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SOLVABILITY OF AN INFINITE SYSTEM OF SINGULAR INTEGRAL EQUATIONS

Mahmoud M. El Borai, Mohamed I. Abbas

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Abstract. Schauder's fixed point theorem is used to establish an existence result for an infinite system of singular integral equations in the form:

$$x_i(t) = a_i(t) + \int_0^t (t-s)^{-\alpha} f_i(s, x_1(s), x_2(s), \dots) \, ds, \tag{1}$$

where $i = 1, 2, \dots$, $\alpha \in (0,1)$ and $t \in I = [0,T]$.

The result obtained is applied to show the solvability of an infinite system of differential equation of fractional orders.

1. Introduction. The theory of infinite systems of integral equations is an important branch of nonlinear analysis. In fact, infinite systems of integral equations are natural generalizations of infinite systems of differential equations of fractional order and are applied to numerous real world problems (cf. [1]).

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On the other hand, infinite systems of integral equations are particular cases of integral equations in Banach spaces which have been considered in many research papers, cf. [4], [5]. For infinite systems of integral equations in abstract Banach spaces, see also [2], [6], [13], [16] and the references therein, where infinite systems of singular integral equations appear in the study of semilinear parabolic and dissipative PDEs with strongly singular initial data and Lipschitz nonlinear terms in the framework of L^p type weighted spaces.

The aim of the present paper is to formulate and prove an existence result of solutions of a class of infinite systems of singular integral equations which allows us to study the solvability of infinite systems of differential equations of fractional orders as an application.

We shall prove the existence of solutions $x = (x_1, x_2, \dots)$ such that $x(t) \in c_0$ for every $t \in I$. Here c_0 is the Banach sequence space consisting of real sequences converging to zero.

In fact, our results in this paper are motivated by the extensions of the work of J. Banaś and M. Lecko, see [5].

2. Notation and auxiliary facts. In this section, we collect a few auxiliary facts which will be needed further on.

Let E be a real infinite-dimensional Banach space with the norm $\|.\|_E$. For a given interval $I = [0, T]$, denote by $C = C(I, E)$ the space of all continuous functions defined on I and taking values in the space E. The norm in the space C is defined in the standard way, i.e.,

$$\|x\|_C = \max\{\|x(t)\|_E : t \in I\}.$$

If X is a set in $C(I, E)$, then for a fixed $t \in I$, we denote by $X(t)$ the following set in E:

$$X(t) = \{x(t) : x \in X\}.$$

Let us point out that under the hypotheses of the Schauder fixed point theorem some kind of compactness assumptions are required. In infinite-dimensional Banach spaces we should not expect such assumptions to be verified easily. Compactness in infinite-dimensional space is always provided by a deep Theorem (see [9]).

In the sequel we shall use the generalized theorem of Arzelà (see [9], for example) which gives a criterion of compactness in the space $C(I, E)$.

Theorem 2.1. *A bounded subset X of the space $C(I, E)$ is relatively compact if and only if all functions belonging to X are equicontinous on I and the set $X(t)$ is relatively compact in E for each $t \in I$.*

Theorem 2.2 (Schauder's Fixed Point Theorem)**.** *Let K be a closed, bounded and convex subset of a Banach space E and A be a completely continuous mapping of K into itself. Then A has at least one fixed point in K.*

In the sequel, we shall work in the Banach space c_0 consisting of real sequences converging to zero with the standard norm:

$$\|x\|_{c_0} = \max\{|x_i| : i = 1, 2, \dots\}$$

for $x = (x_1, x_2, \dots)$.

Let us recall [9] that a bounded subset X of c_0 is relatively compact if and only if

$$\lim_{i\to\infty}\left[\sup_{x\in X}\left[\max\{|x_k| : k \geq i\}\right]\right] = 0.$$

3. Main result. In this section, system (1) will be investigated under the following set of hypotheses:

(i) The operator f defined on the space $I \times c_0$ in the following way:

$$(t, x) \to (fx)(t) = (f_1(t, x), f_2(t, x), \dots)$$

transforms the space $I \times c_0$ into c_0 and is such that the family of functions $\{(fx)(t)\}_{t\in I}$ is equicontinuous at every point of the space c_0, i.e., for any $x_0 \in c_0$ and for any arbitrarily fixed $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|(fx)(t) - (fx_0)(t)\|_{c_0} \leq \varepsilon,$$

for each $t \in I$ and for each $x \in c_0$ such that $\|x - x_0\|_{c_0} \leq \delta$ (cf. [15]).

(ii) There exist nonnegative functions $\beta_i(t)$ and $\gamma_i(t)$ defined, integrable and uniformly bounded on I. Moreover, the functions $\beta_i(t)$ are continuous and converge monotonically to zero and the function sequence $\{\gamma_i(t)\}_{i=1}^{\infty}$ is nondecreasing at each $t \in I$ and the following estimate is satisfied:

$$|f_i(t, x_1, x_2, \dots)| \leq \beta_i(t) + \gamma_i(t) . \sup\{|x_k| : k \geq i\},$$

for each $t \in I, i \in \mathbb{N}$ and for each $x = (x_i) \in c_0$.

(iii) The functions $a_i : I \to R$ are continuous on I and the sequence $(|a_i(t)|)$ converges monotonically to zero at each point $t \in I$.

Remark 3.1. Observe that the functions $\beta_i(t)$ and $|a_i(t)|$, $(i = 1, 2, \dots)$, appearing in assumptions (ii) and (iii) are continuous on I, so in view of Dini Theorem [9], the function sequences $\{\beta_i(t)\}_{i=1}^{\infty}$ and $(|a_i(t)|)$ converge uniformly on I to the function vanishing identically on I.

We define the number $T_{cr} > 0$ by

$$\frac{T_{cr}^{1-\alpha}}{1-\alpha}\sup\{\gamma_i(t) : t \in I, i \in \mathbb{N}\} = 1$$

with the convention $T_{cr} = +\infty$ if $\gamma_i \equiv 0, i \in \mathbb{N}$, i.e.,

$$T_{cr} = \left(\frac{1-\alpha}{\sup \{\gamma_i (t) : t \in I, i \in \mathbb{N}\}} \right)^{\frac{1}{1-\alpha}}.$$

Now, we can formulate our main result.

Theorem 3.1. *Suppose that assumptions* $(i)-(iii)$ *Then, the infinite system (1) has at least one solution* $x(t) = (x_i(t))$ *if* $T < T_{cr}$ *such that* $x(t) \in c_0$ *for each* $t \in I$ *and* $\alpha \in (0,1)$.

Proof. Let X_0 be the subset of the space $C = C(I, c_0)$ consisting of all functions $x(t) = (x_i(t))$ such that:

$$\sup \{|x_k(t)| : k \geq i\} \leq u_i(t) + v_i(t),$$

for $i = 1, 2, \dots$ and $t \in I$, where $u_i(t)$ and $v_i(t)$ are defined in the following way:

$$u_i(t) = \frac{\frac{T^{1-\alpha}}{1-\alpha} . \sup \{\beta_i(t) : t \in I\}}{\left[1 - \frac{T^{1-\alpha}}{1-\alpha} . \sup \{\gamma_i(t) : t \in I\}\right]},$$

$$v_i(t) = \frac{\sup \{|a_i(s)| : 0 \leq s \leq t\}}{\left[1 - \frac{T^{1-\alpha}}{1-\alpha} . \sup \{\gamma_i(t) : t \in I\}\right]},$$

for each $i = 1, 2, \dots$.

Observe that the functions $v_i(t)$ and $v_i(t)$ are nondecreasing on the interval I. Moreover, from the assumptions, it follows that the function sequence $(v_i(t))$ and $(v_i(t))$ converge uniformly on I to the function vanishing identically on I (cf. Remark 3.1).

Further, let us consider the operator F defined on the space $C(I, c_0)$ as follows:

$$(Fx)(t) = ((Fx)_i(t)) = \left(a_i(t) + \int_0^t (t-s)^{-\alpha} f_i(s, x_1(s), x_2(s), \dots) \, ds \right).$$

The operator F maps the set X_0 into itself. In fact, fix arbitrarily i and

$x_0 \in X$, then for $k \geq i$, we have

$$\begin{aligned}
|(Fx)_k(t)| &\leq |a_k(t)| + \left| \int_0^t (t-s)^{-\alpha} f_k\left(s, x_1(s), x_2(s), \dots\right) ds \right| \\
&\leq |a_i(t)| + \int_0^t (t-s)^{-\alpha} \left[\beta_k(s) + \gamma_k(s) . \sup\{|x_n(s)| : n \geq k\}\right] ds \\
&\leq |a_i(t)| + \int_0^t (t-s)^{-\alpha} \left[\beta_k(s) + \gamma_k(s) . (u_k(s) + v_k(s))\right] ds \\
&\leq |a_i(t)| + \int_0^t (t-s)^{-\alpha} \left[\beta_i(s) + \gamma_i(s) . (u_i(s) + v_i(s))\right] ds \\
&\leq \sup\{|a_i(s)| : 0 \leq s \leq t\} \\
&\quad + \frac{T^{1-\alpha}}{1-\alpha} \left[\sup\{\beta_i(t) : t \in I\} + (u_i(t) + v_i(t)) . \sup\{\gamma_i(t) : t \in I\}\right] \\
&\leq u_i(t) + v_i(t),
\end{aligned}$$

which implies that F maps X_0 into itself.

Now, we show that the operator F is continuous on the set X_0.

Let us fix arbitrarily $\varepsilon > 0$ and $x_0 \in X_0$. Next, choose $\delta = \delta(x_0, \varepsilon)$ according to assumption(i), i.e., for $x \in X_0$ such that $\|x - x_0\|_{c_0} \leq \delta$, we have

$$\|(fx)(t) - (fx_0)(t)\|_{c_0} \leq \varepsilon,$$

for any $t \in I$.

Then we obtain:

$$\begin{aligned}
\|(Fx)(t) - (Fx_0)(t)\|_{c_0} &= \max\left\{\left|(Fx)_i(t) - (Fx_0)_i(t)\right| : i = 1, 2, \dots\right\} \\
&\leq \max\left\{\int_0^t (t-s)^{-\alpha} \left| f_i\left(s, x_1(s), x_2(s), \dots\right)\right.\right. \\
&- \left.\left. f_i\left(s, x_1^0(s), x_2^0(s), \dots\right)\right| ds : i = 1, 2, \dots\right\} \\
&\leq \frac{T^{1-\alpha}}{1-\alpha} \varepsilon,
\end{aligned}$$

which gives the desired assertion.

Next, let us consider the set $X_1 = FX_0$. Notice that this set consists of equicontinuous functions on I.

Indeed, taking an arbitrarily $x = (x_i) \in X_0$ and keeping in mind our assumptions, we obtain:

$$
\begin{aligned}
&\left|(Fx)_i(t_2) - (Fx)_i(t_1)\right| \\
&\leq \left|a_i(t_2) - a_i(t_1)\right| + \left|\int_0^{t_2} (t_2 - s)^{-\alpha} f_i(s, x_1(s), x_2(s), \dots)\, ds \right. \\
&\quad \left. - \int_0^{t_1} (t_1 - s)^{-\alpha} f_i(s, x_1(s), x_2(s), \dots)\, ds\right| \\
&\leq \left|a_i(t_2) - a_i(t_1)\right| + \left|\int_0^{t_1} \left[(t_2 - s)^{-\alpha} - (t_1 - s)^{-\alpha}\right] f_i(s, x_1(s), x_2(s), \dots)\, ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} (t_2 - s)^{-\alpha} f_i(s, x_1(s), x_2(s), \dots)\, ds\right| \\
&\leq \left|a_i(t_2) - a_i(t_1)\right| + \left\{\int_0^{t_1} \left|(t_2 - s)^{-\alpha} - (t_1 - s)^{-\alpha}\right| ds + \int_{t_1}^{t_2} (t_2 - s)^{-\alpha}\, ds\right\} \\
&\quad \times \left[\sup\{\beta_i(t) : t \in I\} + \sup\{(u_i(t) + v_i(t))\gamma_i(t)\} : t \in I\right] \\
&\leq \left|a_i(t_2) - a_i(t_1)\right| \\
&\quad + \frac{1}{1-\alpha}\left\{2|t_2 - t_1|^{1-\alpha} + \left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|\right\} \\
&\quad \times \left[\sup\{\beta_i(t) : t \in I\} + \sup\{(u_i(t) + v_i(t))\gamma_i(t)\} : t \in I\right].
\end{aligned}
$$

Taking into account that the function sequences $(\beta_i(t)), (\gamma_i(t)), (u_i(t))$, and $(v_i(t))$ are uniformly bounded on I and the function sequence $(a_i(t))$ is equicontinuous on I, from the above estimate, we deduce that the set $X_1 = FX_0$ is equicontinuous on I (notice that, the last inequality is valid for all $t_1, t_2 \in I$ regardless whether or not $t_1 \leq t_2$).

Now, let us consider the set $Y = Conv\, X_1$ (i.e., the closed convex hull of the set X_1). Obviously, Y is closed, bounded and equicontinuous on I. Moreover, $Y \subset X_0$ and $FY \subset Y$.

On the other hand, for $x \in X_0$, we have

$$\left|(Fx)_i(t)\right| \leq u_i(t) + v_i(t)$$

$(i = 1, 2, \dots, t \in I)$. Since the sequence $(u_i(t) + v_i(t))$ converges uniformly on I to the function vanishing identically on I, we infer that for each $\varepsilon > 0$, there exists a natural number n_0 such that:

$$\left|(Fx)_i(t)\right| \leq \varepsilon,$$

for each $i \geq n_0$ and for each $t \in I$.

Hence, by virtue of the criterion of compactness in the space c_0 (cf. Section 2), we deduce that for any $t \in I$, the set $X_1(t)$ is relatively compact in the space c_0. The above statements allow us to conclude that the set Y is relatively compact in the space $C(I, c_0)$ (cf. Theorem 2.1). Moreover, the closedness of Y implies that it is compact. Hence, keeping in mind that F maps continuously the set Y into itself, we infer (by Schauder's fixed point theorem) that the operator F has a fixed point in the set Y being a solution of our problem. Thus, the proof is completed. $\square$

Remark 3.2. One can generalize Theorem 2.1 replacing $(t-s)^{1-\alpha}$ with $\psi(t-s)$, where $\psi \in L^1(0,T)$ and $\frac{T^{1-\alpha}}{1-\alpha} \sup\{\gamma_i(t) : t \in I, i \in \mathbb{N}\} < 1$ with $\left(\int_0^T |\psi(r)|dr\right) \sup\{\gamma_i(t) : t \in I, i \in \mathbb{N}\} < 1$. Moreover, one can extend Theorem 2.1 in the framework of the spaces of sequences of C^k maps $C^k(I,E)$, provided $\alpha < 1-k$.

4. An application. In this section, we use the existence result in the previous section to study the solvability of infinite systems of differential equations of fractional order of the form:

$$\frac{d^\theta x_i(t)}{dt^\theta} = f_i(t, x_1(t), x_2(t), \dots) \tag{2}$$

with the initial condition

$$x_i(0) = x_i^0 \tag{3}$$

where $i = 1, 2, \dots$ and $\theta \in (0,1]$.

The most widely known definition of the fractional derivative and fractional integral is called the *Riemann-Liouville* definition (see [1], [14], [17]).

Definition 4.1. *Let $\phi \in L_1(a,b)$, $0 \le a < b < \infty$, and let $\theta > 0$. The Riemann-Liouville fractional derivative and fractional integral of order θ are given by*

$$\begin{aligned} D^\theta \phi(t) &= \frac{1}{\Gamma(k-\theta)} \frac{d^k}{dt^k} \int_0^t \frac{\phi(s)}{(t-s)^{1-k+\theta}} ds, \qquad (k-1 \le \theta < k) \\ D^{-\theta} \phi(t) &= \frac{1}{\Gamma(\theta)} \int_0^t \frac{\phi(s)}{(t-s)^{1-\theta}} ds, \end{aligned}$$

respectively.

We also mention that other type of fractional derivatives appear in the study of nonlocal equations related to PDEs appearing in applied sciences (e.g., see [3], [7], [8] and the references therein).

The most important property of the *Riemann-Liouville* fractional derivative is that for $\theta > 0$

$$D^{\theta}\left(D^{-\theta}\phi\left(t\right)\right) = \phi\left(t\right),$$

and in general, for $p, q > 0$, we get

$$D^{p}\left(D^{q}\phi\left(t\right)\right) = D^{p+q}\phi\left(t\right).$$

According to these facts, we can easily show that the system(2), (3) is equivalent to the following system of integral equations:

$$x_i\left(t\right) = x_i^0 + \frac{1}{\Gamma\left(\theta\right)}\int_0^t \left(t-s\right)^{\theta-1} f_i\left(t, x_1\left(s\right), x_2\left(s\right), \dots\right) ds. \tag{4}$$

We define the number $M_{cr} > 0$ by

$$\frac{M_{cr}^{\theta}}{\Gamma(\theta+1)} \sup\left\{\gamma_i\left(t\right) : t \in I, i \in \mathbb{N}\right\} = 1$$

with the convention $M_{cr} = +\infty$ if $\gamma_i \equiv 0, i \in \mathbb{N}$, i.e.,

$$M_{cr} = \left(\frac{\Gamma(\theta+1)}{\sup\left\{\gamma_i\left(t\right) : t \in I, i \in \mathbb{N}\right\}}\right)^{\frac{1}{\theta}}.$$

Proposition 4.1. *Assume that the hypotheses* $(i)-(iii)$ *are satisfied. Then the infinite system* (4) *has at least one solution* $x(t) = (x_i(t))$ *if* $T < M_{cr}$ *such that* $x\left(t\right) \in c_0$ *for each* $t \in I$ *and* $\theta \in (0, 1]$.

Let us consider the operator F' defined on the space $C\left(I, c_0\right)$ as follows:

$$\left(F'x\right)\left(t\right) = \left(\left(F'x\right)_i\left(t\right)\right) = \left(x_i^0 + \frac{1}{\Gamma\left(\theta\right)}\int_0^t \left(t-s\right)^{\theta-1} f_i\left(s, x_1\left(s\right), x_2\left(s\right), \dots\right) ds\right).$$

The operator F' maps the set X_0 into itself. In fact, fix arbitrarily i and $x_0 \in X$, then for $k \geq i$, we have

$$
\begin{aligned}
\left|\left(F'x\right)_k(t)\right| &\le \left|x_k^0\right| + \left|\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} f_k\left(s, x_1(s), x_2(s), \dots\right) ds\right| \\
&\le \left|x_k^0\right| + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left[\beta_k(s) + \gamma_k(s).\sup\left\{|x_n(s)| : n \ge k\right\}\right] ds \\
&\le \left|x_k^0\right| + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left[\beta_k(s) + \gamma_k(s).\left(u_k'(s) + v_k'(s)\right)\right] ds \\
&\le \left|x_k^0\right| + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left[\beta_i(s) + \gamma_i(s).\left(u_i'(s) + v_i'(s)\right)\right] ds \\
&\le \max\left|x_i^0 : i \in \mathbb{N}\right| + \frac{T^\theta}{\Gamma(\theta+1)}.\left[\sup\left\{\beta_i(t) : t \in I\right\} + \left(u_i'(t) + v_i'(t)\right)\right. \\
&\quad \left.\times \sup\left\{\gamma_i(t) : t \in I\right\}\right] \\
&\le u_i'(t) + v_i'(t),
\end{aligned}
$$

where

$$
u_i'(t) = \frac{\frac{T^\theta}{\Gamma(\theta+1)}.\sup\left\{\beta_i(t) : t \in I\right\}}{\left[1 - \frac{T^\theta}{\Gamma(\theta+1)}.\sup\left\{\gamma_i(t) : t \in I\right\}\right]},
$$

$$
v_i'(t) = \frac{\max\left|x_i^0 : i \in \mathbb{N}\right|}{\left[1 - \frac{T^\theta}{\Gamma(\theta+1)}.\sup\left\{\gamma_i(t) : t \in I\right\}\right]},
$$

for each $i = 1, 2, \dots$.

Now, we show that the operator F' is continuous on the set X_0.

Let us fix arbitrarily $\varepsilon > 0$ and $x_0 \in X_0$. Next, choose $\delta = \delta(x_0, \varepsilon)$ according to assumption(i), i.e., for $x \in X_0$ such that $\|x - x_0\|_{c_0} \le \delta$, we have $\|(fx)(t) - (fx_0)(t)\|_{c_0} \le \varepsilon$, for any $t \in I$.

Then we obtain:

$$
\begin{aligned}
\left\|\left(F'x\right)(t) - \left(F'x_0\right)(t)\right\|_{c_0} &= \max\left\{\left|\left(F'x\right)_i(t) - \left(F'x_0\right)_i(t)\right| : i = 1, 2, \dots\right\} \\
&\le \max\left\{\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left|f_i\left(s, x_1(s), x_2(s), \dots\right)\right.\right. \\
&\quad \left.\left. - f_i\left(s, x_1^0(s), x_2^0(s), \dots\right)\right| ds : i = 1, 2, \dots\right\} \\
&\le \frac{T^\theta}{\Gamma(\theta+1)}\varepsilon,
\end{aligned}
$$

which gives the desired assertion.

Next, also as in Section 3, let us consider the set $X_1' = F'X_0$. Notice that this set consists of equicontinuous functions on I. Indeed, taking an arbitrarily $x = (x_i) \in X_0$ and keeping in mind our assumptions, we obtain:

$$
\begin{aligned}
&\left|(F'x)_i(t_2) - (F'x_0)_i(t_1)\right| \\
&\quad \le \left| \frac{1}{\Gamma(\theta)} \int_0^{t_2} (t_2-s)^{\theta-1} f_i(s, x_1(s), x_2(s), \dots)\, ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(\theta)} \int_0^{t_1} (t_1-s)^{\theta-1} f_i(s, x_1(s), x_2(s), \dots)\, ds \right| \\
&\quad \le \left| \frac{1}{\Gamma(\theta)} \int_0^{t_1} \left[(t_2-s)^{\theta-1} - (t_1-s)^{\theta-1}\right] f_i(s, x_1(s), x_2(s), \dots)\, ds \right| \\
&\qquad + \left| \frac{1}{\Gamma(\theta)} \int_{t_1}^{t_2} (t_2-s)^{\theta-1} f_i(s, x_1(s), x_2(s), \dots)\, ds \right| \\
&\quad \le \frac{1}{\Gamma(\theta)} \left\{ \int_0^{t_1} \left|(t_2-s)^{\theta-1} - (t_1-s)^{\theta-1}\right| ds + \int_{t_1}^{t_2} (t_2-s)^{\theta-1}\, ds \right\} \\
&\qquad \times \left[\sup\{\beta_i(t) : t \in I\} + \sup\left\{\left(u_i'(t) + v_i'(t)\right)\gamma_i(t)\right\} : t \in I\right] \\
&\quad \le \frac{1}{\Gamma(\theta+1)} \left\{ 2|t_2 - t_1|^\theta + \left|t_2^\theta - t_1^\theta\right| \right\} \\
&\qquad \times \left[\sup\{\beta_i(t) : t \in I\} + \sup\left\{\left(u_i'(t) + v_i'(t)\right)\gamma_i(t)\right\} : t \in I\right].
\end{aligned}
$$

Taking into account that the function sequences $(\beta_i(t)), (\gamma_i(t)), (u_i'(t))$, and $(v_i'(t))$ are uniformly bounded on I , from the above estimate, we deduce that the set $X_1' = F'X_0$ is equicontinuous on I.

Now, let us consider the set $Y' = ConvX_1'$ (i.e., the closed convex hull of the set X_1'). Obviously, Y' is closed, bounded and equicontinuous on I. Moreover, $Y' \subset X_0$ and $F'Y' \subset Y'$.

On the other hand, for $x \in X_0$, we have

$$\left|(F'x)_i(t)\right| \le u_i'(t) + v_i'(t),$$

$(i = 1, 2, \dots, t \in I)$. Since the sequence $(u_i'(t) + v_i'(t))$ converges uniformly on I to the function vanishing identically on I, we infer that for each $\varepsilon > 0$, there exists a natural number n_0 such that:

$$\left|(F'x)_i(t)\right| \le \varepsilon,$$

for each $i \ge n_0$ and for each $t \in I$.

Hence, by virtue of the criterion of compactness in the space c_0 (cf. Section 2), we deduce that for any $t \in I$, the set $X_1'(t)$ is relatively compact in the space c_0. The above statements allow us to conclude that the set $Y^{'}$ is relatively compact in the space $C(I, c_0)$ (cf. Theorem 2.1). Moreover, the closedness of Y' implies that it is compact. Hence,keeping in mind that F' maps continuously the set Y' into itself, we infer (by Schauder's fixed point theorem) that the operator F' has a fixed point in the set Y' being a solution of our problem.

Thus, the proof is completed. □

Remark 4.1. The result for the solvability of (4) can be extended for arbitrary fractional orders by assigning initial conditions $D_r^t x_i(0) = \xi_i^r$ for $r = 0, 1, \ldots, [\theta]$, where $k := [\theta]$ stands for the integer part of θ, since we are reduced to

$$x_i(t) = \sum_{r=0}^{[\theta]} \frac{t^r}{r!}\xi_i^r + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f_i(t, x_1(s), x_2(s), \ldots)\, ds.$$

In that case the solutions are in $C^k(I, E)$, $k = [\theta]$.

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Department of Mathematics
Faculty of Science
Alexandria University
Alexandria, Egypt
e-mail: m_m_elborai@yahoo.com
e-mail: m_i_abbas77@yahoo.com

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