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A NOTE ON DIV-CURL LEMMA

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Abstract. We prove two results concerning the div-curl lemma without assuming any sort of exact cancellation, namely the divergence and curl need not be zero, and $\mathrm{div}\,(\overrightarrow{u}v) \in \mathcal{H}^1(\mathbb{R}^d)$ which include as a particular case, the result of [3].

1. Introduction. In Coifman, Lions, Meyer and Semmes [3], it was shown that the Hardy spaces can be used to analyze the regularity of the various nonlinear quantities by the compensated compactness theory due to L. Murat [12] and F. Tartar [15]. Recently, Müller [11], Helein [9], [10], Evans [5], Evans and Müller [6], and others have shown that certain nonlinear quantities arising in the theory of compensated compactness and in the study of harmonic maps belong to the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ (see also [8]). Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations. Quite recently, some new, deep endpoint regularity results for div-curl problems have been proved by J. Bourgain and H. Brezis [2] (see also [1], [17]). In particular,

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it was shown that for exponents p,q with $1<p<\infty$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, and vector fields $\overrightarrow{u}\in L^p(\mathbb{R}^d)^d$, $\overrightarrow{v}\in L^q(\mathbb{R}^d)^d$ with $\operatorname{div}\overrightarrow{u}=0$, $\operatorname{curl}\overrightarrow{v}=0$ in the sense of distributions, the scalar product $\overrightarrow{u}.\overrightarrow{v}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that

$$\|\overrightarrow{u}.\overrightarrow{v}\|_{\mathcal{H}^1(\mathbb{R}^n)}\le C\,\|\overrightarrow{u}\|_{L^p}\,\|\overrightarrow{v}\|_{L^q}.$$

The main purpose of the note is to prove two facts about div-curl lemma without assuming any sort of exact cancellation, namely the divergence and curl need not be zero, and which lead to $\operatorname{div}(\overrightarrow{u}v)$ being in the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$.

The proof will be divided into two parts. In part 1, we consider the case $\overrightarrow{u}$ and v are supported on the ball $|x|\le R_0$ where $R_0>1$ is a positive constant to be determined later, while in Part 2, the general case follows by partition of unity. In order to simplify the presentation, we take $p=q=2$.

The Sobolev space $H_p^1\left(\mathbb{R}^d\right)$, $1\le p<\infty$, consists of functions $f\in L^p\left(\mathbb{R}^d\right)$ such that $|\nabla f|\in L^p\left(\mathbb{R}^d\right)$. It is a Banach space with respect to the norm

$$\|f\|_{H_p^1}=\|f\|_{L^p}+\|\nabla f\|_{L^p}.$$

Specifically, we will prove

Theorem 1. *Let* $\overrightarrow{u}\in H_p^1\left(\mathbb{R}^d\right)^d$ *and* $v\in H_q^1\left(\mathbb{R}^d\right)$, $p>1$, $\dfrac{1}{p}+\dfrac{1}{q}=1$. *Then there exists a positive constant* $C(d)$ *such that*

$$\|\operatorname{div}(\overrightarrow{u}v)\|_{\mathcal{H}^1\left(\mathbb{R}^d\right)}\le C\left(\|\overrightarrow{u}\|_{L^p}\|\nabla v\|_{L^q}+\|\operatorname{div}\overrightarrow{u}\|_{L^p}\|v\|_{L^q}\right). \tag{1.1}$$

This result is similar to that in [3] where it is assumed additionally that $\operatorname{div}\overrightarrow{u}=0$.

Remark 1. Such inequalities and their generalizations are useful in hydrodynamics. Reader is refered, in particular to [3], [4].

Theorem 2. *Let* $1<p<\infty$, $\dfrac{1}{p}+\dfrac{1}{q}=1$. *Suppose* $\overrightarrow{u}=(u_1,\dots,u_d)$, $u_j\in L^p(\mathbb{R}^d)$, $1\le j\le d$ *be a vector field satisfying*

$$\operatorname{div}\overrightarrow{u}=\partial_1u_1+\cdots+\partial_du_d\in L^p(\mathbb{R}^d).$$

Assume that the scalar function $v(x)$ *belongs to* $L^q(\mathbb{R}^d)$. *We also suppose that* $\nabla v\in L^q(\mathbb{R}^d)$. *Then we have*

$$\operatorname{div}(v\overrightarrow{u})=\partial_1(vu_1)+\cdots+\partial_d(vu_d)\in\mathcal{H}^1(\mathbb{R}^d).$$

It is a generalized version of the "div-curl" lemma ([3], Theorem II.1). Observe that when div $\overrightarrow{u} = 0$, Theorem 2 reduces to the classical div-curl lemma [3].

For the sake of completeness, we recall the definition and some of the main properties of Hardy spaces $\mathcal{H}^p(\mathbb{R}^d)$ introduced by E. Stein and G. Weiss [14] (for more facts on these spaces see C. Fefferman and E. Stein [7]).

Definition 1 ([7]). *Let $0 < p < \infty$, and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\int\limits_{\mathbb{R}^n} \varphi dx = 1$. A tempered distribution f belongs to the Hardy space $\mathcal{H}^p(\mathbb{R}^d)$ if*

$$f^*(x) = \sup_{t>0} |(\varphi_t * f)(x)| \in L^p(\mathbb{R}^d), \tag{1.2}$$

where $\varphi_t(x) = t^{-d}\varphi\left(t^{-1}x\right)$.

It is known that if $f \in \mathcal{H}^p(\mathbb{R}^d)$, then (1.2) holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\int\limits_{\mathbb{R}^d} \varphi dx = 1$. The (quasi)-norm of $\mathcal{H}^p(\mathbb{R}^d)$ is defined, up to equivalence, by

$$\|f\|_{\mathcal{H}^p(\mathbb{R}^d)} = \|f^*(x)\|_{L^p(\mathbb{R}^d)} = \left(\int\limits_{\mathbb{R}^d} |f^*(x)|^p \, dx\right)^{\frac{1}{p}}.$$

We know by ([7], [13]) that if $1 \le p < \infty$, then $\mathcal{H}^p$ is a Banach space:

$$\begin{aligned} \mathcal{H}^p(\mathbb{R}^d) &= L^p(\mathbb{R}^d) \quad \text{for} \quad 1 < p < \infty, \\ \mathcal{H}^1(\mathbb{R}^d) &\subset L^1(\mathbb{R}^d) \quad \text{with continuous injection,} \end{aligned}$$

and that $\mathcal{H}^p(\mathbb{R}^d)$, $0 < p < 1$, are quasi-Banach spaces in the quasi-norm $\|.\|_{\mathcal{H}^p(\mathbb{R}^d)}$.

The crucial fact for our purpose is the boundedness of the Riesz transforms R_j on all of the spaces $\mathcal{H}^p$. Furthermore, an L^1-function f on $\mathbb{R}^d$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$ if and only if its Riesz transforms $R_j f$ all belong to $L^1(\mathbb{R}^d)$ and

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^d)} \cong \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \quad \text{(equivalent norms).}$$

Notice that all function $f \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy

$$\int\limits_{\mathbb{R}^d} f(x)dx = 0. \tag{1.3}$$

Indeed, the assumption $f \in \mathcal{H}^1(\mathbb{R}^d)$ implies that the Fourier transforms

$$\widehat{f}(\xi) = \int f(x)e^{-ix\xi}dx \quad \text{and} \quad \widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|}\widehat{f}(\xi), \quad (j = 1, \ldots, d),$$

are all continuous on $\mathbb{R}^d$, so $\widehat{f}(0) = 0$, and (1.3) is proved. We know also that if $f \in \mathcal{H}^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for some $0 < p < 1$, then

$$\int |x|^{|\alpha|} |f(x)| \, dx < +\infty \quad \text{and} \quad \int x^\alpha f(x) dx = 0 \tag{1.4}$$

for every multi-index α such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq d\left(\frac{1}{p} - 1\right)$.

We are going to show

Lemma 1. *Let $f \in L^1(\mathbb{R}^d)^d$ and $\nabla.f = 0$. Then*

$$\int f(x)dx = 0.$$

P r o o f. Let $f \in L^1(\mathbb{R}^d)^d$ and $\nabla.f = 0$. Applying the Fourier transform gives

$$\xi.\widehat{f}(\xi) = 0 \quad \text{for all } \xi \in \mathbb{R}^d.$$

We write $\xi = r\omega$ with $r = |\xi|$ and $|\omega| = 1$, to obtain

$$\omega.\widehat{u}(r\omega) = 0.$$

Since $f \in L^1(\mathbb{R}^d)^d$, the function $\widehat{f}$ is continuous on $\mathbb{R}^d$. So letting $r \to 0$ gives

$$\omega.\widehat{f}(0) = 0 \quad \text{for all } \omega \quad \text{with} \quad |\omega| = 1.$$

Hence

$$\widehat{f}(0) = 0$$

and this completes the proof. □

Let $\gamma > 1$. We define the maximal function of f depending on γ,

$$M_\gamma f(x) = \sup_{t>0} \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |f(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}}.$$

We begin by establishing the following result which is a variant of the Hardy-Littlewood maximal theorem. We need

Lemma 2. *If $\gamma < p \leq \infty$, then*

$$M_\gamma : L^p\left(\mathbb{R}^d\right) \to L^p\left(\mathbb{R}^d\right) \quad \textit{is bounded.}$$

See [13] for the proof.

The following result due to [3], shows the importance of the Hardy space theory in estimating the non-linear term $u.\nabla v$ attached to the Navier-Stokes equations and this produces a useful tool for PDE.

Lemma 3. *Let* $1 < p < \infty$, $1 < q < d$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + 1$. *If* $\overrightarrow{u} \in L^p\left(\mathbb{R}^d\right)^d$ *with* $\nabla . \overrightarrow{u} = 0$ *and* $\nabla v \in L^q\left(\mathbb{R}^d\right)$, *then*

$$\overrightarrow{u}.\nabla v \in \mathcal{H}^r\left(\mathbb{R}^d\right),$$

and

$$\left\|\overrightarrow{u}.\nabla v\right\|_{\mathcal{H}^r\left(\mathbb{R}^d\right)} \leq C \left\|\overrightarrow{u}\right\|_{L^p} \left\|\nabla v\right\|_{L^q}.$$

Proof. The result is due to [3]; but we give here a detailed proof for the reader' s convenience. Since $\nabla . \overrightarrow{u} = 0$, we have

$$f = \overrightarrow{u}.\nabla v = \nabla .\left(\overrightarrow{u} \otimes (v - c)\right)$$

for an arbitrary constant vector c. So we get

$$\left(\varphi_t * f\right)(x) = t^{-(d+1)} \int_{B_t(x)} \left(\nabla \varphi\right)\left(t^{-1}(x-y)\right) \overrightarrow{u}(y)\left(v(y) - m_B(v)\right) dy$$

where

$$m_B(v) = \frac{1}{|B_t(x)|} \int_{B_t(x)} v(y) dy.$$

Take

$$1 < \gamma < \infty, \quad 1 < \beta < d, \quad \text{with} \quad \frac{1}{\gamma} + \frac{1}{\beta} = 1 + \frac{1}{d},$$

and denote

$$\frac{1}{\beta^*} = \frac{1}{\beta} - \frac{1}{d}.$$

We recall the Sobolev-Poincaré inequality

$$\left(\int_{B_t(x)} \left|v(y) - m_B(v)\right|^{\beta^*} dy\right)^{\frac{1}{\beta^*}}$$

$$\overset{\text{Sobolev}}{\leq} C(d,\beta) \left(\int_{B_t(x)} \left|\nabla v(y)\right|^{\beta} dy + t^{-\beta} \int_{B_t(x)} \left|v(y) - m_B(v)\right|^{\beta} dy\right)^{\frac{1}{\beta}}$$

$$\overset{\text{Poincaré}}{\leq} C(d,\beta) \left(\int_{B_t(x)} \left|\nabla v(y)\right|^{\beta} dy\right)^{\frac{1}{\beta}}$$

where $\beta^* = \dfrac{\beta d}{d-\beta} > \beta$ is the Sobolev-exponent. Using Hölder and Sobolev-Poincaré inequalities we get

$$\begin{aligned}
|(\varphi_t * f)(x)| &\le \frac{C}{t^{d+1}} \left(\int\limits_{B_t(x)} |\overrightarrow{u}(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\int\limits_{B_t(x)} |v(y) - m_B(v)|^{\beta^*} \, dy \right)^{\frac{1}{\beta^*}} \\
&\le \frac{C}{t^{d+1}} \left(\int\limits_{B_t(x)} |\overrightarrow{u}(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\int\limits_{B_t(x)} |\nabla v(y)|^{\beta} \, dy \right)^{\frac{1}{\beta}} \\
&= C \left(\frac{1}{|B_t(x)|} \int\limits_{B_t(x)} |\overrightarrow{u}(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\frac{1}{|B_t(x)|} \int\limits_{B_t(x)} |\nabla v(y)|^{\beta} \, dy \right)^{\frac{1}{\beta}} \\
&\le C \left(M_\gamma \overrightarrow{u} \right)(x). \left(M_\beta (\nabla v) \right)(x).
\end{aligned}$$

We thus obtain

$$\sup_{t>0} |(\varphi_t * f)(x)| \le C \left(M_\gamma \overrightarrow{u} \right)(x). \left(M_\beta (\nabla v) \right)(x).$$

Since we can take γ and β so that

$$1 < \gamma < p, \quad 1 < \beta < q < d,$$

it follows from Lemma 2 that

$$\| M_\gamma \overrightarrow{u} \|_{L^p} \le C \| \overrightarrow{u} \|_{L^p}, \qquad \| M_\beta (\nabla v) \|_{L^q} \le C \| \nabla v \|_{L^q}.$$

Lemma 3 now follows from Hölder' s inequality :

$$\| f.g \|_{L^r} \le \| f \|_{L^p} \| g \|_{L^q} \qquad \left(0 < p < \infty, 0 < q < \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \right).$$

This finishes the proof of the lemma. □

2. Proof of Theorem 1. Without loss of generality, we may assume $\overrightarrow{u} \in C_0^\infty \left(\mathbb{R}^d \right)^d$ and $v \in C_0^\infty \left(\mathbb{R}^d \right)$. Take a nonnegative function $\varphi \in C_0^\infty \left(\mathbb{R}^d \right)$ so that

$$\text{supp } \varphi \subset \{ |x| \le 1 \}, \qquad \int \varphi dx = 1.$$

and set

$$\varphi_t(x) = t^{-d} \varphi \left(t^{-1} x \right) \quad \text{for} \quad t > 0.$$

Then

$$\|\mathrm{div}\,(\overrightarrow{u}\,v)\|_{\mathcal{H}^1\left(\mathbb{R}^d\right)} \approx \left\|\sup_{t>0}|\mathrm{div}\,(\overrightarrow{u}\,v)*\varphi_t|\right\|_{L^1\left(\mathbb{R}^d\right)}.$$

A simple calculation gives

$$\mathrm{div}\,(\overrightarrow{u}\,v)*\varphi_t(x) = -t^{-d-1}\int_{B_t(x_0)} \nabla\varphi\left(t^{-1}(x-y)\right)\overrightarrow{u}(y)\left(v(y)-m_B(v)\right)dy+$$
$$+\,t^{-d}m_B(v)\int_{B_t(x_0)} \varphi_t\left(x-y\right)\mathrm{div}\,\overrightarrow{u}(y)dy.$$

Following the proof of Lemma II.1 in [3], we take γ, β so that

$$1\le\gamma<p,\quad 1<\beta<q,\quad \text{with}\quad \frac{1}{\gamma}+\frac{1}{\beta}=1+\frac{1}{d}.$$

Using Hölder and Sobolev-Poincaré inequalities we get

$$\begin{aligned}
|\varphi_t*\mathrm{div}\,(\overrightarrow{u}\,v)\,(x)| &\le C\left\{\frac{1}{t^d}\int_{B_t(x_0)}|\overrightarrow{u}(y)|^{\beta}dy\right\}^{\frac{1}{\beta}}\left\{\frac{1}{t^{d+\beta'}}\int_{B_t(x_0)}|v(y)-m_B(v)|^{\beta'}dy\right\}^{\frac{1}{\beta'}}\\
&\quad + C\frac{1}{t^d}|m_B(v)|\int_{B_t(x_0)}|\mathrm{div}\ \overrightarrow{u}(y)|\,dy\\
&\le C\left\{\frac{1}{t^d}\int_{B_t(x_0)}|\overrightarrow{u}(y)|^{\beta}dy\right\}^{\frac{1}{\beta}}\left\{\frac{1}{t^{d}}\int_{B_t(x_0)}|\nabla v(y)|^{\gamma}dy\right\}^{\frac{1}{\gamma}}\\
&\quad + \frac{C}{t^d}|m_B(v)|\int_{B_t(x_0)}|\mathrm{div}\ \overrightarrow{u}(y)|\,dy\\
&\le C\left\{\frac{1}{t^d}\int_{B_t(x_0)}|\overrightarrow{u}(y)|^{\beta}dy\right\}^{\frac{1}{\beta}}\left\{M\,|\nabla v(x)|^{\gamma}\right\}^{\frac{1}{\gamma}}\\
&\quad + \frac{C}{t^d}|m_B(v)|\int_{B_t(x_0)}|\mathrm{div}\ \overrightarrow{u}(y)|\,dy.
\end{aligned}$$

where the various constants $C>0$ are independent of t or x_0. Here M is the

Hardy-Littlewood maximal function. Also, we have

$$\frac{1}{t^d}\left|m_B(v)\right| \int\limits_{B_t(x_0)} \left|\mathrm{div}\ \overrightarrow{u}(y)\right| dy \leq Mv(x)\ M\left(\mathrm{div}\ \overrightarrow{u}\right)(x)$$

Combining these estimates, we obtain

$$\sup_{t>0}\left|\varphi_t * \mathrm{div}\left(\overrightarrow{u}v\right)(x)\right| \leq C\left(M\left|\overrightarrow{u}(x)\right|^{\beta}\right)^{\frac{1}{\beta}}\left(M\left|\nabla v(x)\right|^{\gamma}\right)^{\frac{1}{\gamma}} + C\,Mv(x)\,M\left(\mathrm{div}\ \overrightarrow{u}\right)(x).$$

By Hölder' s inequality together with the maximal inequality, Theorem 1 is proved. □

3. Proof of Theorem 2. To prove the result, we distinguish three cases.

Case A. Let us assume first that

$$\mathrm{div}\ \overrightarrow{u} = \nabla.\overrightarrow{u} = 0.$$

In this case we get

$$\begin{aligned} \mathrm{div}\left(v\overrightarrow{u}\right) &= (\nabla v).\overrightarrow{u} + v\,\mathrm{div}\,\overrightarrow{u} \\ &= \overrightarrow{u}.\nabla v. \end{aligned}$$

Then we have $\overrightarrow{u} \in L^p(\mathbb{R}^d)^d$, $\nabla v \in L^q(\mathbb{R}^d)$ with $\mathrm{div}\,\overrightarrow{u} = 0$, curl $(\nabla v) = 0$ in the sense of distributions. It follows from Lemma 3 that

$$\overrightarrow{u}.\nabla v \in \mathcal{H}^1(\mathbb{R}^d)$$

and there exists an absolute constant C such that

$$\left\|\mathrm{div}\left(v\overrightarrow{u}\right)\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\left\|\overrightarrow{u}\right\|_{L^p}\left\|\nabla v\right\|_{L^q}.$$

Case B. We may of course suppose under additional assumptions that $\overrightarrow{u}$ and v are supported on the ball $|x| \leq R_0$. In order to simplify the presentation, we take $p = q = 2$. We shall write Ω for the ball in $\mathbb{R}^d$ of radius R_0 centered at the origin. By $H_0^1(\Omega)$ we denote the closed subspace of $H^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in the H^1 norm. Let

$$g = \mathrm{div}\,\overrightarrow{u} \in L^2(\mathbb{R}^d).$$

By the classical result (see e.g. [16]) we know that

$$g = \partial_1 g_1 + \cdots + \partial_n g_n,$$

where $g_1, \ldots, g_d$ belong to $H_0^1(\Omega)$. Setting

$$\overrightarrow{G} = (g_1, \ldots, g_n) \quad \text{and} \quad \overrightarrow{r} = \overrightarrow{u} - \overrightarrow{G}.$$

Then it follows

$$\operatorname{div} \overrightarrow{r} = 0 \quad \text{and} \quad \overrightarrow{r} \in L^2(\Omega).$$

Using Lemma 3 we infer

$$\operatorname{div}(\overrightarrow{r} v) \in \mathcal{H}^1(\mathbb{R}^n).$$

Further we set

$$f = \operatorname{div}\left(\overrightarrow{G} v\right).$$

For this purpose, we use Lemma 4 below, thus it follows that $f \in \mathcal{H}^1(\mathbb{R}^d)$.

Case C. The general case. We call φ a smooth bump function with compact support such that

$$1 = \sum_{k \in \mathbb{Z}^d} \varphi^2(x-k).$$

We have thus, if f and g are two functions,

$$\begin{aligned} f(x)g(x) &= \sum_{k \in \mathbb{Z}^d} f(x)\varphi^2(x-k)g(x) \\ &= \sum_{k \in \mathbb{Z}^d} f_k(x)g_k(x) \end{aligned}$$

where

$$f_k(x) = \varphi(x-k)f(x) \quad \text{and} \quad g_k(x) = \varphi(x-k)g(x).$$

Now set

$$\overrightarrow{u}_k(x) = \varphi(x-k)\overrightarrow{u}(x) \quad \text{and} \quad v_k(x) = \varphi(x-k)v(x)$$

for $k \in \mathbb{Z}^d$. We then have

$$\operatorname{div}(\overrightarrow{u} v) = \sum_{k \in \mathbb{Z}^d} (\overrightarrow{u}_k v_k) = \sum_{k \in \mathbb{Z}^d} w_k, \quad w_k = \operatorname{div}(\overrightarrow{u}_k v_k).$$

We are going to check that

$$\sum_{k \in \mathbb{Z}^d} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^d)} < \infty.$$

To do this, we apply the local version (**Case A**) and it follows

$$\begin{aligned} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C\left(\|u_k\|_{L^2} + \|\operatorname{div} u_k\|_{L^2}\right)\left(\|v_k\|_{L^2} + \|\operatorname{div} v_k\|_{L^2}\right) \\ &= \epsilon_k \in l^1\left(\mathbb{Z}^d\right). \end{aligned}$$

Up to now we have proved

$$\|\operatorname{div}(\overrightarrow{u} v)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\left(\|\overrightarrow{u}\|_{L^2} + \|\operatorname{div} \overrightarrow{u}\|_{L^2}\right)\left(\|v\|_{L^2} + \|\operatorname{div} v\|_{L^2}\right). \tag{3.1}$$

This automatically yields the estimate

$$\|\operatorname{div}(\overrightarrow{u}v)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\left(\|\overrightarrow{u}\|_{L^2}\|\nabla v\|_{L^2} + \|v\|_{L^2}\|\operatorname{div}\overrightarrow{u}\|_{L^2}\right). \tag{3.2}$$

To see this, we may replace $\overrightarrow{u}$ in the inequality above by

$$\overrightarrow{u}_\delta = \delta^{\left(\frac{1}{2}-\frac{d}{2}\right)}\overrightarrow{u}\left(\frac{x}{\delta}\right), \qquad \text{whenever } \ 0<\delta<\infty.$$

and similarly v by

$$v_\delta = \delta^{\left(\frac{1}{2}-\frac{d}{2}\right)}v\left(\frac{x}{\delta}\right), \qquad \text{whenever } \quad 0<\delta<\infty.$$

Thus the left-hand side of (3.1) fortunately does not change, while at right-hande we get rid the undesirable terms by letting δ either to 0, or to $+\infty$. This completes the proof. □

Now we turn to the proof of Lemma 4. One can show that every function $f \in L^p(\mathbb{R}^n)$, $p \in (1,+\infty]$, with compact support and $\int f dx = 0$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$. In particular,

Lemma 4. *If* $n^* = \dfrac{n}{n-1}$, $f \in L^{n^*}$, $\operatorname{supp} f \subset \overline{\Omega}$ *and*

$$\int f dx = 0,$$

then $f \in \mathcal{H}^1(\mathbb{R}^n)$.

Proof.

$$f = \operatorname{div}\left(\overrightarrow{G}\right)v + \overrightarrow{G}.\nabla v$$

and we have to prove that the two terms belong to L^{n^*}. We consider the first term on the right. Since $\nabla v \in L^2$, we have

$$\operatorname{div}\left(\overrightarrow{G}\right) \in L^2 \ \text{ and } \ v \in L^q \quad \text{where } \ \frac{1}{2}-\frac{1}{q}=\frac{1}{n}$$

Thus,

$$v\operatorname{div}\left(\overrightarrow{G}\right) \in L^{n^*}.$$

A similar argument works in the second term and this completes the proof of the lemma. □

Remark 2. It should be added that at the time the paper was finished, the author learnt that J. Y. Chemin has also obtained similar results. These are contained in his book "Perfect Incompressible Fluids, Asterisque 1995". His proofs which use a paradifferential approach, are quite different from the ones in this note.

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