{2.22}$$

for $n = 1, 2, \ldots$.

The assumption for a second-order recurrence formula of the form (2.13), which do not depend on the wave number yields to the relations

$$M_l = K_l = 0, \quad l = 2, 3, \ldots, \tag{2.23}$$

$$\deg(\mathbf{P}_{S,2}) = 2, \quad \mathbf{P}_{S,l} \equiv 0, \quad l = 3, 4, \ldots. \tag{2.24}$$

Now, if we substitute the obtained terms into the series expansions (2.16), we arrive at the following sequence of conditions

$$\begin{gathered} \sum_{n=0}^{\infty} \frac{K_n(\xi,\eta)}{r^n} = 1, \qquad \sum_{n=0}^{\infty} \frac{M_n(\xi,\eta)}{r^n} = \frac{2}{r}, \\ \sum_{n=0}^{\infty} \frac{\mathbf{P}_{S,n}(\xi,\eta,\partial_\xi,\partial_\eta)}{r^n} = \frac{\mathbf{P}_{S,2}(\xi,\eta,\partial_\xi,\partial_\eta)}{r^2}, \end{gathered} \tag{2.25}$$

which, in terms of the scale factors $h_\alpha : \alpha = r, \xi, \eta$, becomes

$$\begin{aligned} h_r^2 &= 1, \\ \frac{1}{h_\xi h_\eta}\frac{\partial}{\partial r}(h_\xi h_\eta) &= \frac{2}{r}, \\ \frac{1}{h_\xi^2}\frac{\partial^2}{\partial \xi^2} + \frac{1}{h_\xi h_\eta}\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi}\right)\frac{\partial}{\partial \xi} + \frac{1}{h_\eta^2}\frac{\partial^2}{\partial \eta^2} & \\ + \frac{1}{h_\xi h_\eta}\frac{\partial}{\partial \eta}(h_\xi h_\eta)\frac{\partial}{\partial \eta} &= \frac{\mathbf{P}_{S,2}(\xi, \eta, \partial_\xi, \partial_\eta)}{r^2}. \end{aligned} \tag{2.26}$$

The second relation in (2.26) is a first order ODE with respect to $h_\xi h_\eta$, which immediately implies $h_\xi h_\eta = r^2 C(\xi, \eta)$. Since the last relation in (2.26) is symmetric in h_ξ and h_η, we get

$$h_r^2 = 1, \qquad h_\xi^2 = r^2 A(\xi, \eta), \qquad h_\eta^2 = r^2 B(\xi, \eta). \tag{2.27}$$

If we introduce now the vectors

$$\begin{aligned} \mathrm{e}_r &= \frac{\partial}{\partial r}\mathrm{x}(r, \xi, \eta), \\ \mathrm{e}_\xi &= \frac{\partial}{\partial \xi}\mathrm{x}(r, \xi, \eta), \\ \mathrm{e}_\eta &= \frac{\partial}{\partial \eta}\mathrm{x}(r, \xi, \eta), \end{aligned} \tag{2.28}$$

we obtain new mutually orthogonal vector system $(\mathrm{e}_r, \mathrm{e}_\xi, \mathrm{e}_\eta)$ for which, using (2.27), we have

$$\mathrm{e}_r^2 = 1, \qquad \mathrm{e}_\xi^2 = r^2 A(\xi, \eta), \qquad \mathrm{e}_\eta^2 = r^2 B(\xi, \eta). \tag{2.29}$$

Therefore, the vector $\frac{\partial}{\partial r}\mathrm{e}_r$ may be expanded in terms of $\mathrm{e}'\mathrm{s}$ as

$$\frac{\partial}{\partial r}\mathrm{e}_r = a\mathrm{e}_r + b\mathrm{e}_\xi + c\mathrm{e}_\eta, \tag{2.30}$$

where a, b and c are unknown coefficients.

It follows immediately from the orthogonality of our e-system that

$$\begin{aligned} a &= \left(\frac{\partial}{\partial r}\mathrm{e}_r \cdot \mathrm{e}_r\right) = 0, \\ r^2 A(\xi, \eta)\, b &= \left(\frac{\partial}{\partial r}\mathrm{e}_r \cdot \mathrm{e}_\xi\right) = 0, \\ r^2 B(\xi, \eta)\, c &= \left(\frac{\partial}{\partial r}\mathrm{e}_r \cdot \mathrm{e}_\eta\right) = 0, \end{aligned} \tag{2.31}$$

which gives that $\frac{\partial}{\partial r}\mathrm{e}_r = 0$. This means that the base vector e_r does not depend on the radial variable r and may be written as $\mathrm{e}_r = \mathrm{E}(\xi,\eta)$, with $|\mathrm{E}| = 1$.

Integrating the first identity in (2.28), we obtain that

$$\mathrm{x}(r,\xi,\eta) = r\mathrm{E}(\xi,\eta) + \hat{\mathrm{E}}(\xi,\eta) \tag{2.32}$$

with $\hat{\mathrm{E}}$ being a vector independent of the 'radial' variable r.

The last two relations of (2.28) give

$$\begin{aligned}
\mathrm{e}_\xi^2 &= r^2 A(\xi,\eta) \\
&= r^2\left(\frac{\partial}{\partial\xi}\mathrm{E}(\xi,\eta)\right)^2 + 2r\frac{\partial}{\partial\xi}\mathrm{E}(\xi,\eta)\,\frac{\partial}{\partial\xi}\hat{\mathrm{E}}(\xi,\eta) + \left(\frac{\partial}{\partial\xi}\hat{\mathrm{E}}(\xi,\eta)\right)^2, \\
\mathrm{e}_\eta^2 &= r^2 B(\xi,\eta) \\
&= r^2\left(\frac{\partial}{\partial\eta}\mathrm{E}(\xi,\eta)\right)^2 + 2r\frac{\partial}{\partial\eta}\mathrm{E}(\xi,\eta)\,\frac{\partial}{\partial\eta}\hat{\mathrm{E}}(\xi,\eta) + \left(\frac{\partial}{\partial\eta}\hat{\mathrm{E}}(\xi,\eta)\right)^2
\end{aligned}$$

or

$$\frac{\partial}{\partial\xi}\hat{\mathrm{E}}(\xi,\eta) = 0, \qquad \frac{\partial}{\partial\eta}\hat{\mathrm{E}}(\xi,\eta) = 0, \tag{2.33}$$

which implies that $\hat{\mathrm{E}}$ is a constant vector.

Thus, we find that in curvilinear coordinates r, ξ and η the position vector x has the final form

$$\mathrm{x}(r,\xi,\eta) = r\mathrm{E}(\xi,\eta) + \hat{\mathrm{E}}, \quad |\mathrm{E}| = 1, \tag{2.34}$$

which leads to the result, that the surface $S = \{\mathrm{x}(r,\xi,\eta); r = r_0\}$ is a sphere $S^2_{r_0}$ and the proof is completed.

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