Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **33** (2007), 125–162

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON SOME EXTREMAL PROBLEMS OF LANDAU

Szilárd Révész

Communicated by V. Drensky

Dedicated to the memory of Lubomir Tschakaloff, 1886–1963

Abstract. The prime number theorem with error term presents itself as $\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(xe^{-K\log^L x}\right)$. In 1909, Edmund Landau provided a systematic analysis of the proof seeking better values of L and K. At a key point of his 1899 proof de la Vallée Poussin made use of the nonnegative trigonometric polynomial $\frac{2}{3}(1+\cos x)^2 = 1 + \frac{4}{3}\cos x + \frac{1}{3}\cos 2x$. Landau considered more general positive definite nonnegative cosine polynomials $1 + a_1\cos x + \cdots + a_n\cos nx \geq 0$, with $a_1 > 1, a_k \geq 0$ $(k = 1, \ldots, n)$, and deduced the above error term with $L = 1/2$ and any $K < 1/\sqrt{2V(\mathbf{a})}$,

2000 *Mathematics Subject Classification:* Primary: 42A05. Secondary: 42A82, 11N05.

Key words: Prime number formula, positive trigonometric polynomials, positive definite functions, extremal problems, Borel measures, convexity, duality.

Supported in part in the framework of the Hungarian-French Scientific and Technological Governmental Cooperation, Project # F-10/04.

The author was supported in part by the Hungarian National Foundation for Scientific Research, Project #s T-049301, T-049693 and K-61908.

This work was accomplished during the author's stay in Paris under his Marie Curie fellowship, contract # MEIF-CT-2005-022927.

where $V(\mathbf{a}) := \dfrac{a_1 + a_2 + \dots a_n}{(\sqrt{a_1} - 1)^2}$. Thus the extremal problem of finding $V := \min V(\mathbf{a})$ over all admissible coefficients, i.e. polynomials, arises.

The question was further studied by Landau and later on by many other eminent mathematicians. The present work surveys these works as well as current questions and ramifications of the theme, starting with a long unnoticed, but rather valuable Bulgarian publication of Lubomir Chakalov.

1. Introduction

1.1. In his famous book "Handbuch der Lehre von der Verteilung der Primzahlen" [15] Edmund Landau provided a systematic analysis of the proof of the prime number theorem developed by de la Vallée Poussin [26] ten years before the appearance of the monograph. The prime number formula with error term presents itself as

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\big(x \cdot \exp(-K \log^L x)\big), \tag{1.1}$$

and Landau sought better values of L an K, which can be deduced using the method of de la Vallée Poussin. A key point of the original proof is the application of the nonnegative trigonometric polynomial

$$\frac{2}{3}(1+\cos x)^2 = 1 + \frac{4}{3}\cos x + \frac{1}{3}\cos 2x. \tag{1.2}$$

Landau's idea was to improve upon the error term by finding even better polynomials in place of (1.2). Considering positive definite nonnegative cosine polynomials

$$f(x) = 1 + a_1 \cos x + \dots + a_n \cos nx \ge 0, \qquad a_k \ge 0 \quad (k = 1, \dots, n) \tag{1.3}$$

he proved in §65 of his book that (1.1) holds true with $K = 1$ and for any L satisfying

$$L < \frac{1}{U+2} \tag{1.4}$$

where

$$U = \inf\left\{ \frac{f(0)}{a_1 - 1} : f \in \mathcal{T} \quad \text{satisfies (1.3) with } a_1 > 1 \right\}. \tag{1.5}$$

Here and elsewhere in the paper $\mathcal{T}$ stands for the set of all trigonometric polynomials periodic by 2π, and $\mathcal{T}_n$ denotes the set of polynomials from $\mathcal{T}$ having degree not exceeding n.

We also use $C(\mathbb{T})$ for the set of all continuous functions on the one dimensional torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (i.e. the circle group) which is identical to the set of continuous 2π-periodic functions on $\mathbb{R}$. Note that $\mathbb{T}$ is compact.

Landau realized that his first proof, although shorter and more direct, did not provide $L = 1/2$ proved by de la Vallée Poussin, as Landau himself found that $5 < U < 6$. In his second proof described in §79 of [15] he deduced for the Riemann ζ function that

$$\zeta(s) \neq 0 \quad \text{if} \quad s = \sigma + it, \quad \sigma \geq 1 - \frac{1}{R \log t}, \quad t \geq 2 \tag{1.6}$$

with R satisfying

$$R > \frac{1}{2} V, \tag{1.7}$$

where

$$V = \inf \left\{ \frac{f(0) - 1}{\left(\sqrt{a_1} - 1\right)^2} : f \in \mathcal{T} \quad \text{satisfies (1.3) with } a_1 > 1 \right\}. \tag{1.8}$$

In §80 Landau finishes his argument proving that (1.6) entails (1.1) with $L = 1/2$ and $K < 1/\sqrt{R}$ for all $x > x_0(K)$. Now this proof gives as good an error term as de la Vallée Poussin's, and even better with respect to K. Note that later improvements pushed L up to $\frac{3}{5} - \varepsilon$, cf. e.g. [6, p. 111], [29] but for practical applications, in particular for computational number theory, the theoretically better asymptotic results all have a defect with respect to the O-constant (and/or the validity range $x > x_0$). Thus Landau's method is still interesting for practical applications, cf. e.g. [22].

1.2. The present work is aimed to deal with the extremal quantities (1.5) and (1.8). As attracting many eminent mathematicians, the determination, or estimation of U and V became a well-known problem independently of its number theoretic applications. To give a historical account of results thus far, let us introduce a more systematic notation. We put for any $a \in \mathbb{R}$

$$\mathcal{F}(a) := \Big\{ f \in C(\mathbb{T}) : f(x) = 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx \geq 0 \quad (\forall x), \qquad a_k \geq 0 \quad (k \in \mathbb{N}) \Big\} \tag{1.9}$$

and denote

$$\begin{gathered} \mathcal{F}_n(a) := \mathcal{F}(a) \cap \mathcal{T}_n, \quad \mathcal{F}^*(a) := \mathcal{F}(a) \cap \mathcal{T}, \quad \mathcal{F} = \bigcup_{a>1} \mathcal{F}(a), \\ \mathcal{F}_n := \bigcup_{a>1} \mathcal{F}_n(a) = \mathcal{F} \cap \mathcal{T}_n, \quad \mathcal{F}^* := \bigcup_{a>1} \mathcal{F}^*(a) = \mathcal{F} \cap \mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{F}_n. \end{gathered} \tag{1.10}$$

One can define

$$\begin{aligned} \alpha(a) &:= \inf \big\{ f(0) : f \in \mathcal{F}(a) \big\} \\ \alpha^*(a) &:= \inf \big\{ f(0) : f \in \mathcal{F}^*(a) \big\} \\ \alpha_n(a) &:= \inf \big\{ f(0) : f \in \mathcal{F}_n(a) \big\}. \end{aligned} \tag{1.11}$$

Note that the definitions (1.11) can be used whenever $\mathcal{F}(a) \neq \emptyset$, $\mathcal{F}^*(a) \neq \emptyset$ or $\mathcal{F}_n(a) \neq \emptyset$, resp. It is easy to see that $\alpha(a) = \alpha^*(a)$ for all $a \in \mathcal{D}(\alpha)$ except possibly for the point A at the left end of the domain of α where $\mathcal{F}^*(A)$ may be empty. It is also easy to see that $[1,2) \subset \mathcal{D}(\alpha)$, $\alpha(a)$ is continuous in $[1,2)$, and that $\mathcal{F}(a) = \emptyset$ for $a \geq 2$; moreover, $\alpha(a) \to +\infty$ as $a \to 2-0$. Finally the infimum in the definition of $\alpha(a)$ is actually a minimum,

$$\begin{aligned} \alpha(a) &= \min \big\{ f(0) : f \in \mathcal{F}(a) \big\} \\ \alpha_n(a) &= \min \big\{ f(0) : f \in \mathcal{F}_n(a) \big\}. \end{aligned} \tag{1.12}$$

These observations can be found in [19] in a more general setting. However, we have to note that most of the facts mentioned here appeared first in [2] where $\chi(a) = \alpha(a) - 1$ and $\chi_n(a) = \alpha_n(a) - 1$ are defined and analyzed (for $a \geq 1$). This analysis is continued (for $a \geq 0$) in [1].

Thus we can omit α^* and $\mathcal{F}^*$ from now on using only α and $\mathcal{F}$ in place of the original, equivalent usage of Landau. With these cleared we can also put

$$\begin{aligned} U &:= \min_{a>1} \frac{\alpha(a)}{a-1}, & U_n &:= \min_{a>1} \frac{\alpha_n(a)}{a-1}, \\ V &:= \min_{a>1} \frac{\alpha(a)-1}{(\sqrt{a}-1)^2}, & V_n &:= \min_{a>1} \frac{\alpha_n(a)-1}{(\sqrt{a}-1)^2}, \end{aligned} \tag{1.13}$$

where the use of min in place of inf is justified later.

Plainly for all a we have $\alpha_n(a) \searrow \alpha(a)$ $(n \to \infty)$ and $U_n \searrow U$, $V_n \searrow V$ $(n \to \infty)$. (Here $\searrow$ means monotonically nonincreasing convergence.) Below is a list of values already determined.

	$U_2 = 7$	Landau, [15] & Chakalov, [4, 5]
	$U_3 = U_4 = U_5 = 6$	Landau, [16, 17] & Chakalov, [4, 5]
	$U_6 = 5.92983\ldots$	Chakalov, [4, 5]
(1.14)	$U_7 = U_8 = U_9 = 5.90529\ldots$	Chakalov, [4, 5]
	$V_2 = 53.1390719\ldots$	French, [11]
	$V_3 = 36.9199911\ldots$	Arestov, [1]
	$V_4 = V_5 = V_6 = 34.8992258\ldots$	Arestov, [1].

Estimates were also deduced for many of the extremal quantities. It follows a list of records to date in estimating these values.

	$U < 5.90529\ldots$	Chakalov, [4, 5]
	$U > 5.8726$	Arestov-Kondrat'ev, [2]
(1.15)	$V < 34.5035864\ldots$	Arestov-Kondrat'ev, [2]
	$V > 34.468305\ldots$	Arestov-Kondrat'ev, [2]
	$V_8 < 34.54461566$	Kondrat'ev, [13].

For historical completeness let us mention a few other results, already improved upon.

$V < 35.074$	Westphal, [28]
$V_2 \le 53.15$	Stechkin, [25]
$U_{11} > 5.792$	Chakalov, [4, 5]
$V_3 \le 37.04$	Landau, [15]
$V_3 < 36.97$	Stechkin, [25]
$V_4 < 35.03264$	Rosser and Schönfeld, [22]

$V > 21.64$	French, [11] referring to an unpublished result of Schoenfeld & V. J. Le Veque
$V > 32.49$	Stechkin, [25]
$U > 5.8642$	B. L. van der Waerden, [27]
$V > 32.5136$	French, [11]
$36.96 > V_3 > 36.59$	Bateman (unpublished, quoted in [11]) and [25], resp.
$34.91 > V_4 > 34.35$	Stechkin, [25]
$V_{5,6,7} > 33.373$	French, [11]
$V_8 > 33.313$	French, [11]
$V_9 > 33.1766$	French, [11]

$$V_n \geq \left(8 - \frac{3\pi - 7}{2\cos\left(\frac{\pi}{n+2}\right) - 1}\right) \cdot \frac{\sqrt{2\cos\left(\frac{\pi}{n+2}\right) + 1}}{\sqrt{2\cos\left(\frac{\pi}{n+2}\right) - 1}} \quad (n \in \mathbb{N}) \qquad \text{Stechkin, [25]}$$

$34.8993 > V_4$	D. Hollenbeck (unpublished, referred to in [23])
$V > 33.58$	Reztsov, [20]
$U > 5.8656$	Révész, unpublished
$33.54 < V < 34.677$	Révész, unpublished.

1.3. In the many investigations of Landau's extremal problems, a number of new relatives were introduced. In his quite elegant and sharp lower estimation for U, van der Waerden [27] used the construction of a measure

$$d\kappa(x) \sim b_0 + 2\sum_{k=1}^{\infty} b_k \cos kx \geq 0 \tag{1.16}$$

with the properties

$$\kappa \geq 0, \quad b_0 + b_1 \leq 2, \quad b_k \leq 1 \quad (k \in \mathbb{N}_2), \tag{1.17}$$

where $\mathbb{N}_2 := \mathbb{N} \cap [2, \infty]$. Actually van der Waerden sought minimal b_1 ($b_1 < 0$ with maximal absolute value) and could prove that $U \geq 1 - b_1$. Formulating this as an extremal problem, van der Waerden treated

$$\Omega := \sup\left\{1 - b_1 : \exists\kappa \in BM(\mathbb{T}), \quad \kappa \geq 0 \quad \text{with (1.16)–(1.17)}\right\}. \tag{1.18}$$

Finding a measure with (1.16)–(1.17) and with $b_1 = -4.8642\ldots$, van der Waerden showed actually

$$U \geq \Omega \geq 5.8642\ldots\ . \tag{1.19}$$

S. B.Š̃techkin [25] used a different method aiming mainly the estimation of V. In the course of proof he defined an intermediate quantity between U and V when introducing

(1.20)

$$W := \inf\left\{\frac{f(0)-1}{a_1-1} : f \in \mathcal{T} \quad \text{satisfies (1.3) with } a_1 > 1\right\} = \min_{a>1} \frac{\alpha(a)-1}{a-1},$$

$$W_n := \inf\left\{\frac{f(0)-1}{a_1-1} : f \in \mathcal{T}_n \quad \text{satisfies (1.3) with } a_1 > 1\right\} = \min_{a>1} \frac{\alpha_n(a)-1}{a-1}.$$

As in case of U and V, we again have $W_n \searrow W$, and the determination of W and W_n is a problem of a similar sort.

Stechkin himself could estimate W as follows.

$$\begin{aligned} &W_2 = \frac{1}{2}\left(5+\sqrt{17}\right) = 4.56\ldots, \\ &W_4 \leq W_3 \leq \frac{1}{2}\left(5+\sqrt{13}\right) = 4.30\ldots, \\ &W \geq 4.159. \end{aligned} \tag{1.21}$$

1.4. We also introduce some more extremal quantities. Denote by λ and δ_z ($z \in \mathbb{T}$) the normalized measures (the Lebesgue and (essentially) the Dirac measures at $z \in \mathbb{T}$)

$$\begin{aligned} &d\lambda(x) \sim 1, \\ &d\delta_z(x) \sim 1 + 2\sum_{b=1}^{\infty}\left(\cos kz\cos kx + \sin kz \sin kx\right), \\ &\delta := \delta_0. \end{aligned} \tag{1.22}$$

We consider the measure sets

$$\mathcal{M}(0) := \Bigg\{ \tau \in BM(\mathbb{T}) : d\tau(x) \sim \sum_{k=1}^{\infty} t_k \cos kx, \qquad t_1 \in \mathbb{R}, \quad t_k \le 0 \quad (k \in \mathbb{N}_2) \Bigg\},$$

$$\mathcal{M}_n(0) := \Bigg\{ \tau \in BM(\mathbb{T}) : d\tau(x) \sim \sum_{k=1}^{\infty} t_k \cos kx, \qquad t_1 \in \mathbb{R}, \quad t_k \le 0 \quad (2 \le k \le n) \Bigg\},$$

$$\begin{aligned} \mathcal{M}(a) &:= \Bigg\{ \tau \in BM(\mathbb{T}) : d\tau(x) \sim b\left(1 - \frac{2}{a}\cos x\right) + \sum_{k=2}^{\infty} t_k \cos kx, \qquad b \in \mathbb{R}, \quad t_k \le 0 \quad (k \in \mathbb{N}_2) \Bigg\} \\ &= \left\{ \tau \in BM(\mathbb{T}) : \tau = -\frac{a}{2} t_1(\tau_0) \cdot \lambda + \tau_0, \quad \tau_0 \in \mathcal{M}(0) \right\}, \\ &\qquad \Big(t_1(\tau_0) := \langle \tau_0, 2\cos x \rangle \Big), \end{aligned} \tag{1.23}$$

$$\begin{aligned} \mathcal{M}_n(a) &:= \Bigg\{ \tau \in BM(\mathbb{T}) : d\tau(x) \sim b\left(1 - \frac{2}{a}\cos x\right) + \sum_{k=2}^{\infty} t_k \cos kx, \qquad b \in \mathbb{R}, \quad t_k \le 0 \quad (2 \le k \le n) \Bigg\} \\ &= \left\{ \tau \in BM(\mathbb{T}) : \tau = -\frac{a}{2} t_1(\tau_0) \cdot \lambda + \tau_0, \quad \tau_0 \in \mathcal{M}_n(0) \right\}, \\ &\qquad \Big(t_1(\tau_0) := \langle \tau_0, 2\cos x \rangle \Big), \end{aligned}$$

where the last two definitions are valid for any $a \in \mathbb{R}$ and $a \neq 0$. Also we put for arbitrary $y \in \mathbb{R}$

$$\begin{aligned}\mathcal{N}(y) :=& \Bigg\{\nu \in BM(\mathbb{T}) : \nu \geq 0, \quad d\nu(x) \sim 1 + \sum_{k=1}^{\infty} y_k \cos kx, \\ & \qquad y_1 \in \mathbb{R}, \quad y_k \leq y \quad (k \in \mathbb{N}_2)\Bigg\}, \\ \mathcal{N}_n(y) :=& \Bigg\{\nu \in BM(\mathbb{T}) : \nu \geq 0, \quad d\nu(x) \sim 1 + \sum_{k=1}^{\infty} y_k \cos kx, \\ & \qquad y_1 \in \mathbb{R}, \quad y_k \leq y \quad (2 \leq k \leq n)\Bigg\}.\end{aligned} \tag{1.24}$$

Finally let us introduce for all $b \in (-2, 2)$ the square-integrable function set

$$\mathcal{G}(b) := \left\{ g \in L^2(\mathbb{T}) : g \geq 0, \quad g(x) \sim 1 + b \cos x + \sum_{k=2}^{\infty} b_k \cos kx \right\}. \tag{1.25}$$

To these sets we define the following extremal quantities.

$$\begin{aligned}\omega(a) &:= \sup\big\{t : \exists \tau \in \mathcal{M}(a), \quad \tau + \delta \geq t \cdot \lambda\big\}, \\ \omega_n(a) &:= \sup\big\{t : \exists \tau \in \mathcal{M}_n(a), \quad \tau + \delta \geq t \cdot \lambda\big\}, \\ \beta(y) &:= \sup\big\{ - y_1 : \exists \nu \in \mathcal{N}(y), \quad y_1 = \langle \nu, 2\cos x\rangle\big\}, \\ \beta_n(y) &:= \sup\big\{ - y_1 : \exists \nu \in \mathcal{N}_n(y), \quad y_1 = \langle \nu, 2\cos x\rangle\big\}, \\ \vartheta(y) &:= \sup\big\{y_1 : \exists \nu \in \mathcal{N}(y), \quad y_1 = \langle \nu, 2\cos x\rangle\big\}, \\ \vartheta_n(y) &:= \sup\big\{ - y_1 : \exists \nu \in \mathcal{N}_n(y), \quad y_1 = \langle \nu, 2\cos x\rangle\big\}, \\ \gamma(b) &:= \inf\big\{||g||_2 : g \in \mathcal{G}(b)\big\}.\end{aligned} \tag{1.26}$$

1.5. I would like to thank Professor Andrzej Schinzel for calling my attention to the problem and giving the first sources for the study of the subject. Gábor Halász suggested the investigation of (1.25) and $\gamma(b)$ in (1.26) which led to nice findings. After presenting the matter in Zakopane on the Number Theory Conference dedicated to the sixtieth birthday of Professor Andrzej Schinzel, Sergei Konyagin provided important references [1], [13], [20], [2] on the work published in the Russian mathematical literature. That helped me to correct

and extend this work at many places, as previously I did not know about, and hence did not refer to these very valuable works. In particular, many of my results in [18], [19] are greatly overlap with these, and my estimates announced in Zakopane, were preceded by even better ones of Arestov and Kondrat'ev. That led me a complete revision of this work and now I hope to give due credit to all researchers who contributed to the subject. I also hope that the independent and sometimes more or less different formulations and proofs could add to the value of the results in [1], [2], [13], [20]. The overlapping results are pointed out in due course.

2. Preliminaries

2.1. As we will extensively use sets of Borel measures and extremal quantities defined on these sets, we summarize a few facts of the structure of $BM(\mathbb{T})$ at the outset.

Let us recall that $BM(\mathbb{T}) = C(\mathbb{T})^*$, the topological dual of the Banach space $C(\mathbb{T})$ with the norm of the total variation norm

$$\|\mu\|_{BM(\mathbb{T})} = \int_{\mathbb{T}} |d\mu|. \tag{2.1}$$

We know that $C(\mathbb{T})$ is not reflexive, and $BM(\mathbb{T})^* \supsetneq C(\mathbb{T})$. Hence the weak, and the weak $*$ topology of $BM(\mathbb{T})$ are different, the weak topology being the weakest topology so that all functionals from $BM(\mathbb{T})^*$ be continuous linear functionals on $BM(\mathbb{T})$, while the weak $*$ topology is the weakest topology so that the functionals belonging to $C(\mathbb{T})$ be continuous on $BM(\mathbb{T})$. Thus the weak $*$ topology is even weaker than the weak one.

In a topological vector space convex and closed sets remain convex and closed when considering the weak topology in place of the original topology. However, in dual spaces like $BM(\mathbb{T})$ closedness is not necessarily saved when considering the weak $*$ topology instead of the weak topology. On the other hand we have the Banach-Alaoglu Theorem ([9], 4.10.3. Theorem, p. 205) stating that all the closed balls in the dual space $BM(\mathbb{T})$ are weak $*$ compact.

Our application of these structural facts will have the following pattern. Usually we define a set of measures in $BM(\mathbb{T})$ and wish to extremalize some quantity on that set. Using the definition, we can pass on to a decreasing sequence of closed, bounded and convex sets $F_n \subset BM(\mathbb{T})$, and to show that there exists an extremal measure, we are entitled to show that $F := \bigcap_{n=1}^{\infty} F_n \neq \emptyset$. This is a Cantor type property, and can be guaranteed for decreasing and nonempty sequences of compact sets. Now usually $F_n \subset BM(\mathbb{T})$ will be convex and closed, but not compact. To save the idea, we pass on to the weak $*$ topology. First, the nonempty sets $F_n \subset BM(\mathbb{T})$ remain convex in any topology. They will be bounded in the norm of $BM(\mathbb{T})$ usually because for nonnegative measures

$$\|\mu\|_{BM(\mathbb{T})} = \int_{\mathbb{T}} |d\mu| = \int_{\mathbb{T}} d\mu = 2\pi\langle 1, \mu\rangle \quad (\mu \in BM(\mathbb{T}), \quad \mu \geq 0). \tag{2.2}$$

Hence F_n will be conditionally compact in the weak $*$ topology according to the Banach-Alaoglu Theorem. To show that F_n are weak $*$ compact, the key point is to show that F_n are weak $*$ closed, too.

It is obvious that any closed and convex sets F_n can be represented as the intersection of a set of closed halfspaces defined by continuous linear functionals *from the bidual space*. However, such level sets of linear functionals can be proved to be even weak $*$ closed only if the functionals themselves are weak $*$ continuous, i.e. if the functionals belong to $C(\mathbb{T})$. Thus we will look for a representation of F_n as an intersection of level sets of the type

$$X(f, c) := \left\{\mu \in BM(\mathbb{T}) : \langle f, \mu\rangle \leq c\right\} \tag{2.3}$$

with $f \in C(\mathbb{T})$. Having such a representation, we can claim F_n to be even weak $*$ closed, hence we get that F_n is not only conditionally compact, but it is also compact in the weak $*$ topology. Finally, we can refer to the Cantor type property that the intersection of the compact, decreasing and nonempty sets F_n must be nonempty. To formalize this argument, we can state the following.

Lemma 2.1. *Suppose that F_n $(n \in \mathbb{N})$ is a sequence of subsets of $BM(\mathbb{T})$ with the following properties.*

i) $F_n \neq \emptyset$ $(n \in \mathbb{N})$.

ii) $F_{n+1} \subset F_n$ $(n \in \mathbb{N})$.

iii) *F_n is bounded in the total variation norm of $BM(\mathbb{T})$ (perhaps for $n > n_0$).*

iv) *F_n can be represented as the intersection of a number of closed halfspaces of the form* (2.3) *with the generating functionals belonging to $C(\mathbb{T})$.*

Then the intersection

$$F := \bigcap_{n=1}^{\infty} F_n \tag{2.4}$$

is a norm-bounded, closed, convex, weak $*$-compact and nonempty subset of $BM(\mathbb{T})$.

Consider the sets

$$
\begin{aligned}
BM(\mathbb{T})_C := &\big\{\mu \in BM(\mathbb{T}) : \mu \\
&\quad \text{is even (i.e. } \mu(H) = \mu(-H) \quad (\forall H \subset \mathbb{T}, \text{ measurable}))\big\} \\
= &\big\{\mu \in BM(\mathbb{T}) : \langle \sin kx, \mu\rangle = 0 \quad (k \in \mathbb{N})\big\} \\
= &\big\{\mu \in BM(\mathbb{T}) : \langle f, \mu\rangle = 0 \quad \forall f \in C(\mathbb{T}), \\
&\qquad\qquad f(x) \equiv -f(-x)(x \in \mathbb{T})\big\},
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
BM(\mathbb{T})_S := &\big\{\mu \in BM(\mathbb{T}) : \mu \\
&\quad \text{is odd } (\mu(H) = -\mu(-H) \quad (\forall H \subset \mathbb{T}, \text{ measurable}))\big\} \\
= &\big\{\mu \in BM(\mathbb{T}) : \langle \cos kx, \mu\rangle = 0 \quad (k \in \mathbb{N})\big\} \\
= &\big\{\mu \in BM(\mathbb{T}) : \langle f, \mu\rangle = 0 \quad \forall f \in C(\mathbb{T}), \\
&\qquad\qquad f(x) \equiv -f(-x)(x \in \mathbb{T})\big\},
\end{aligned} \tag{2.6}
$$

and the set

$$
\begin{aligned}
BM(\mathbb{T})_P := &\big\{\mu \in BM(\mathbb{T}) : \mu \\
&\quad \text{is nonnegative (i.e. } \mu(H) \geq 0 \quad (\forall H \subset \mathbb{T}, \text{ measurable}))\big\} \\
= &\big\{\mu \in BM(\mathbb{T}) : \langle f, \mu\rangle \geq 0 \qquad \forall f \in C(\mathbb{T}), \quad f \geq 0\big\}.
\end{aligned} \tag{2.7}
$$

Here we use the sets of even, odd and nonnegative functions

$$
\begin{aligned}
E &:= \Big\{f \in C(\mathbb{T}) : f(x) \equiv f(-x) \quad (x \in \mathbb{T})\Big\}, \\
O &:= \Big\{f \in C(\mathbb{T}) : f(x) \equiv -f(-x) \quad (x \in \mathbb{T})\Big\}, \\
P &:= \Big\{f \in C(\mathbb{T}) : f \geq 0\Big\},
\end{aligned} \tag{2.8}
$$

to establish a representation of the type iv) for the sets (2.5)–(2.7). Namely,

$$BM(\mathbb{T})_C = \bigcap_{f \in O} (X(f,0) \cap X(-f,0)), \tag{2.8}$$

$$BM(\mathbb{T})_S = \bigcap_{f \in E} (X(f,0) \cap X(-f,0)), \tag{2.9}$$

$$BM(\mathbb{T})_P = \bigcap_{f \in P} (X(-f,0). \tag{2.10}$$

Thus in the following we can use property iv) for the sets (2.5), (2.6) and (2.7).

2.2. Let $0 < a < b$, and $k : [a,b] \to \mathbb{R}$ be any continuous, strictly increasing and concave function on the interval.

We define the "tangential function to k" and the "extremal tangential curve" to k as follows.

Definition. *For $t \in \mathbb{R}$ let us consider the points (x,t), $(0,t)$, $(x,k(x))$ in this order for all $x \in [a,b]$ and denote $\varphi(t,x)$ the angle (measured from the positive x direction to the counterclockwise sense) of the chord drawn from $(0,t)$ to $(x,k(x))$. As $0 < a \le x \le b$, and the vector $((0,t),\ (x,t))$ is horizontal, we plainly have $-\frac{\pi}{2} < \varphi(t,x) < \frac{\pi}{2}$.*

We introduce the "extremal tangential curve"

$$\Gamma := \Gamma_k := \left\{ (t_0, x_0) \in \mathbb{R}^2 : \varphi(t_0,x_0) = \max_{a \le x \le b} \varphi(t_0,x) \right\}; \tag{2.12}$$

we also introduce the "tangential function to k"

$$f(t) := f_k(t) := \max_{a \le x \le b} \frac{k(x)-t}{x} = \max_{x\in[a,b]} \tan\varphi(t,x) = \tan\varphi(t,x^*) \qquad \big((t,x^*) \in \Gamma_k\big). \tag{2.13}$$

Plainly we may also consider

$$\varphi(t) := \arctan f(t) = \max_{a \le x \le b} \varphi(t,x) = \varphi(t,x^*) \quad ((t,x^*) \in \Gamma_k). \tag{2.14}$$

Geometrically $\varphi(t)$ is the oriented angle, $f(t)$ is the slope of the tangential straight line drawn from the point $(0,t)$ to the curve $\{(x,k(x)) : a \le x \le b\}$.

Note that $f_k(t)$ is just the well-known Legendre transform of the function k; as properties of the Legendre transform are well-known, see e.g. [21], in the following assertions we will omit the proofs.

Lemma 2.2.

i) *The function $f(t) : \mathbb{R} \to \mathbb{R}$ is continuous and strictly decreasing.*

ii) *The curve Γ is "oriented positively" in the sense that for any two points (t',x'), $(t'',x'') \in \Gamma$ $\quad t' < t''$ entails $x' \le x''$.*

iii) *The point set $I(t) := \{x : (t,x) \in \Gamma\}$ is a convex closed set $\subset [a,b]$.*

Now let us define

$$
\begin{aligned}
\underline{x}(t) &:= \min\left\{x : (t,x) \in \Gamma\right\} = \min I(t);\\
\overline{x}(t) &:= \max\left\{x : (t,x) \in \Gamma\right\} = \max I(t);\\
x(t) &:= \underline{x}(t) \quad \text{whenever} \quad \underline{x}(t) = \overline{x}(t);\\
T(x) &:= \left\{t \in \mathbb{R} : (t,x) \in \Gamma\right\};\\
\overline{t}(x) &:= \max\left\{t : (t,x) \in \Gamma\right\} = \max T(x);\\
\underline{t}(x) &:= \min\left\{t : (t,x) \in \Gamma\right\} = \min T(x);\\
t(x) &:= \underline{t}(x) \quad \text{whenever} \quad \underline{t}(x) = \overline{t}(x).
\end{aligned}
\tag{2.15}
$$

The existence and nature of T, $\overline{t}$, $\underline{t}$ are similar to I, $\overline{x}$, $\underline{x}$, by the very same Lemma 2.2 ii) and iii).

Lemma 2.3.

i) *The concave function k is differentiable iff $t(x)$ exists for all $a < x < b$. Moreover, for any $x \in (a,b)$ we have $\overline{t}(x) = \underline{t}(x)$ iff $k'(x-0) = k'(x+0)$.*

ii) *We always have*

$$f'(t+0) = \frac{-1}{\overline{x}(t)}, \quad f'(x-0) = \frac{-1}{\underline{x}(t)}.$$

Corollary 2.1. *If f is the tangential function defined to k, then f is always a continuous, strictly decreasing convex function. Moreover, f is differentiable (and then also continuously) iff k is strictly concave. Conversely, f is strictly convex iff k is differentiable iff $k \in C^1[a,b]$.*

3. Analysis of the extremal quantities

3.1. First of all let us record some basic properties of the functions defined in Section 1. As for the domain of definition (where the corresponding definition yields a finite value) we use the notation $\mathcal{D}$; similarly, the range of a function is denoted by $\mathcal{R}$.

Proposition 3.1.

i) $\mathcal{D}(\alpha) = (A,2)$ *or* $[A,2)$, *where* $-\sqrt{3} \le A \le -\sqrt{2}$.

ii) $\mathcal{D}(\alpha_n) = [A_n, B_n]$, *where* $A_n \le -\sqrt{2}$ $(n \ge 2)$ *and* $B_n = 2\cos\frac{\pi}{n+2}$.

iii) $\mathcal{D}(\gamma) = (-2,2)$.

P r o o f. i) See Proposition 4.1 of [19].

ii) The first estimate follows from the example (4.2) of [19], and the second statement is a consequence of a theorem of Fejér [10] and Szász [24] who determined the corresponding extremal polynomials. See also [25], Lemma 1.

iii) Similar to i) but essentially trivial. □

Proposition 3.2.

i) *In the definition* (1.11) *of* $\alpha(a)$ *the infimum is actually a minimum, i.e.*

$$\alpha(a) = \min\left\{f(0) : f \in \mathcal{F}(a)\right\}$$

for all $a \in \mathcal{D}(\alpha)$.

ii) *If* $A \in \mathcal{D}(\alpha)$, *then* $\lim_{a\to A+} \alpha(a) = \alpha(A)$, *and if* $A \notin \mathcal{D}(\alpha)$, *then* $\lim_{a\to A+} \alpha(a) = \infty$.

iii) $\lim_{a\to 2-} \alpha(a) = \infty$.

iv) $\alpha(a)$ *is a convex function on* $\mathcal{D}(\alpha)$.

v) $\alpha_n(a)$ *is a convex function on* $\mathcal{D}(\alpha_n)$.

P r o o f. These can be found in [19], Propositions 4.2 and 4.3 or, in a somewhat more general form, in [18], 2.3, 2.5 and 2.6 Propositions. Note that i) and v) appeared already in Theorem 1 3) of [2], while iii) was proved first in [20], see also the comments to Proposition 3.5. iii). □

Our knowledge about the actual function values of $\alpha(a)$ is summarized in the next three propositions.

Proposition 3.3.

i) $\alpha(a) = 1 + a$ *for* $-1 \le a \le 1$.

ii) $\alpha(a) = 2a$ *for* $1 \le a \le 4/3$.

iii) $\alpha(a) = 0$ *for* $-4/3 \le a \le -1$.

Proposition 3.4. $\alpha(a) > 0$ *for* $a < -4/3$, $a \in \mathcal{D}(\alpha)$.

Proposition 3.5. *In the range* $4/3 < a < 2$ *we have the following lower estimates for the function* $\alpha(a)$.

i) $\alpha(a) > \delta(a) := 2a$,

ii) $\alpha(a) \geq \varphi(a) := 8a - 3\pi$,

iii) $\alpha(a) \geq \tau(a) := \sqrt{\dfrac{2+a}{2-a}}$.

Proof. For a proof of the claims in Proposition 3.3, see e.g. Proposition 4.4 i), ii) and iv) in [19]. Note that i) is (0.17) of [1] and ii) is covered by (0.15) of [1] or Theorem 1 1) in [2].

For Proposition 3.4 see Proposition 4.4 v) in [19].

Lastly, consider Proposition 3.5. First, i) can be found in Proposition 4.4 iii) of [19]. ($\alpha(a) \geq \delta(a)$ is trivial from Proposition 3.3 ii) and Proposition 3.2 iii).)

Proposition 3.5 ii) is an estimate of Stechkin, see Lemma 3 in [25].

Finally, the nontrivial estimate of Proposition 3.5 iii) is proved e.g. in [19], Theorem 5.1. A very similar proof of a very similar, but somewhat more elaborated (and thus slightly better) nonlinear estimate was given first in [20]. Actually, the key lemma to the result was attributed to Yudin (oral communication) in [20], while in [19] an independent and different proof was given which precisely characterizes also the extremal cases of the lemma. □

3.2. For the functions introduced in (1.26) their use and relevance to the problems studied can be best seen from the relation between α and ω. Let $f \in \mathcal{F}(a)$ be arbitrary and take any $\tau \in \mathcal{M}(a)$ satisfying $\tau + \delta \geq t \cdot \lambda$ with some $t \geq 0$. (Such t and τ must exist since the zero measure, $\mathbf{0} \in \mathcal{M}(a)$.) We have from the nonnegativity of f and a_k ($k \in \mathbb{N}_2$), the nonpositivity of t_k ($k \in \mathbb{N}_2$) and from $\delta \geq t \cdot \lambda - \tau$, that

(3.1)
$$\begin{aligned} f(0) &= \langle f, \delta \rangle \geq \langle f, t \cdot \lambda - \tau \rangle \\ &= t - \langle f, \tau \rangle = t - \left\{ b + \frac{1}{2} \cdot a \cdot b \cdot \left(\frac{-2}{a} \right) + \sum_{k=2}^{\infty} a_k t_k \right\} = t - \sum_{k=2}^{\infty} a_k t_k \geq t. \end{aligned}$$

Taking supremum over all τ and t on the right, and then infimum at the left-hand side, we obtain the inequality

$$\alpha(a) \geq \omega(a). \tag{3.2}$$

That estimate was essentially at the heart of van der Waerden's estimate, as we shall see later. This estimate is not only close numerically, but actually it is theoretically exact.

Theorem 3.1 (Duality).

i) $\mathcal{D}(\alpha) = \mathcal{D}(\omega)$, *and the* sup *in the first definition of* (1.26) *is actually a maximum.*

ii) *For all* $a \in \mathcal{D}(\alpha) \quad \alpha(a) = \omega(a)$.

iii) *For all* $n \in \mathbb{N} \quad \mathcal{D}(\alpha_n) = \mathcal{D}(\omega_n)$, *and the* sup *in the second definition of* (1.26) *is actually a maximum.*

iv) *For all* $n \in \mathbb{N}$ *and* $a \in \mathcal{D}(\alpha_n) \quad \alpha_n(a) = \omega_n(a)$.

Proof. The easy part is $\alpha(a) \geq \omega(a)$ and its relatives $\alpha_n(a) \geq \omega_n(a)$, as shown above. That also entails $\mathcal{D}(\alpha) \subset \mathcal{D}(\omega)$, $\mathcal{D}(\alpha_n) \subset \mathcal{D}(\omega_n)$. The converse is nontrivial, and the proof applies functional analysis. For the whole argument we refer to [18], especially 3.4 Theorem and 3.5 Proposition. Note that here the index sets M and L of [18] are $\mathbb{N}_2$ and $\emptyset$ or $[2, n]$ and $\emptyset$, and thus also 2.6 Proposition of [18] applies. That covers the border cases $a = A$ and $a = A_n$ or B_n, not included in the even more general setting of 3.4 Theorem of [18]. The existence of extremal measures ω and ω_n follows from the argument as pointed out in section 3.6 of [18].

3.3. With the above duality theorem at hand, let us also define the functions

$$
\begin{aligned}
U(a) &:= \frac{\alpha(a)}{a-1} = \frac{\omega(a)}{a-1}, \\
V(a) &:= \frac{\alpha(a)-1}{(\sqrt{a}-1)^2} = \frac{\omega(a)-1}{(\sqrt{a}-1)^2}, \\
W(a) &:= \frac{\alpha(a)-1}{a-1} = \frac{\omega(a)-1}{a-1} \qquad (a \in (1,2)),
\end{aligned}
\tag{3.3}
$$

and for any $n \in \mathbb{N}$ their finite degree counterparts

$$
\begin{aligned}
U_n(a) &:= \frac{\alpha_n(a)}{a-1} = \frac{\omega_n(a)}{a-1}, \\
V_n(a) &:= \frac{\alpha_n(a)-1}{(\sqrt{a}-1)^2} = \frac{\omega_n(a)-1}{(\sqrt{a}-1)^2}, \\
W_n(a) &:= \frac{\alpha_n(a)-1}{a-1} = \frac{\omega_n(a)-1}{a-1} \qquad (a \in (1, B_n]).
\end{aligned}
\tag{3.4}
$$

Since the functions (3.3)-(3.4) are the products of one of the positive convex functions $\alpha(a)$, $\alpha(a) - 1$, $\alpha_n(a)$, or $\alpha_n(a) - 1$ and one of the strictly convex and positive functions $\dfrac{1}{a-1}$ or $\dfrac{1}{(\sqrt{a}-1)^2}$, all functions are positive and strictly convex. Note also that all the six functions tend to $+\infty$ as $a \to 1+0$ as the denominators tend to $+0$ and the numerators are finite and positive. Similarly, as Proposition 3.5 iii) entails, $\alpha(a) \to +\infty$ $(a \to 2-0)$, and that implies $U(a) \to +\infty$, $V(a) \to +\infty$ and $W(a) \to +\infty$ $(a \to 2-0)$. Hence we see that the functions (3.3)–(3.4) all have minimum points where the extremal quantities (1.13) and (1.20) are attained, and also that these points are unique due to strict convexity. Thus we have

Proposition 3.6.

i) *All the functions* (3.3)–(3.4) *are strictly positive and strictly convex in their domain of definition.*

ii) *All the functions* (3.3)–(3.4) *have limit* $+\infty$ *at* $1+0$.

iii) *The functions* (3.3) *have limit* $+\infty$ *at* $2-0$ *while the functions* (3.4) *are continuous and finite at* B_n.

iv) *The functions* (3.3)–(3.4) *have unique minimum points* a_U, a_V, a_W *and* $a_{U,n}$, $a_{V,n}$, $a_{W,n}$, *respectively, where we have*

$$\begin{array}{lll} U = U(a_U), & V = V(a_V), & W = W(a_W), \\ U_n = U_n(a_{U,n}), & V = V(a_{V,n}), & W = W(a_{W,n}). \end{array}$$

3.4. Proposition 3.7.

i) $\mathcal{N}(y) = \emptyset$ *for* $y < 0$.

ii) $\mathcal{N}(y) = \mathcal{N}(2) = \{\nu \in BM(\mathbb{T}) : \langle \nu, 1\rangle = 1$ *and* $\nu \geq 0\}$ *for* $y \geq 2$.

iii)
$$\emptyset \neq \Big\{ \nu \in BM(\mathbb{T}) : d\nu(x) = 1 + \sum_{k=1}^{\infty} y_k \cos kx, \\ \sum_{k=1}^{\infty} |y_k| \leq 1,\ y_k \leq 0\ (k \in \mathbb{N}_2) \Big\} \subset \mathcal{N}(0) = \\ = \Big\{ \nu \in BM(\mathbb{T}) : \nu \geq 0, d\nu(x) = 1 + \sum_{k=1}^{\infty} y_k \cos kx, \\ 1 + \sum_{k=1}^{\infty} y_k \geq 0, y_k \leq 0 \quad (k \in \mathbb{N}_2) \Big\}.$$

iv) *For any two values* $0 \leq y' < y'' \leq 2$ *we have* $\mathcal{N}(y') \subsetneq \mathcal{N}(y'')$.

v) *For all* $y \geq 0$ $\mathcal{N}(y)$ *is a convex, closed and bounded set in* $BM(\mathbb{T})$.

Proof. i) Suppose that $y < 0$ and $\nu \in \mathcal{N}(y)$. Consider the convolution $\nu * F_N = f_N \in \mathcal{T}_N$ for any $N \in \mathbb{N}$ where F_N denotes the usual Fejér kernel. On the one hand $f_N \geq 0$, on the other hand $f_N(0) = 1 + \left(1 - \frac{1}{N+1}\right) y_1 + \sum_{k=2}^{N} \left(1 - \frac{k}{N+1}\right) y_k \leq 1 + 2 + y \sum_{k=2}^{N} \left(1 - \frac{k}{N+1}\right)$. As the right-hand side tends to $-\infty$ with $N \to \infty$ by $y < 0$, we have proved i) by contradiction.

ii) For all $\nu \in BM(\mathbb{T})$, $\langle \nu, 1 \rangle = 1$, $\nu \geq 0$ we have $\langle \nu, 2 \cos kx \rangle \leq \langle \nu, 2 \rangle = 2$.

iii) It suffices to prove the last equation, the others being easy consequences. Plainly the conditions on the right-hand side are exceeding the defining conditions (1.24) for $\mathcal{N}(0)$ in two respect: by prescribing convergence of the Fourier representation, and by supposing

$$1 + \sum_{k=1}^{\infty} y_k \geq 0. \tag{3.5}$$

This last condition, together with $y_k \leq 0$ ($k \in \mathbb{N}_2$), entails absolute convergence of the series (3.5), hence the Fourier representation must be absolutely uniformly convergent, and the measure ν is an absolutely continuous measure with a derivative having absolutely uniformly convergent series representation. The only thing to show that (3.5) holds for all $\nu \in \mathcal{N}(0)$. We can use the Fejér kernel F_N and the convolution $F_N * \nu$, already used in part i) to get for arbitrary $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq f_N(0) = (\nu * F_N)(0) &= 1 + \sum_{k=1}^{N} \left(1 - \frac{k}{N+1}\right) y_k \\ &\leq 1 + y_1 + \sum_{k=2}^{\infty} \left(1 - \frac{k}{N+1}\right) y_k. \end{aligned}$$

Using also $y_k \leq 0$ ($k \in \mathbb{N}_2$), we can take limits with respect to $N \to \infty$, what yields (3.5).

iv) The inclusion is trivial. $\nu = \left(1 - \frac{y''}{2}\right) \cdot \lambda + \frac{y''}{2} \cdot \delta \in \mathcal{N}(y'')$ but $\nu \notin \mathcal{N}(y')$ shows that $\mathcal{N}(y') \neq \mathcal{N}(y'')$.

v)

$$\mathcal{N}(y) = \mathcal{N}(2) \cap \bigcap_{k=2}^{\infty} \left\{\nu \in BM(\mathbb{T}) : \langle 2 \cos kx, \nu \rangle \leq y\right\}$$

$\mathcal{N}(2)$ is the intersection of a hyperplane defined by $\langle 1, \nu \rangle = 1$, and the (closed and convex) set of nonnegative measures. The other intersection is defined as the intersection of closed halfspaces. Therefore $\mathcal{N}(y)$ is convex and closed. Note that for any $\nu \in \mathcal{N}(y)$ $||\nu||_{BM(\mathbb{T})} = 2\pi$, cf. (2.2), hence $\mathcal{N}(y)$ is also bounded. $\square$

Proposition 3.8.

i) *For all $n \in \mathbb{N}$ there exists a unique C_n, $-2 \leq C_n < 0$, so that $\mathcal{N}_n(y) = \emptyset$ for $y < C_n$, but not for $y \geq C_n$.*

ii) *$C_n \nearrow 0$ as $n \to +\infty$.*

iii) *$\mathcal{N}_n(y) = \mathcal{N}_n(2) = \mathcal{N}(2)$ for $y \geq 2$.*

iv) *For any two values $C_n \leq y' < y'' \leq 2$ we have*

$$\mathcal{N}_n(y') \subsetneq \mathcal{N}_n(y'').$$

v) *For all $y \geq C_n$ $\mathcal{N}_n(y)$ is a convex, closed and bounded set in $BM(\mathbb{T})$.*

P r o o f. i) Plainly, as the inclusion part is trivial from statement iv), $C_n := \inf\{y : \exists \nu \in \mathcal{N}_n(y)\} = \sup\{y : \mathcal{N}_n(y) = \emptyset\}$.

First we prove $C_n < 0$, or more precisely, $C_n \leq -\dfrac{1}{n-1}$. To this end let us consider the function $g_n(x) := 1 - \dfrac{1}{n-1} \displaystyle\sum_{k=2}^{n} \cos kx$ and the measure $d\nu(x) = g_n(x)dx$. Plainly $\nu \in \mathcal{N}\left(-\frac{1}{n-1}\right)$ showing $C_n \leq -\dfrac{1}{n-1}$. On the other hand let $y < 0$ be arbitrary with $\mathcal{N}(y) \neq 0$ and let $\nu \in \mathcal{N}(y)$. If we choose $N = n$ in the construction of the Proof of Proposition 2.7 i), we obtain $0 \leq g_n(0) \leq 3 + y\left(\dfrac{N}{2} - 1 + \dfrac{1}{N+1}\right) \leq 3 - |y| \cdot \dfrac{n-2}{2}$. Thus $|y| \leq \dfrac{6}{n-2}$, proving also $C_n \to 0$ $(n \to +\infty)$. For small n the above estimate can be substituted by the easier one

$$0 \leq \langle 1 + \cos 2x, \nu \rangle = 1 + \frac{y_2}{2} \leq 1 + \frac{y}{2},$$

showing $y \geq -2$; the measure $\nu_{\frac{\pi}{2}} := \dfrac{1}{2}\left(\delta_{\frac{\pi}{2}} + \delta_{-\frac{\pi}{2}}\right) \in \mathcal{N}_2(-2)$ shows that this estimate is sharp for $n = 2$.

We now prove $\mathcal{N}_n(C_n) \neq \emptyset$. This statement is a Cantor-type one, as $M_m := \mathcal{N}\left(C_n + \dfrac{1}{m}\right)$ are closed (also convex) and nonempty sets of $BM(\mathbb{T})$ and plainly $\mathcal{N}_n(C_n) = \bigcap_{m=1}^{\infty} M_m$, where $M_m \supset M_M$ for all $m < M$.

To apply Cantor's Lemma, we only have to show that the sets M_m are compact sets. That is not true in the original topology of $BM(\mathbb{T})$, but it holds true in the weak $*$ topology of $BM(\mathbb{T})$. Indeed, $\mathcal{N}_n(y)$ is bounded in view of (1.24) and (2.1), and all bounded sets of $BM(\mathbb{T})$ are conditionally compact in the weak $*$ topology. Moreover, $\mathcal{N}(y)$ is also closed, since by (2.7) $BM(\mathbb{T})_P$ is closed, and we have, using notation (2.3),

$$\mathcal{N}_n(y) = BM(\mathbb{T})_P \cap \left(\bigcap_{k=2}^{n} X(\cos kx, y) \right) \cap X(-1,-1) \cap X(1,1). \tag{3.6}$$

Thus Lemma 2.1 can be applied to show $\mathcal{N}_n(C_n) \neq \emptyset$.

ii) Monotonicity is obvious from definition, and $C_n \to 0$ is already proved.

iii) Follows from Proposition 3.7 ii) trivially.

iv) The inclusion is obvious. If $y'' > 0$, the example in Proposition 3.7 iv) fits here, too, showing $\mathcal{N}_n(y'') \setminus \mathcal{N}_n(y') \neq \emptyset$, while $\lambda \in \mathcal{N}_n(0)$ belongs to no $\mathcal{N}_n(y')$ with $y' < 0$. In case $y' < y'' < 0$ by the same way any $\nu' \in \mathcal{N}_n(y')$ with $\min\limits_{2\le k\le n} y_k(\nu') = y \le y'$ can be a starting point to define $\nu'' = \dfrac{y''}{y}\cdot\nu' + \left(1 - \dfrac{y''}{y}\right)\lambda \in \mathcal{N}_n(y'')$ with $\nu'' \notin \mathcal{N}_n(y')$ since $\min\limits_{2\le k\le n} y_k(\nu'') = y'' > y'$.

v) Clear. $\square$

Proposition 3.9. *In the definitions* (1.26) *for* β, ϑ, β_n, ϑ_n $(n \in \mathbb{N})$, *the supremum can be substituted by maximum since in case* $\mathcal{N}(y) \neq \emptyset$, *resp.* $\mathcal{N}_n(y) \neq \emptyset$, *the supremum is actually attained by some measure of* $\mathcal{N}(y)$, *resp.* $\mathcal{N}_n(y)$.

P r o o f. One proof can be given following the proof of $\mathcal{N}_n(C_n) \neq \emptyset$. However, it is easier to refer to the fact that $\mathcal{N}(y)$ and all $\mathcal{N}_n(y)$ $(n \in \mathbb{N})$ are closed sets, and thus the continuous functional $\nu \to \langle 2\cos x, \nu\rangle$ maps these sets to some closed sets of $\mathbb{R}$. (We may also note that the sets are convex and bounded, too, hence the image sets must be finite closed intervals.) Taking the supremums as in (1.26), we actually extremalize on these image sets of $\mathbb{R}$, and that concludes the argument. $\square$

Proposition 3.10.

i) $\vartheta_2(y) = \sqrt{2+y} \quad (C_2 = -2 \le y \le 2)$.

ii) $\beta_2(y) = \sqrt{2+y} \quad (C_2 = -2 \le y \le 2)$.

P r o o f. We already know that $C_2 = -2$ (cf. the end of the proof of part i) of Proposition 3.8.) Since $\beta_2(y) \ge 0$ and $\vartheta_2(y) \ge 0$ and both functions are nondecreasing in their domain of definition, it is enough to prove the statements for all $-2 < y \le 2$. Let us fix one particular y, and let us choose two extremal measures $\mu, \nu \in \mathcal{N}_2(y)$ with Fourier series

$$d\mu(x) \sim 1 + \sum_{k=1}^{\infty} z_k \cos kx, \quad d\nu(x) \sim 1 + \sum_{k=1}^{\infty} y_k \cos kx \tag{3.7}$$

so that

$$z_1 = -\beta_2(y), \quad y_1 = \vartheta_2(y). \tag{3.8}$$

The extremal measures exist according to Proposition 3.9. Let us estimate y_2 and z_2 by the values of y_1 and z_1! That kind of estimation was already worked out in [19], Theorem 2.2 (See also the Remark after it.). We get from this Theorem that

$$z_2 \ge 2\cos\left(2\arccos\left(\frac{-z_1}{2}\right)\right), \quad y_2 \ge 2\cos\left(2\arccos\left(\frac{y_1}{2}\right)\right). \tag{3.9}$$

Combining (3.7) and (3.8) with the inequalities $z_2 \le y$, $y_2 \le y$, coming from $\mu, \nu \in \mathcal{N}(y)$, we are led to

$$y \ge 2\cos\left(2\arccos\left(\frac{\beta_2(y)}{y}\right)\right), \quad y \ge 2\cos\left(2\arccos\left(\frac{\vartheta_2(y)}{2}\right)\right). \tag{3.10}$$

After some calculation this yields the estimates

$$\beta_2(y) \le \sqrt{2+y}, \quad \vartheta_2(y) \le \sqrt{2+y}. \tag{3.11}$$

Now we only have to show that this upper estimate is sharp. Let us consider the measure

$$\eta := \frac{1}{2}\left(\delta_w + \delta_{-w}\right) \quad \left(w := \arccos\left(\frac{z}{2}\right)\right), \quad \left(z := \sqrt{2+y}\right).$$

We have $\eta \ge 0$, and

$$d\eta(x) \sim 1 + \sum_{k=1}^{\infty} 2\cos(kw)\cos kx,$$

and thus the coefficient of $\cos x$ is z, and the coefficient of $\cos 2x$ is $2\cos(2w) = 2(2\cos^2(w) - 1) = z^2 - 2 = y$, verifying $\eta \in \mathcal{N}_2(y)$ and $\vartheta_2(y) \ge z = \sqrt{2+y}$. The translated measure $d\eta(x+\pi)$ shows by the same way $\beta_2(y) \ge \sqrt{2+y}$. These and (3.11) together concludes the proof of the Proposition. $\square$

Proposition 3.11.

i) *β and ϑ are concave functions on $\mathcal{D}(\beta) = \mathcal{D}(\vartheta) = [0,\infty)$.*

ii) *For all $n \in \mathbb{N}$ β_n and ϑ_n are concave functions on $\mathcal{D}(\beta_n) = \mathcal{D}(\vartheta_n) = [C_n,\infty)$.*

iii) *$\beta(y) = \vartheta(y) = \beta_n(y) = \vartheta_n(y) = 2$ for all $y \geq 2$ and $n \in \mathbb{N}$.*

iv) *For all $m > n$, $m,n \in \mathbb{N}$ and $y \geq C_m$, we have*

$$\beta_m(y) \leq \beta_n(y), \quad \vartheta_m(y) \leq \vartheta_n(y).$$

v) *For all $y \geq 0$ we have $\beta_n(y) \to \beta(y)$, $\vartheta_n(y) \to \vartheta(y)$ $(n \to \infty)$, uniformly in y.*

vi) *$\beta(0) = 1$, $\vartheta(0) \geq \dfrac{2}{\sqrt{3}}$.*

vii) *β and ϑ are strictly increasing in $[0,2]$; β_n and ϑ_n are strictly increasing in $[C_n,2]$ $(n \in \mathbb{N})$.*

Proof. i) Let $0 \leq y' \leq y \leq y''$ be arbitrary and $y = \lambda y' + (1-\lambda)y''$ be the representation of y. Note that here we have $0 \leq \lambda \leq 1$. Suppose that $\nu' \in \mathcal{N}(y')$ and $\nu'' \in \mathcal{N}(y'')$ and consider the measure

$$\nu := \lambda\nu' + (1-\lambda)\nu'' \in BM(\mathbb{T}) \tag{3.12}$$

which is nonnegative as $\lambda \geq 0$ and $1 - \lambda \geq 0$.

Plainly

$$\begin{aligned} \langle 1,\nu\rangle =& \lambda\langle 1,\nu'\rangle + (1-\lambda)\langle 1,\nu''\rangle = 1, \\ y_1 =& \langle 2\cos x,\nu\rangle = \lambda y_1' + (1-\lambda)y_1'', \\ y_k =& \langle 2\cos kx,\nu\rangle = \lambda y_k' + (1-\lambda)y_k'', \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} d\nu(x) &\sim 1 + \sum_{k=1}^{\infty} y_k \cos kx, \\ d\nu'(x) &\sim 1 + \sum_{k=1}^{\infty} y_k' \cos kx, \\ d\nu''(x) &\sim 1 + \sum_{k=2}^{\infty} y_k'' \cos kx. \end{aligned} \tag{3.14}$$

Now the first and the third lines of (3.13) prove $\nu \in \mathcal{N}(y)$ as $y_k = \lambda y'_k + (1-\lambda) y''_k \leq \lambda y' + (1-\lambda) y'' = y$ $(k \in \mathbb{N}_2)$. The second equation of (3.13) entails that

$$\begin{aligned}\vartheta(y) &= \sup\{y_1 : \nu \in \mathcal{N}(y)\} \\ &\geq \lambda \sup\{y'_1 : \nu' \in \mathcal{N}(y')\} + (1-\lambda) \sup\{y''_1 : \nu'' \in \mathcal{N}(y'')\} \\ &= \lambda \vartheta(y') + (1-\lambda)\vartheta(y'')\end{aligned} \tag{3.15}$$

and similarly to (3.15) we also have

$$\beta(y) = \sup\{-y_1 : \nu \in \mathcal{N}(y)\} \geq \lambda \beta(y') + (1-\lambda)\beta(y''), \tag{3.16}$$

proving concavity.

ii) Similar to i).

iii) Proposition 3.7 ii) and Proposition 3.8 iii) entail that all functions are constant for $y \geq 2$. The same coefficient estimate

$$|\langle 2\cos kx, \nu\rangle| \leq \langle 2, \nu\rangle = 2 \quad (k \in \mathbb{N},\ \nu \in \mathcal{N}(2)), \tag{3.17}$$

already used in the proof of Proposition 3.8 iii), shows that $|y_1| \leq 2$. Now $\beta_n(2) = \beta(2) = 2$ is shown by $\delta_\pi \in \mathcal{N}(2)$, and $\vartheta_n(2) = \vartheta(2) = 2$ is shown by $\delta \in \mathcal{N}(2)$.

iv) Trivial in view of $C_n \leq C_m \leq y$ and $\emptyset \neq \mathcal{N}_m(y) \subseteq \mathcal{N}_n(y)$.

v) For any fixed particular $y \geq 0$ we have $\mathcal{N}(y) = \bigcap_{n=2}^\infty \mathcal{N}_n(y)$. Therefore $\beta(y) \leq \beta_n(y)$ $(n \in \mathbb{N})$ is trivial. To prove convergence of $\beta_n(y)$ to $\beta(y)$ at the point y, let us denote for all $n \in \mathbb{N}$

$$N_n := \Big\{\nu_n \in \mathcal{N}_n(y) : \langle 2\cos x, \nu_n\rangle \leq -\beta_n(y)\Big\} = \mathcal{N}_n(y) \cap X\big(2\cos x, \beta_n(y)\big). \tag{3.18}$$

Note that in view of Proposition 3.9 $N_n \neq \emptyset$, and the sets N_n satisfy all the conditions of Lemma 2.1 in view of (3.6) and (3.18). Hence $N = \bigcap\limits_{n=2}^{\infty} N_n$ is nonempty. One can easily see that any $\nu \in N$ belongs to $\mathcal{N}(y)$ and

$$\langle 2\cos x, \nu\rangle \leq -\lim_{n\to\infty} \beta_n(y), \tag{3.19}$$

proving $\beta(y) \geq \lim\limits_{n\to\infty} \beta_n(y)$. Now we have $\beta_n(y) \to \beta(y)$ monotonically nonincreasingly in the pointwise sense on the whole $[0,2]$. But for the concave and hence continuous functions β_n and β that entails also uniform convergence on $[0,2]$ by Dini's monotone convergence criteria (cf. e.g. [7], (7.2.2), p. 129). With part iii) that settles uniform convergence, too. A similar argument works for ϑ as well.

vi) The easy examples $d\nu_+(x) = (1+\cos x)dx$, $d\nu_-(x) = (1-\cos x)dx$ show that $\vartheta(0) \geq 1$ and $\beta(0) \geq 1$. To show that $\vartheta(0) \geq 2/\sqrt{3}$, one may consider the trigonometric polynomial

$$h(x) = 1 + \frac{1}{\cos\frac{\pi}{6}}\cos x - \frac{\tan\frac{\pi}{6}}{3}\cos 3x = 1 + \frac{2}{\sqrt{3}}\cos x - \frac{1}{3\sqrt{3}}\cos 3x \tag{3.20}$$

and the corresponding measure $d\nu(x) = h(x)dx$. The only thing to check is $h \geq 0$, which can be done directly, or we can refer to the $k = 3$ case of Proposition 2.1 of [19]. On the other hand, Proposition 3.7 iii) entails that for $\nu \in \mathcal{N}(0)$ we can not have $\langle 2\cos x, \nu\rangle = y_1 < -1$, and this proves $\beta(0) = 1$.

vii) Follows from parts i), ii), iii) and Proposition 3.10. □

3.5. Proposition 3.12.

i) *For $1 \leq a < 2$ we have*

$$\omega(a) = a \cdot \max_{y>0} \frac{\beta(y) - 2/a}{y} + a + 1. \tag{3.21}$$

ii) *For $a \leq -1$, $a \in \mathcal{D}(\omega)$ we have*

$$\omega(a) = (-a)\max_{y>0} \frac{\vartheta(y) + 2/a}{y} + a + 1. \tag{3.22}$$

Remark. For $-1 \leq a \leq 1$ we have $\omega(a) = \alpha(a) = 1 + a$ according to Propositions 3.3 i) and Theorem 3.1 i). Also for $-1 \leq a \leq 1$

$$\sup_{y>0} \frac{a\cdot\beta(y) - 2}{y} = \lim_{y\to+\infty} \frac{a\cdot\beta(y) - 2}{y} = 0 \tag{3.23}$$

and

$$\sup_{y>0} \frac{(-a)\vartheta(y) - 2}{y} = \lim_{y\to+\infty} \frac{(-a)\vartheta(y) - 2}{y} = 0, \tag{3.24}$$

since $|a| \leq 1$, and $0 \leq \beta(y) \leq 2$, $0 \leq \vartheta(y) \leq 2$ (see Proposition 3.11) and hence the numerators of these functions are always nonpositive. In this sense the statement is valid for all a, but to emphasize the given forms, where sup is changed to max and a has been brought out, we used the above formulation.

P r o o f. i) For $a = 1$ the statement is trivial according to the above Remark. For $a > 1$ let us take two extremal measures $\tau \in \mathcal{M}(a)$ and $\nu_0 \in \mathcal{N}(y_0)$

with $\max_{y>0} \frac{\beta(y)-2/a}{y} = \frac{\beta(y_0)-2/a}{y_0}$. Since $a>1$ and $\beta(0)=1$ we see that $\varphi_a(y)=\varphi(y)=\frac{\beta(y)-2/a}{y}$ is negative for small $y>0$, while for $y=2$ $\varphi(2)=1-1/a>0$, and for $y>2$ $\quad \varphi(y)<\varphi(2)$ and $\varphi(y)\to 0$ $(y\to+\infty)$. Hence there exists a $y_0\notin(0,2]$, depending on a, where $\varphi_a(y_0)$ is a maximum. Also τ and ν_0 must exist in view of Proposition 3.9 and Theorem 3.1 i). First, we recall

$$\tau+\delta\geq t\cdot\lambda;\quad t=\omega(a) \tag{3.25}$$

and define

$$\mu:=\tau+\delta-t\cdot\lambda\geq 0, \tag{3.26}$$

where with the Fourier expansion in (1.22) and (1.23) we are led to

$$\begin{aligned} d\mu(x)\sim(b+1-t)+\left(2-\frac{2b}{a}\right)\cos x+\sum_{k=2}^{\infty}(2+t_k)\cos kx,\\ b\in\mathbb{R},\ t_k\leq 0\ (k\in\mathbb{N}_2). \end{aligned} \tag{3.27}$$

Now by $\mu\geq 0$ we have also $b+1-t\geq 0$. In case of $b+1-t=0$ the trivial argument (2.2) would give $\mu\equiv 0$, leading in view of the coefficient of $\cos x$ to the equation $b=a$ and thus $a+1-t=0$. But $t=\omega(a)=\alpha(a)\geq 2a>1+a$ for $a>1$ according to Propositions 3.3 ii) and 3.5 i), thus excluding $b+1-t=0$ for $a>1$. We get

$$b+1-t>0\quad\text{if}\quad a>1, \tag{3.28}$$

and we can introduce the new normalized measure

$$\nu:=\frac{1}{b+1-t}\mu,\quad d\nu(x)\sim 1+\frac{2(a-b)}{a(b+1-t)}\cos x+\sum_{k=2}^{\infty}\frac{2+t_k}{b+1-t}\cos kx. \tag{3.29}$$

Denoting

$$y_1:=-\frac{2(b-a)}{a(b+1-t)},\quad y_k:=-\frac{2+t_k}{b+1-t}\quad(k\in\mathbb{N}_2),\quad y:=\frac{2}{a(b+1-t)}, \tag{3.30}$$

we immediately get that $\nu\in\mathcal{N}(y)$ with the parameters and coefficients in (3.30). Consequently, we have by definition

$$\frac{2(b-a)}{a(b+1-t)}\leq\beta(y)\quad\left(y-\frac{2}{b+1-t}>0\right). \tag{3.31}$$

Let us use the definition of y and $t = \omega(a)$ in the left-hand side to express (3.31) by y and $\omega(a)$ as

$$\frac{y}{a}\left\{\left(\frac{2}{y} - 1 + \omega(a)\right) - a\right\} \le \beta(y), \tag{3.32}$$

or, after some calculation,

$$\omega(a) \le a \cdot \frac{\beta(y) - 2/a}{y} + a + 1. \tag{3.33}$$

That proves that the left-hand side of (3.21) can not exceed the right-hand side. Next we start by considering the extremal measure $\nu_0 \in \mathcal{N}(y_0)$ and define

(3.34)

$$b := a\left(\frac{\beta(y_0)}{y_0} + 1\right), \quad t := \frac{-2}{y_0} + 1 + b, \quad t_k := \frac{2(y_k - y_0)}{y_0} \quad (k \in \mathbb{N}_2),$$

$$\tau_0 := \frac{2}{y_0}\nu - \delta + t \cdot \lambda \in BM(\mathbb{T}), \quad d\tau_0(x) \sim b + \left(\frac{2}{y_0}y_1 - 2\right)\cos x + \sum_{k=2}^{\infty} t_k \cos kx.$$

We immediately have $t_k \le 0$ $(k \in \mathbb{N}_2)$ and from the extremality of $\nu_0 \in \mathcal{N}(y_0)$ we also have $y_1 = -\beta(y_0)$. Moreover, in view of the definition of b, we have for t_1, the coefficient of $\cos x$ in the Fourier expansion of τ_0, the equation

$$t_1 = \frac{-2\beta(y_0)}{y_0} - 2 = b\left(-\frac{2}{a}\right). \tag{3.35}$$

Now (3.34)–(3.35) yield $\tau_0 \in \mathcal{M}(a)$, and, as $\nu_0 \in \mathcal{N}(y_0)$ entails $\nu_0 \ge 0$, we immediately get $\tau_0 + \delta \ge t \cdot \lambda$ proving that

$$\omega(a) \ge t. \tag{3.36}$$

Now let us substitute the parameters (3.34) in (3.36) to obtain

$$\omega(a) \ge -\frac{2}{y_0} + 1 + a\left(\frac{\beta(y_0)}{y_0} + 1\right) = a \cdot \frac{\beta(y_0) - 2/a}{y_0} + a + 1. \tag{3.37}$$

Comparing (3.33) and (3.37) proves the assertion.

ii) The proof is very similar to i), hence we omit a few details and give here only the main steps and formulas. Again we suppose $a < -1$, and check that $\psi_a(y) := \psi(y) := \dfrac{\vartheta(y) + 2/a}{y}$ has a positive maximum attained for some y in $0 < y \le 2$. The $\le$ part will be proved by taking an extremal $\tau \in \mathcal{M}(a)$ and following the preceding argument from (3.23) up to (3.30) with the only alteration that here in place of (3.28) we have

$$b + 1 - t > 0 \quad \text{if} \quad a < -1 \tag{3.38}$$

because of the relations $t = \omega(a) \geq 0 > 1 + a$ $(a < -1)$. Now in place of (3.31) we will obtain from the extremality of τ that

$$\frac{2(b-a)}{(-a)(b+1-t)} \leq \vartheta(y) \qquad \left(y = \frac{2}{b+1-t} > 0 \right), \tag{3.39}$$

and similarly to (3.32)–(3.34), some calculation leads to

$$\omega(a) = t \leq (-a)\frac{\vartheta(y) + 2/a}{y} + a + 1. \tag{3.40}$$

The converse direction goes like (3.34)–(3.36) with the only change that here we take $y_1 = \vartheta(y_0)$ in place of $-\beta(y_0)$. Hence the same change occurs in (3.37) and we get the $\geq$ part. $\square$

Remark. We have to note here that implicitly we used that $\vartheta(y) + 2/a$ is positive only for $y > 0$, i.e. $\vartheta(0) \leq 2/|a|$. Now we really have

$$\vartheta(0) = \frac{2}{-A}, \tag{3.41}$$

a duality-type relation between different extremal problems, cf. [18], in particular the discussion around (2.9)–(2.14).

Note that this settles the existence of maximum for $\psi_a(y)$ in $y > 0$ for all $a > A$, but leaves the question open if $A \in \mathcal{D}(\alpha)$ and $a = A$. In this case for small y $(\vartheta(y) + 2/A)/y$ has a small, but positive numerator and the denominator is also positive. Thus we can extend ψ_A to 0 as

$$\begin{aligned} \psi_A(0) &= (-A) \lim_{y \to 0+} \frac{\vartheta(y) + 2/A}{y} + A + 1 \\ &= (-A) \lim_{y \to 0+} \frac{\vartheta(y) - \vartheta(0)}{y} + A + 1 = (-A)\vartheta'(0+) + A + 1 \end{aligned} \tag{3.42}$$

in case it is finite. In turn, if (3.42) is finite, by concavity we conclude that the maximum is attained at 0, and we conclude

$$\alpha(A) = \omega(A) = (-A)\vartheta'(0+) + A + 1. \tag{3.43}$$

On the other hand, if $\vartheta'(0+) = +\infty$, similarly to (3.37) it is easy to show that we will have $\lim_{a\to A+} \alpha(a) = +\infty$, and hence $A \notin \mathcal{D}(\alpha)$.

Similarly, from $\alpha(a) \to +\infty$ $(a \to 2-)$ we can conclude that

$$\beta'(0+) = +\infty. \tag{3.44}$$

Later even the asymptotic order of β will be specified, so we leave this question for the moment.

Let us point out the geometric interpretation of the maximum in (3.21). The concave curve $\{(y, \beta(y)) : y \geq 0\}$ defines a convex domain of points lying below the curve. The maximum is just the slope of one of the tangent straight lines drawn from the outer point $(0, 2/a)$ to this convex domain. (The other tangent is just the second coordinate axis.)

Proposition 3.13. *We have for all $n \in \mathbb{N}$ the relations*

i) $\beta_n(0) = \dfrac{2}{B_n}$;

ii) $\vartheta_n(0) = \dfrac{-2}{A_n}$;

iii) *For all $a \in [0, B_n]$*

$$\omega_n(a) = a \sup_{y>0} \frac{\beta_n(y) - 2/a}{y} + a + 1;$$

in particular,

$$\omega_n(B_n) = B_n \cdot \beta_n'(0+) + B_n + 1;$$

iv) *For all $a \in [A_n, 0]$*

$$\omega_n(a) = (-a) \sup_{y>0} \frac{\vartheta_n(y) + 2/a}{y} + a + 1;$$

in particular

$$\omega_n(A_n) = (-A_n) \cdot \vartheta_n'(0+) + A_n + 1.$$

P r o o f. Follows similarly to the argument of Proposition 3.12 and the Remark after it. We omit the details. □

Next we define another extremal quantity as follows.

$$
(3.45)\qquad \begin{aligned} Z := \inf\Big\{ y>0 : \exists\zeta \in BM(\mathbb{T}), \qquad \zeta \ge 0, \quad z \ge 2(1-y), \\ d\zeta(x) \sim 1 - z\cos x + \sum_{k=2}^{\infty} z_k \cos kx, \quad z_k \le y \quad (k \in \mathbb{N}_2) \Big\}. \end{aligned}
$$

Proposition 3.14.

i) *There exists a unique point y_U in $(0,2]$ so that*

$$\beta(y_U) = 2(1-y_U).$$

ii) *For the point y_U we have $Z = y_U$.*

iii) *We have $\Omega = \frac{2}{y_U} - 1 = \dfrac{2}{Z} - 1$.*

Proof. i) The functions $\beta(y)$ and $2(1-y)$ are continuous and strictly monotonous in the opposite direction from 0 to 2 and from 2 to -2 in the domain $[0,2]$. Hence there exists a unique solution of the equation $\beta(y) = 2(1-y)$ in the interval $(0,2)$.

ii) Denote the set of measures used in the definition of Z as $\mathcal{Z}(y)$. Then

$$
(3.46)\qquad Z := \inf\big\{ y > 0 : \mathcal{Z}(y) \ne \emptyset \big\}.
$$

Now if $y > Z$, we have $\mathcal{Z}(y) \ne \emptyset$, and, as $\mathcal{Z}(y) \subset \mathcal{N}(y)$, we find that $\beta(y) = \max\{-\langle 2\cos x, \zeta\rangle : \zeta \in \mathcal{Z}(y)\} \ge 2(1-y)$. Hence, in view of the definition of y_U and the monotonicity of $\beta(y)$ and $2(1-y)$, we conclude $y \ge y_U$ and a fortiori $Z \ge y_U$. Conversely, if $y > y_U$, then $\beta(y) > \beta(y_U) = \max\{\langle -2\cos x, \zeta\rangle : \zeta \in \mathcal{N}(y_U)\}$, and for any extremal measure $\zeta_0 \in \mathcal{N}(y_U)$, we have $\langle -2\cos x, \zeta_0\rangle = \beta(y_U) = 2(1-y_U)$, hence $\zeta_0 \in \mathcal{Z}(y_U)$ and $Z \le y_U$.

iii) Let $\mathcal{K}$ be the measure set in (1.18) where the defining supremum for Ω is defined. Note that $\mathcal{K}$ contains δ_π, hence $\mathcal{K} \ne \emptyset$. Moreover we have for any $\kappa \in \mathcal{K}$

$$
(3.47)\qquad \begin{aligned} 0 \le ||\kappa||_{BM(\mathbb{T})} = \int |d\kappa| = \int d\kappa = 2\pi \cdot b_0 \le 2\pi(2-b_1) = \\ = 2\pi\big(1 + (1-b_1)\big) \le 2\pi(1+\Omega), \end{aligned}
$$

and using the estimate $\Omega \le U$ (stated already in (1.19) as a result implicitly contained already in [27], and proven in Corollary 3.1 below) we immediately get that $\mathcal{K}$ is bounded. Note that $\mathcal{K}$ is also closed and convex, and can be represented in the form of the intersection of a set of closed halfspaces generated by functionals from $C(\mathbb{T})$, hence $\mathcal{K}$ is also weakly $*$ compact and the sup in (1.18) is actually a maximum. Now for the extremal measure $\kappa \in \mathcal{K}$ we consider its Fourier series (1.16) and prove that $b_0 > 1$ and $b_0 + b_1 = 2$ for κ. Indeed, in case $b_0 \le 1$ we must have $|b_1| \le 1$, $\Omega = 1 - b_1 \le 2$, and the known examples are much better than that. Also if $b_0 + b_1 < 2$, one can consider $d\kappa^*(x) = d\kappa(x) + \frac{2}{3}(2 - b_0 - b_1) \cdot (1 - \cos x)dx$, $\quad b_0^* + b_1^* = \langle 1 - \cos x, \kappa^* \rangle = b_0 + b_1 + \frac{2}{3}(2 - b_0 - b_1)\left(1 + \frac{1}{2}\right) = 2$, hence $\kappa^* \in \mathcal{K}$, and $b_1^* < b_1$ would provide a contradiction.

Now let us define the measure

$$\nu = \frac{1}{b_0} \cdot \kappa \ge 0.$$

Plainly $\nu \in \mathcal{N}\left(\frac{2}{b_0}\right)$, hence $\beta\left(\frac{2}{b_0}\right) \ge \frac{-2b_1}{b_0} = \frac{2(b_0-2)}{b_0} = 2\left(1 - \frac{2}{b_0}\right)$. Let y_0 be $\frac{2}{b_0}$, then we see $\beta(y_0) \ge 2(1 - y_0)$, hence $y_0 \ge y_U$. From this we get $b_0 = \frac{2}{y_0} \le \frac{2}{y_U}$, hence $\Omega = 1 - b_1 = b_0 - 1 \le \frac{2}{y_U} - 1$. Similarly, for y_U we can take any β-extremal measure $\nu \in \mathcal{N}(y_U)$ and consider the measure

$$\kappa = \frac{2}{y_U} \nu \in \mathcal{K}$$

proving $\Omega \ge 1 - b_1 = 1 + \frac{2}{y_U} \cdot \frac{\beta(y_U)}{2} = 1 + \frac{2}{y_U}(1 - y_U) = \frac{2}{y_U} - 1$. $\square$

Putting $[2, n]$ in place of $\mathbb{N}_2$ one can also introduce $\mathcal{Z}_n$ and $\mathcal{K}_n$, and the corresponding extremal quantities Z_n and Ω_n ($n \in \mathbb{N}_2$). It is no surprise now that we have the analogous

Proposition 3.15. *For arbitrary $n \in \mathbb{N}_2$ the following statements hold true.*

i) *There exists a unique point $y_{U,n}$ in $(0, 2]$ so that $\beta_n(y_{U,n}) = 2(1 - y_{U,n})$.*

ii) *For the point $y_{U,n}$ we have $Z_n = y_{U,n}$.*

iii) *We have $\Omega_n = \frac{2}{y_{U,n}} - 1 = \frac{2}{Z_n} - 1$.*

iv) *$Z_n \to Z$ monotonically increasingly, and $\Omega_n \to \Omega$ a nonincreasing way.*

Corollary 3.1. *We have $U = \Omega$ and also $U_n = \Omega_n$ $(n \in \mathbb{N}_2)$.*

Proof. As the proofs are very similar, we prove only $U = \Omega$. The easy part is $U \geq \Omega$, essentially already proved by van der Waerden [27] the idea dating back to Landau [15]. Indeed, let $f \in \mathcal{F}(a)$ and $\kappa \in \mathcal{K}$ be any particular elements, we then have by $f \geq 0$, $\kappa \geq 0$ and using $b_k \leq 1$, $a_k \geq 0$ $(k \in \mathbb{N}_2)$ that

$$0 \leq \langle f, \kappa \rangle = b_0 + ab_1 + \sum_{k=2}^{\infty} a_k b_k \leq (b_0 - 1) + a(b_1 - 1) + \left(1 + a + \sum_{b=2}^{\infty} a_k\right) = b_0 - 1 + a(b_1 - 1) + f(0). \tag{3.48}$$

We also apply $b_0 + b_1 \leq 2$ for $\kappa \in \mathcal{K}$, and get for $a > 1$ the inequalities

$$(1 - b_1)(a - 1) \leq b_1 - 1 + b_0 - 1 + f(0) \leq f(0). \tag{3.49}$$

Now let us take supremum over $\mathcal{K}$ at the left, and infimum over $\mathcal{F}(a)$ at the right-hand side to get

$$\Omega(a - 1) \leq \alpha(a). \tag{3.50}$$

Dividing by $a - 1$ (> 0) and minimizing $U(a) = \dfrac{\alpha(a)}{a-1}$, we get $\Omega \leq U$. (Note that the minimum place is $a = a_U$, cf. Proposition 3.6 iv) for the definition and uniqueness.)

Now let us prove the converse! We start with noting that by Proposition 3.14 iii) $\Omega = \dfrac{2}{y_U} - 1$, and choose $a = a_\Omega$ such that the maximum at the right-hand side of (3.21) is attained at $y = y_U$. Note that $0 < y_U < 0.5$ is trivial, and for (any one of the) tangential lines of β at the point $(y_U, \beta(y_U))$ the intersection point of the straight line with the second coordinate axis defines such an a_Ω by $\left(0, \dfrac{2}{a}\right)$ being the intersection point. Hence we conclude the existence of such an a_Ω. Consequently, with $a = a_\Omega$ and using the Duality Theorem (Theorem 3.1 i)), we get

$$U \leq \frac{\alpha(a_\Omega)}{a_\Omega - 1} = \frac{\omega(a_\Omega)}{a_\Omega - 1} = \frac{1}{a_\Omega - 1}\left(a_\Omega \frac{\beta(y_U) - 2/a_\Omega}{y_U} + a_\Omega + 1\right) = \tag{3.51}$$
$$= \frac{1}{a-1}\left(a\frac{2(1-y)}{y} - \frac{2}{y} + a + 1\right) = \frac{1}{a-1}\left((a-1)\frac{2}{y} - a + 1\right) = \frac{2}{y_U} - 1 = \Omega.$$

□

We may note that the above mentioned duality relation enables us to give another form of $\omega(a)$, which has the interesting feature that only the goal function to be maximalized is dependent on a, but not the set of measures on what the maximization takes place. Namely, we have

$$\alpha(a) = \omega(a) = \sup\big\{(1-b_1)(a-1) + (2-b_0-b_1) : \exists\kappa\in\mathcal{K} \quad (\text{with } (1.16)\text{–}(1.17))\big\}. \tag{3.52}$$

Indeed, let us define the right-hand side as $\zeta(a)$, and define also the auxiliary quantity

$$w(b) := \sup\big\{1-b_1 : \exists\kappa\in\mathcal{K}, \quad b_0 = b \quad (\text{with } (1.16)\text{–}(1.17))\big\}. \tag{3.53}$$

Plainly with $y := \dfrac{2}{b}$ the function $w(b)$ is related to $\beta(y)$ as

$$w(b) = 1 + \frac{b}{2}\beta(y) \qquad \left(y := \frac{2}{b}\right), \tag{3.54}$$

since for $\kappa\in\mathcal{K}$ with $b_0 = b$ the measure $\nu := \frac{1}{b}\kappa\in\mathcal{N}(y)$, and for $\nu\in\mathcal{N}(y)$ the measure $\kappa := \frac{2}{y}\nu\in\mathcal{K}$. Plainly

$$\begin{aligned}\zeta(a) &= \sup\big\{(1-b_1)a + (1-b_0) : \exists\kappa\in\mathcal{K} \quad (\text{with } (1.16)\text{–}(1.18))\big\} = \\ &= \sup\big\{(1-b_0) + aw(b_0) : \kappa\in\mathcal{K}\big\} = \sup_{b_0>0}\left\{1 - b_0 + a\left(1 + \frac{b_0}{2}\beta\left(\frac{2}{b_0}\right)\right)\right\} \\ &= \sup_{y>0}\left\{a\left(1+\frac{\beta(y)}{y}\right) + 1 - \frac{2}{y}\right\} = a\sup_{y>0}\frac{\beta(y)-2/a}{y} + a + 1 = \omega(a)\end{aligned}$$

by (3.22), Proposition 3.12 i).

4. Concluding remarks and further questions. One can ask if Landau's extremal problems are interesting even now. We have already mentioned that they can be of interest for practical applications, in particular for computational number theory, as in [22]. Let us mention a further point and comment connections to recent publications like [8] and [12].

Already Landau proved the Prime Ideal Theorem, and later on further generalizations of Dirichlet's and Riemann's approach (use of multiplicative generating functions, i.e. Dirichlet series, in the study of multiplicative problems) appeared. A quite general setup is the *Beurling theory of prime distribution.* Now for Beurling primes the de la Vallée Poussin-Landau method works, but no other refined techniques can be utilized, since there are counterexamples: Diamond, Montgomery and Vorhauer [8] has constructed recently a Beurling set of primes so that no better zero-free region of $\zeta(s)$, and no better error term of $\pi(x)$, can be established than (1.6) and $xe^{-c\sqrt{\log x}}$, respectively.

Also, these problems are related to, or similar to, and have common generalizations with many other important families of extremal problems. So we are convinced that further research of them has merit not only for the analytical beauty and difficulty of them. Hence let us end this work by listing a few questions.

1.) The asymptotic order of $\alpha(a)$ when $a \to 2-0$ was determined in [19]. It is of interest to obtain more precise descriptions of values of $\alpha(a)$, in particular when $a \to 2-0$.

2.) We have seen that the extremal function in the α-problem is a polynomial when say $-3/4 < a < \sqrt{2}$. (We can calculate this a bit further.) Do we have for all $a \in \mathcal{D}(\alpha)$ that there is an $N := N(a)$ so that $\alpha(a) = \alpha_N(a)$? (If so, the "right" (minimal) degree $N(a) \to \infty$ when $a \to 2-0$.) Having $N(1.85)$, say, would allow to *exactly* determine U, V, W by finite range computer search.

3.) We have seen (Chakalov) that sometimes $U_{k+1} = U_k$. It seems that in the dual (van der Waerden-type) extremal problem $\omega(a)$ and Ω for measures, we have vanishing Fourier coefficients for exactly those indices $k \in \mathbb{N}$. Prove or disprove!

4.) Determine A and A_n (left endpoints of $\mathcal{D}(\alpha)$ and of $\mathcal{D}(\alpha_n)$, resp.). These lead to extremal problems in themselves: for how large an a can an even Fourier series $g(x)$ (a cosine polynomial of degree n) with $\widehat{g}(0) = 1$ and $\widehat{g}(1) = 0$ be strictly positive definite while $g(x) + a\cos x \geq 0$?

5.) It is possible to consider similar, however not only positive definite, but *signed* Landau-problems, i.e. instead of $a_k \geq 0$ we can assume arbitrary sign conditions on various k's. We already have the duality [18]. Note that considering negative a leads to this question naturally.

6.) Let G be a locally compact Abelian group. Develop the similar theory.

We have emphasized several times that the Landau extremal problems are related to many classical and current extremal problems. In relation to questions 5.) and 6.), let us give an example for the Landau problem with some sign conditions on groups.

Example 4.1. Assume, as sign condition, that $\widehat{f} = 0$ outside D, where $D \subset \widehat{G}$ is a domain in the dual group. (E.g. if $G = \mathbb{T}$ and $\widehat{G} = \mathbb{Z}$, then considering polynomials of degree $\leq n$ is equivalent to assume $\widehat{f} = 0$ outside $[-n, n]$.) Then the problem is about the minimal value of $f(0)$ when $a = a_1 = 2\widehat{f}(1)$ is given.

Changing the role of G and $\widehat{G}$ and taking $\varphi := \widehat{f}$, we obtain the following extremal problem:
i) $\varphi = 0$ on $G \setminus D$, i.e. φ is supported in D;
ii) $\widehat{\varphi} \geq 0$, i.e. φ is positive definite;
iii) $\varphi(0) = 1$ (normalization);
iv) $\varphi(1) = a/2$;
and then we seek to minimize $\int \varphi = \widehat{\varphi}(0)$.

If one only looks for the largest possible value of a so that the problem has a finite solution, (e.g. if we look for B_n), then the extremal problem becomes maximization of $\varphi(1)$ under the conditions (i)–(iii) given. This is called "pointwise Turán problem" (although in $\mathbb{R}$ was already considered by Boas and Kac [3] in the forties). See [12] and the references therein. In the special case of α_n, it is an extremal problem solved by Fejér [10] and Szász [24] – that is the exact value of B_n given above.

REFERENCES

[1] V. V. Arestov. On an extremal problem for nonnegative trigonometric polynomials. *Trudy Inst. Math. Mech. Ukr. Acad. Nauk.* **1** (1992), 50–70 (in Russian).

[2] V. V. Arestov, V. P. Kondrat'ev. On an extremal problem for nonnegative trigonometric polynomials. *Mat. Zametki*, **47**, *1* (1990), 15–28 (in Russian, translated as Mathematical Notes).

[3] R. P. Jr. Boas, M. Kac. Inequalities for Fourier Transforms of positive functions. *Duke Math. J.* **12** (1945), 189–206.

[4] L. Tschakaloff. Trigonometrische Polynome mit einer minimumeigenschaft. *Jahrbuch der Universität Sofia, Phys-Math. Fakultät* Bd. **19** (1923), 355–387 (in Bulgarian).

[5] L. Tschakaloff. Trigonometrische Polynome mit einer Minimumeigenschaft. *Annali di Scuola Normale Superiore di Pisa (2)* **9** (1940), 13–26.

[6] K. Chandrasekharan. Arithmetical Functions. Springer, Berlin–Heidelberg–New York, 1970.

[7] J. Dieudonné. Foundations of modern analysis. Academic Press, New York–London, 1960.

[8] H. G. Diamond, H. L. Montgomery, U. M. Vorhauer. Beurling primes with large oscillation. *Math. Ann.* **334**, *1* (2006), 1–36.

[9] R. E. Edwards. Functional Analysis. Holt, Rinehart and Winston, New York–Toronto–London, 1965.

[10] L. Fejér. Über trigonometrische Polynome. *J. angew. Math.* **146** (1915), 53–82; see also L. Fejér. Gesammelte Arbeiten. Akadémiai Kiadó, Budapest, 1970, Bd. I, 842–872.

[11] S. H. French. Trigonometric polynomials in prime number theory. *Illinois J. Math.* **10** (1966), 240–248.

[12] M. N. Kolountzakis, Sz. Gy. Révész. On pointwise estimates of positive definite functions with given support. *Canad. J. Math.* **58**, *2* (2006), 401–418.

[13] V. P. Kondrat'ev. On some extremal properties of positive trigonometric polynomials. *Mat. Zametki* **22**, *3* (1977), 371–374 (in Russian, translated as Mathematical Notes).

[14] E. Landau. Beitrage zur analitische Zahlentheorie. *Rendiconti del Circolo Matematico di Palermo* **26** (1908), 169–302.

[15] E. Landau. Handbuch der Lehre von der Verteilung der Primzahlen. Teubner, Leipzig–Berlin, 1909.

[16] E. Landau. Eine Frage über trigonometrische Polynome. *Annali di Scuola Normale Superiore di Pisa (2)* **2** (1933), 209–210.

[17] E. Landau. Nachtrag zu meiner Arbeit "Eine Frage über trigonometrische Polynome", *Annali di Scuola Normale Superiore di Pisa (2)* **5** (1936), 141.

[18] Sz. Gy. Révész. The least possible value at zero of some nonnegative cosine polynomials and equivalent dual problems. *Journal of Fourier Analysis and Applications*, Kahane Special Issue (1995), 485–508.

[19] Sz. Gy. Révész. Minimization of maxima of nonnegative and positive definite cosine polynomials with prescribed first coefficients. *Acta Sci. Math. (Szeged)* **60** (1995), 589–608.

[20] A. V. Reztsov. Certain extremal properties of nonnegative trigonometric polynomials. *Mat. Zametki* **39**, *2* (1986), 245–252 (in Russian, translated as Mathematical Notes).

[21] R. T. Rockafellar. Convex Analysis. Reprint of the 1970 original, Princeton University Press, Princeton, 1997.

[22] J. B. Rosser, L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.* **6** (1962), 64–92.

[23] J. B. Rosser, L. Schoenfeld. Sharper Bounds for the Chebyshev Functions $\vartheta(x)$ and $\psi(x)$. *Mathematics of Computation* **29**, *129* (1975), 243–269.

[24] O. Szász. Elementare Extremalprobleme über nichtnegative trigonometrische Polynome. *Sitzungsberichte Bayerische Akad. der Wiss. Math.-Phys. Kl.* (1927), 185–196; see also O. Szász. Collected Mathematical Papers, University of Cincinnati, 1955, 734–735.

[25] S. B. Stechkin. On some extremal properties of nonnegative trigonometric polynomials. *Mat. Zametki* **7**, *4* (1970), 411–422 (in Russian).

[26] Ch.-J. de la Vallée Poussin. Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inferieurs a une limite donnee. Belg. Mem. cour. in 8°, **59** (1899), 1–74.

[27] B. L. van der Waerden. Über Landau's Beweis des Primzahlsatzes. *Math.-Zeitschrift* **52** (1949), 649–653.

[28] H. Westphal. Über die Nullstellen der Riemannschen Zetafunction im kritischen Streifen. Schriften des Math. Seminars und des Instituts für angewandte Math. der Universität Berlin **4** (1938), 1–31.

[29] A. Walfisz. Weylsche Exponentialsummen in der neueren Zahlentheorie. Berlin, 1963.

A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
Reáltanoda u. 13-15
1054 Budapest, Hungary
e-mail: `reveszrenyi.hu`
and
Institut Henri Poincaré
11 rue Pierre et Marie Curie
75005 Paris, France
e-mail: `reveszihp.jussieu.fr`

Received September 30, 2006
Revised March 3, 2007