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EXISTENCE OF SOLUTIONS FOR A CLASS OF QUASI-LINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. An existence theorem is proved for a class of quasi-linear singular integro-differential equations with Cauchy kernel.

1. Introduction. Many classes of singular integral and integro-differential equations are currently used in several fields of applied engineering mechanics like elasticity, plasticity, viscoelasticity and fracture mechanics. Non-linear singular integro-differential equations are further applied in other fields of engineering mechanics like structural analysis. Such structural analysis problems are reduced to the solution of a non-linear singular integro-differential equations connected with the behavior of the stress fields. These types of singular integral equations and singular integro-differential equations form the latest high technology on the solution of very important problems of solid and fluid mechanics and therefore

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in recent years special attention is given to them [1]–[3], [10, 11, 15]. We refer to Ladopoulos E. G. [8] for many applications of singular integral equations and singular integro-differential equations in Engineering and Science. Also, we refer to [4, 5, 12], [15]–[18] for the methods of solution and many other applications.

The aim of this paper is to investigate the existence of solution of a class of quasi-linear singular integro-differential equation with Cauchy kernel in the form:

$$A(s,u(s))u'(s) - B(s,u(s))\frac{1}{\pi i}\int\limits_{\Gamma}\frac{u'(\tau)}{\tau - s}d\tau = g(s,u(s)), \tag{1.1}$$

under the initial condition

$$u(r) = 0, \tag{1.2}$$

where Γ is a closed smooth contour and r is a fixed point on Γ.

Our discussion is based on the applicability of Schauder's fixed point theorem [6, 7, 14, 15, 18] to our problem.

Throughout the paper $L_p(\Gamma)$, $(1 < p < \infty)$, means the Banach space of all measurable functions f on Γ, with the norm:

$$\|f\|_{L_p} = \left(\int_\Gamma |f(t)|^p\, dt\right)^{p^{-1}},$$

and $\mathcal{W}_p^1(\Gamma)$, $(1 < p < \infty)$, means the Sobolev space of all functions $u \in L_p(\Gamma)$ which possess $u' \in L_p(\Gamma)$ [9].

We shall seek the solution $u(s)$ of equation (1.1) in the Sobolev space $\mathcal{W}_p^1$, under the following conditions:

(I) The functions $A(s,u(s))$, $B(s,u(s))$ and $g(s,u(s))$ are continuous and positive on the region:

$D = \{(s,u) : s \in \Gamma,\ |u| < \infty\}$. We assume:

$$A^2(s,u) - B^2(s,u) \geq 1, \tag{1.3}$$

for each (s,u) belongs to D.

(II) The functions $A(s,u(s))$, $B(s,u(s))$ and $g(s,u(s))$ satisfy the following Holder-Lipschtiz conditions:

$$|A(s_2,u_2) - A(s_1,u_1)| \leq \gamma_1(|s_1 - s_2|^\delta + |u_1 - u_2|), \tag{1.4}$$

$$|B(s_2,u_2) - B(s_1,u_1)| \leq \gamma_2(|s_2 - s_1|^\delta + |u_2 - u_1|) \tag{1.5}$$

and

$$|g(s_2,u_2) - g(s_1,u_1)| \leq \gamma_3(|s_2 - s_1|^\delta + |u_2 - u_1|). \tag{1.6}$$

where γ_i, $(i = 1, 2, 3)$ are positive constants and $0 < \delta < 1$.

Theorem 1.1. *Assume that* $A(s, u(s))$, $B(s, u(s))$ *and* $g(s, u(s))$ *satisfy conditions* (1.3)–(1.6). *Then equation* (1.1) *under the initial condition* (1.2), *has at least one solution* u *in the space* $\mathcal{W}_p^1$.

It is worthwhile to notice that our problem (1.1) and (1.2) is a generalization of the problem discussed by Wolfersdorf L. V. in [18].

We shall solve our problem through two steps. The first step, in Section 2, we write the function $u'(s)$ in the form of an integral equation in the unknown function $u(s)$, and we discuss some properties of the kernel of a fixed point equation in $u(s)$. The second step, in Section 3, is to prove the existence theorem for our problem (1.1), (1.2).

2. Reduction to fixed point equation.

In this section we obtain some results which give us the possibility to handle the fixed point equation with the help of Schauder's fixed point theorem.

Theorem 2.1. *If condition* **(I)** *is satisfied, then the quasi-linear singular integro-differential equation* (1.1) *reduces to a fixed point equation under the initial condition* (1.2).

P r o o f. Let us consider the analytic function

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{u'(\sigma)}{\sigma - z}d\sigma, \quad \Phi^-(\infty) = 0 \tag{2.1}$$

which for the following Sokhotski formulae [5] hold:

$$\Phi^\pm(s) = \pm\frac{1}{2}u'(s) + \frac{1}{2\pi i}\int_\Gamma \frac{u't(\tau)}{\tau - s}d\tau. \tag{2.2}$$

Therefore

$$\begin{cases} \Phi^+(s) + \Phi^-(s) = \dfrac{1}{\pi i}\displaystyle\int_\Gamma \frac{u'(\tau)}{\tau - s}d\tau \\ \text{and} \\ \Phi^+(s) - \Phi^-(s) = u'(s). \end{cases} \tag{2.3}$$

Substituting the expressions from equations (2.3) into (1.1), we get the following boundary value problem (B.V.P.):

$$\Phi^+(s) = \left(\frac{A(s,u(s)) + B(s,u(s))}{A(s,u(s)) - B(s,u(s))}\right)\Phi^-(s) + \frac{g(s,u(s))}{A(s,u(s)) - B(s,u(s))}. \tag{2.4}$$

From the theory of linear singular integral equations [5], the index χ of equation (1.1) is the index of the problem (2.4) and it is given by $\chi = \operatorname{ind}\left(\dfrac{A(s,u(s))+B(s,u(s))}{A(s,u(s))-B(s,u(s))}\right)$. Assuming that $\chi = 0$, hence we can put:

$$C(s) = \frac{A(s,u(s))+B(s,u(s))}{A(s,u(s))-B(s,u(s))} = \frac{X^+(s)}{X^-(s)}. \tag{2.5}$$

Therefore,

$$X(z) = \exp(M(z)) \quad \text{with} \quad M(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln C(\tau)}{\tau - z}d\tau, \tag{2.6}$$

where

$$X^+(s) = \sqrt{C(s)}\exp(M(s)) \quad \text{and} \quad X^-(s) = \frac{1}{\sqrt{C(s)}}\exp(M(s)). \tag{2.7}$$

Substituting (2.5) into (2.4), we get:

$$\left(\frac{\Phi^+(s)}{X^+(s)}\right) = \left(\frac{\Phi^-(s)}{X^-(s)}\right) + \frac{g(s,u(s))}{(A(s,u(s))-B(s,u(s)))X^+(s)}. \tag{2.8}$$

From [5], the B.V.P. (2.8) has the following solution:

$$\Phi(z) = (X(z))\left(\frac{1}{2\pi i}\int_\Gamma \left(\frac{g(\tau,u(\tau))}{[A(\tau,u(\tau))-B(\tau,u(\tau))]X^+(\tau)}\right)\left(\frac{1}{\tau - z}\right)d\tau\right).$$

Hence,

$$\Phi^+(s) = \frac{g(s,u(s))}{2(A(s,u(s))-B(s,u(s)))} +$$

$$+\frac{X^+(s)}{2\pi i}\int_\Gamma \frac{g(\upsilon,u(\upsilon))}{X^+(\upsilon)(A(\upsilon,u(\upsilon))-B(\upsilon,u(\upsilon)))(\upsilon - s)}d\upsilon \tag{2.9}$$

and

$$\Phi^-(s) = \frac{-X^-(s)g(s,u(s))}{2X^+(s)(A(s,u(s))-B(s,u(s)))} +$$

$$+\frac{X^-(s)}{2\pi i}\int_\Gamma \frac{g(\upsilon,u(\upsilon))}{X^+(\upsilon)(A(\upsilon,u(\upsilon))-B(\upsilon,u(\upsilon)))(\upsilon - s)}d\upsilon. \tag{2.10}$$

Then,

$$\Phi^+(s) - \Phi^-(s) = \frac{g(s,u(s))}{2(A(s,u(s)) - B(s,u(s)))}\left(1 + \frac{X^-(s)}{X^+(s)}\right) +$$

$$+\frac{(X^+(s) - X^-(s))}{2\pi i}\int_\Gamma \frac{g(\upsilon,u(\upsilon))}{X^+(\upsilon)(A(\upsilon,u(\upsilon)) - B(\upsilon,u(\upsilon)))(\upsilon - s)}d\upsilon. \tag{2.11}$$

From (2.5) and (2.7), we have:

$$X^+(s) - X^-(s) = \left(\frac{2B(s,u(s))}{\sqrt{A^2(s,u(s)) - B^2(s,u(s))}}\right)\exp(M(s)). \tag{2.12}$$

From (2.3), (2.5), (2.11) and (2.12), we obtain:

$$u'(s) = \frac{A(s,u(s))g(s,u(s))}{f^2(s,u(s))} +$$

$$+\frac{B(s,u(s))\exp(M(s))}{\pi i f(s,u(s))}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - s}\right)d\upsilon. \tag{2.13}$$

where:

$$\begin{cases} \mu(\upsilon,u(\upsilon)) = X^+(\upsilon)(A(\upsilon,u(\upsilon)) - B(\upsilon,u(\upsilon))) \\ \\ f(s,u(s)) = \sqrt{A^2(s,u(s)) - B^2(s,u(s))}. \end{cases} \tag{2.14}$$

Integrating the expression (2.13), and applying the initial condition (1.2), we can see that our original problem turns to a fixed point equation for $u(s)$ as follows:

$$u(s) = \int_r^s \left\{\frac{A(\sigma,u(\sigma))g(\sigma,u(\sigma))}{f^2(\sigma,u(\sigma))} +\right.$$

$$\left. +\frac{B(\sigma,u(\sigma))\exp(M(\sigma))}{\pi i f(\sigma,u(\sigma))}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - \sigma}\right)d\upsilon\right\}d\sigma.$$

The above integral equation can be written as the following fixed point equation:

$$Su = u, \tag{2.15}$$

where S is the operator defined as follows:

$$(Su)(s) = \int_r^s T(\sigma,u(\sigma))d\sigma, \tag{2.16}$$

with the kernel function:

$$T(s,u(s)) = \frac{A(s,u(s))g(s,u(s))}{f^2(s,u(s))}+$$

$$+\frac{B(s,u(s))\exp(M(s))}{\pi i f(s,u(s))}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - s}\right)d\upsilon. \tag{2.17}$$

where $s \in \Gamma$. Hence the theorem is verified. □

Lemma 2.1. *The kernel function $T(s,u(s))$ given by (2.17) is bounded in the space L_p, $p>1$.*

Proof. We shall estimate the kernel $T(s,u(s))$ for $p>1$ as follows:

$$\|T(s,u(s))\|_p \le \|N_1(s)\|_p + \|N_2(s)\|_p \tag{2.18}$$

where

$$N_1(s) = \frac{A(s,u(s))g(s,u(s))}{f^2(s,u(s))} \tag{2.19}$$

and

$$N_2(s) = \frac{B(s,u(s))\exp(M(s,u(s)))}{\pi i f(s,u(s))}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - s}\right)d\upsilon. \tag{2.20}$$

By using condition **(I)** and from [7, 13], we have:

$$\|N_1(s)\|_p \le \|A(s,u(s))\|_{p_1}\|g(s,u(s))\|_{p_2}, \tag{2.21}$$

where

$$p_1^{-1} + p_2^{-1} = p^{-1},$$

$$\|A(s,u(s))\|_{p_1} = \left(\int_\Gamma |A(s,u(s))|^{p_1} ds\right)^{p_1^{-1}} \tag{2.22}$$

and

$$\|g(s,u(s))\|_{p_2} = \left(\int_\Gamma |g(s,u(s))|^{p_2} ds\right)^{p_2^{-1}}. \tag{2.23}$$

By using the conditions (1.4)–(1.6), we obtain:

$$|A(s,u(s))| \le \eta_1 \quad \text{with} \quad \eta_1 = \gamma_1 R + \sigma_1, \qquad \sigma_1 = \max_{s\in\Gamma}|A(s,0)|. \tag{2.24}$$

Therefore, from (2.22) and (2.24), we have:

$$\|A(s,u(s))\|_{p_1} \le \eta_1 L^{p_1^{-1}} \tag{2.25}$$

where

$$L = |\Gamma| = \int_\Gamma ds \quad \text{is the length of} \quad \Gamma.$$

Similarly,

$$|g(s,u(s))| \le \eta_2 \quad \text{with} \quad \eta_2 = \gamma_3 R + \sigma_2, \sigma_2 = \max_{s\in\Gamma} |g(s,0)|\,. \tag{2.26}$$

Therefore, from (2.23) and (2.26), we have:

$$\|g(s,u(s))\|_{p_2} \le \eta_2 L^{p_2^{-1}}. \tag{2.27}$$

Substituting from (2.25) and (2.26) into (2.21), we have:

$$\|N_1(s)\|_p \le Q_1 \equiv \text{const}\,, \tag{2.28}$$

where

$$Q_1 = \eta_1\eta_2 L^{p^{-1}}.$$

Also, by using condition **(I)** and [13], we obtain:

$$\|N_2(s)\|_p \le \left\| B(s,u(s))\exp(M(s))\left(\frac{1}{\pi i}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - s}\right)d\upsilon\right)\right\|_p \le$$

$$\le \|B(s,u(s))\|_{q_1}\|\exp(M(s))\|_{q_2}\left\|\frac{1}{\pi i}\int_\Gamma \left(\frac{g(\upsilon,u(\upsilon))}{\mu(\upsilon,u(\upsilon))}\right)\left(\frac{1}{\upsilon - s}\right)d\upsilon\right\|_{q_3}, \tag{2.29}$$

where, $q_1^{-1} + q_2^{-1} + q_3^{-1} = p^{-1}$.

To obtain a bound for the norm given in (2.29), we carry out the calculation in three steps:

(a) Estimation of $\|B(s,u(s))\|_{q_1}$. As in (2.24) it is easy to see that:

$$|B(s,u(s))| \le \eta_3 \quad \text{with} \quad \eta_3 = \gamma_2 R + \sigma_3, \quad \sigma_3 = \max_{s\in\Gamma}|B(s,0)|. \tag{2.30}$$

Therefore,

$$\|B(s,u(s))\|_{q_1} \le \eta_3 L^{q_1^{-1}}. \tag{2.31}$$

(b) Estimation of $\|\exp(M(s))\|_{q_2}$. Since by [4],

$$\|\exp(M(s))\|_{q_2} \le (1 + \|M(s)\|_{q_2})\exp(\|M(s)\|_{q_2}),$$

where, from (2.6) and [5],

$$\|M(s)\|_{q_2} = \left\|\frac{1}{2\pi i}\int_\Gamma \frac{\ln C(\tau)}{\tau - z}d\tau\right\|_{q_2} \le \rho_1\|C(\tau)\|_{q_2} \le \rho_1(\eta_1+\eta_3)^2 L^{q_2^{-1}}, \tag{2.32}$$

one obtain

$$\|\exp(M(s))\|_{q_2} \leq (1+\rho_1 \Lambda L^{q_2^{-1}}) \exp(\rho_1 \Lambda L^{q_2^{-1}}), \tag{2.33}$$

where

$$\Lambda = (\eta_1 + \eta_3)^2.$$

(c) Estimation of $\left\| \frac{1}{\pi i} \int_\Gamma \left(\frac{g(v,u(v))}{\mu(v,u(v))} \right) \left(\frac{1}{v-s} \right) dv \right\|_{q_3}$**.** By using equations (2.7), (2.14) and condition **(I)**, we get:

$$\left\| \frac{1}{\pi i} \int_\Gamma \left(\frac{g(v,u(v))}{\mu(v,u(v))} \right) \left(\frac{1}{v-s} \right) dv \right\|_{q_3} \leq \rho_2 \frac{\|g(s,u(s))\|_{q_3}}{\|\exp(M(s))\|_{q_3}} \leq$$

$$\leq \rho_2 \eta_2 L^{q_3^{-1}} \left(\frac{1}{\|\exp(M(s))\|_{q_3}} \right) \leq \rho_2 \eta_2 L^{q_3^{-1}}. \tag{2.34}$$

From (2.31), (2.33) and (2.34) into (2.29), we obtain:

$$\|N_2(s)\|_p \leq \Omega L^{p^{-1}} = Q_2, \tag{2.35}$$

where

$$\Omega = \left[\rho_2 \eta_2 \eta_3 (L^{-q_2^{-1}} + \rho_1 \Lambda) \exp(\rho_1 \Lambda L^{q_2^{-1}}) \right].$$

Finally, substituting from (2.28) and (2.35) into (2.18), we have:

$$\|T(s,u(s))\|_p \leq Q, \tag{2.36}$$

where $Q = Q_1 + Q_2 \equiv \mathrm{const}$. Lemma 2.1. is proved. $\square$

3. Existence theorem. For non-negative constants R and α, let us define the following compact set:

$$K_{R,\alpha}^{0,\delta} = \left\{ u \in C_0(\Gamma),\ |u| \leq R,\ |u(s_2) - u(s_1)| \leq \alpha |s_2 - s_1|^\delta,\ s_1, s_2 \in \Gamma \right\}, \tag{3.1}$$

where $C_0(\Gamma)$ is the space of all continuous functions $u(s)$ that defined on Γ such that $u(r) = 0$. We are going to prove some assertions about the set $K_{R,\alpha}^{0,\delta}$, its image under the operator S, i.e. $S(K_{R,\alpha}^{0,\delta})$ and the continuity of the operator S defined in (2.23). It is easy to see that the set $K_{R,\alpha_u}^{0,\delta}$ is a convex set.

In fact, we shall show that the operator S defined by (2.26) transforms the function u into a function which belongs to the class $K_{R,\alpha}^{0,\delta}$.

Therefore, for each $u \in K^{0,\delta}_{R,\alpha}$, let us have:

$$(Su)(s) = v(s), \quad v(s) \in C_0(\Gamma), \tag{3.2}$$

where

$$(Su)(s) = \int_r^s T(\sigma, u(\sigma))d\sigma.$$

Then, for any $s \in \Gamma$, we have from (2.26) and (2.43) that:

$$|v(s)| \leq \int_\Gamma |T(\sigma, u)|d\sigma \leq \|T(s, u(s))\|_p L^{\kappa^{-1}} \leq R, \tag{3.3}$$

where $R = QL^{\kappa^{-1}}$ and $p^{-1} + \kappa^{-1} = 1$.

Now, we evaluate $|v(s_2) - v(s_1)|$ for $p > 1$, $p^{-1} + \kappa^{-1} = 1$, $\kappa < \delta^{-1}$, $s_1, s_2 \in \Gamma$. We have:

$$|v(s_2) - v(s_1)| \leq \left| \int_r^{s_1} T(\sigma, u)d\sigma - \int_r^{s_2} T(\sigma, u)d\sigma \right| \leq$$
$$\leq \left| \int_{s_1}^{s_2} T(\sigma, u)d\sigma \right| \leq \|T(s, u)\|_p |s_2 - s_1|^{\kappa^{-1}}.$$

If $\|T(s, u)\|_p \leq Q \leq \alpha$, the operator S maps the whole space $C_0(\Gamma)$ into its convex compact subset $K^{0,\delta}_{R,\alpha}$. In particular, S maps $K^{0,\delta}_{R,\alpha}$ into itself. Then all the transformed functions $v(s)$ belong to the set $K^{0,\delta}_{R,\alpha}$. Hence, the following lemma is valid.

Lemma 3.1. *Let the functions $A(s, u(s))$, $B(s, u(s))$ and $g(s, u(s))$ satisfy conditions* (1.3)–(1.6). *Then, for arbitrary $u \in K^{0,\delta}_{R,\alpha}$, the transformed points $(Su)(s) = v(s)$ belong to the set $K^{0,\delta}_{R,\alpha}$.*

Lemma 3.2. *The operator S, defined in* (3.2), *which transforms the set $K^{0,\delta}_{R,\alpha}$ into itself, is continuous.*

P r o o f. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements of the set $K^{0,\delta}_{R,\alpha}$ converges uniformly to the element $u \in K^{0,\delta}_{R,\alpha}$.

Let us assume that:

$$C_n(s) = C(s, u_n(s)) = \frac{A(s, u_n(s)) + B(s, u_n(s))}{A(s, u_n(s)) - B(s, u_n(s))},$$

$$M_n(s) \equiv M(s, u_n(s)) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln C(\tau, u_n(\tau))}{\tau - s} d\tau,$$

$$X_n^{\pm}(s) = X^{\pm}(s, u_n(s)),$$

$$\mu(\upsilon, u_n(\upsilon)) = X_n^{+}(\upsilon)(A(\upsilon, u_n(\upsilon)) - B(\upsilon, u_n(\upsilon)))$$

and

$$f(s, u_n(s)) = \sqrt{A^2(s, u_n(s)) - B^2(s, u_n(s))}.$$

We consider the following difference:

$$|v_n(s) - v(s)| \leq \int_r^s |T(\sigma, u_n(\sigma)) - T(\sigma, u(\sigma))| d\sigma,$$

where:

$$|T(s, u_n(s)) - T(s, u(s))| \leq |(N_n^1)(s)| + |(N_n^2)(s)|,$$

such that:

$$(N_n^1)(s) = \frac{A(s, u_n(s)) g(s, u_n(s))}{f^2(s, u_n(s))} - \frac{A(s, u(s)) g(s, u(s))}{f^2(s, u(s))}$$

and

$$(N_n^2)(s) = \frac{B(s, u_n(s)) \exp(M_n(s))}{\pi i f(s, u_n(s))} \int_\Gamma \left(\frac{g(\upsilon, u_n(\upsilon))}{\mu(\upsilon, u_n(\upsilon))} \right) \left(\frac{1}{\upsilon - s} \right) d\upsilon -$$

$$- \frac{B(s, u(s)) \exp(M(s))}{\pi i f(s, u(s))} \int_\Gamma \left(\frac{g(\upsilon, u(\upsilon))}{\mu(\upsilon, u(\upsilon))} \right) \left(\frac{1}{\upsilon - s} \right) d\upsilon.$$

Now, we show that: $\lim\limits_{n \to \infty} |v_n(s) - v(s)| = 0.$

Since $\|u_n\|$ is uniformly bounded, using the conditions (1.3)–(1.6), we have

$$\lim_{n \to \infty} |A(s, u_n(s)) - A(s, u(s))| = 0,$$

$$\lim_{n \to \infty} |B(s, u_n(s)) - B(s, u(s))| = 0$$

and

$$\lim_{n \to \infty} |g(s, u_n(s)) - g(s, u(s))| = 0.$$

Therefore,

$$\lim_{n \to \infty} |(N_n^1)(s)| = 0. \tag{3.4}$$

In the following , we estimate $(N_n^2)(s)$. To carry out our investigation we consider

$$|I_n| = |M(s, u_n(s)) - M(s, u(s))| :$$

$$|I_n| = |M(s, u_n(s)) - M(s, u(s))| \le \frac{1}{2}|(\ln C(s, u_n(s)) - \ln C(s, u(s)))| +$$

$$(3.5)\quad + \left| \frac{1}{2\pi i} \int_\Gamma \left(\frac{\ln C(\tau, u_n(\tau)) - \ln C(s, u_n(s))}{\tau - s} - \frac{\ln C(\tau, u(\tau)) - \ln C(s, u(s))}{\tau - s} \right) d\tau \right|.$$

Since $A(s, u(s))$ and $B(s, u(s))$ are uniformly continuous, then

$$\lim_{n\to\infty} |C(s, u_n(s)) - C(s, u(s))| = 0,$$

therefore

$$(3.6)\qquad \lim_{n\to\infty} \left| \sqrt{C(s, u_n(s))} - \sqrt{C(s, u(s))} \right| = 0,$$

$$(3.7)\qquad \lim_{n\to\infty} |\ln C(s, u_n(s)) - \ln C(s, u(s))| = 0,$$

and

$$\lim_{n\to\infty} |f(s, u_n(s)) - f(s, u(s))| = 0.$$

The difference in the right hand side of the inequality (3.5) tends uniformly to zero as n tends to ∞.

To handle the integral in the same inequality (3.5), we draw a circle of center z and radius ω, so small that inside the circle lies the single arc of the curve Γ.

We decompose this integral into two parts $I_n^l(s) + I_n^{\Gamma - l}(s)$ where l is the the part of Γ inside the circle and $\Gamma - l$ is the remaining part.

By using equation (2.5) and condition (1.3), we have:

$$|C(s, u(s))| = \left| \frac{(A(s, u(s)) + B(s, u(s)))^2}{A^2(s, u(s)) - B^2(s, u(s))} \right| \le (A(s, u(s)) + B(s, u(s)))^2$$

Due to the continuity of the functions $A(s, u(s))$, $B(s, u(s))$ and from the inequalities (1.4), (1.6), we obtain:

$$|C(\tau, u_n(\tau)) - C(s, u_n(s))| \le \tilde{\gamma}(1 + \alpha_u)|\tau - s|^\delta,$$

where $\tilde{\gamma} = 2(\eta_1 + \eta_3)(\gamma_1 + \gamma_2)$.

Therefore, we get the following inequality

$$\left| \frac{1}{2\pi i} \int_l \frac{\ln C(\tau, u_n(\tau)) - \ln C(s, u_n(s))}{\tau - s} d\tau \right| \le \tilde{\gamma}(1 + \alpha_u) \int_l |\tau - s|^{\delta - 1} |d\tau|$$

$$\le 2\tilde{\gamma} m(1 + \alpha_u) \int_0^\omega t^{\delta - 1} dt < \frac{\varepsilon}{3},$$

$$\varepsilon \equiv \text{ small positive number},$$

where

$$t = |\tau - s|, \quad |d\tau| = mdt; \qquad m \equiv \text{positive constant, (see [5])},$$

and

$$\omega \leq \left(\frac{\delta\varepsilon}{6\tilde{\gamma}m(1+\alpha_u)} \right)^{\delta^{-1}}.$$

Consequently,

$$\left| \frac{1}{2\pi i} \int_l \frac{\ln C(\tau, u(\tau)) - \ln C(s, u(s))}{\tau - s} d\tau \right| < \frac{\varepsilon}{3}.$$

Then

$$|I_n^l(s)| < \frac{2\varepsilon}{3}.$$

Since the point $s \in l$ lies outside the arc of integration $\Gamma - l$ and $I_n^{\Gamma-l}(s)$ are continuous, then we can select a positive integer J_ε such that, for each s, the inequality

$$\left| I_n^{\Gamma-l}(s) \right| < \frac{\varepsilon}{3},$$

holds for each $n > J_\varepsilon$. Hence it follows that:

$$|I_n| < \varepsilon \quad \text{for} \quad n > J_\varepsilon$$

and

$$\lim_{n\to\infty} |M(s, u_n(s)) - M(s, u(s))| = 0.$$

Therefore,

$$\lim_{n\to\infty} |\exp M(s, u_n(s)) - \exp M(s, u(s))| = 0. \tag{3.8}$$

From (2.7), (3.6) and (3.8), we obtain:

$$\lim_{n\to\infty} |X_n^+(s) - X^+(s)| = 0, \tag{3.9}$$

therefore,

$$\lim_{n\to\infty} |\mu(s, u_n(s)) - \mu(s, u(s))| = 0. \tag{3.10}$$

By using the properties of the convergent sequence, we have:

$$\lim_{n\to\infty} |(N_n^2)(s)| = 0. \tag{3.11}$$

Hence , by using (3.4) and (3.11), we obtain:

$$\lim_{n\to\infty} |v_n(s) - v(s)| = 0.$$

Then, the operator S is continuous. Lemma 3.2 is proved. □

Proof of Theorem 1.1. From the preceding lemmas and Arzela-Ascoli theorem [9, 15], the image of $K^{0,\delta}_{R,\alpha}$ is compact. Therefore, we can use Schauder's fixed point theorem to show that the operator S has at least one fixed point. Our main result (Theorem 1.1.) is proved. □

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