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A NEW CHARACTERIZATION OF WEIGHTED PEETRE *K*-FUNCTIONALS (II)*

Borislav R. Draganov†, Kamen G. Ivanov

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Abstract. Certain types of weighted Peetre K-functionals are characterized by means of the classical moduli of smoothness taken on a proper linear transforms of the function. The weights with power-type asymptotic at the ends of the interval with *arbitrary real* exponents are considered. This paper extends the method and results presented in [3].

1. Introduction. Let I be an open interval on the real line and let the weights w and φ on I be defined in Table 1, where the γ's, λ's, a and b are arbitrary real numbers.

We denote the weighted L_p-space by $L_p(w)(I) = \big\{f : wf \in L_p(I)\big\}$, $1 \le p \le \infty$. The set of the absolutely continuous functions on $[a_1, b_1]$ is denoted by

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Table 1. Weights

I	$w(x)$	$\varphi(x)$
(a,b)	$(x-a)^{\gamma_a}(b-x)^{\gamma_b}$	$(x-a)^{\lambda_a}(b-x)^{\lambda_b}$
(a,∞)	$(x-a)^{\gamma_a}(x-a+1)^{\gamma_\infty-\gamma_a}$	$(x-a)^{\lambda_a}(x-a+1)^{\lambda_\infty-\lambda_a}$
$\mathbb{R}=(-\infty,\infty)$	$\begin{cases} \lvert x\rvert^{\gamma_{-\infty}}, & x<-1, \\ 1, & -1\le x\le 1, \\ x^{\gamma_{+\infty}}, & x>1. \end{cases}$	$\begin{cases} \lvert x\rvert^{\lambda_{-\infty}}, & x<-1, \\ 1, & -1\le x\le 1, \\ x^{\lambda_{+\infty}}, & x>1. \end{cases}$

$AC[a_1,b_1]$ and $AC^k_{loc}(I)=\{g\,:\,g,g',\dots,g^{(k)}\in AC[a_1,b_1]\ \forall a_1,b_1\in I,\ a_1<b_1\}$. By D we denote the first derivative, $D=\frac{d}{dx}$, and $D^r g$ means the r-th derivative of the function g.

The weighted Peetre K-functional is given by

$$
\begin{aligned}
(1.1)\quad K(f,t^r;L_p(w)(I),AC^{r-1}_{loc},\varphi^r D^r) = \inf\big\{&\|w(f-g)\|_{p(I)}+t^r\|w\,\varphi^r D^r g\|_{p(I)} \\
&: g\in AC^{r-1}_{loc}(I),\ wg, w\varphi^r D^r g\in L_p(I)\big\}.
\end{aligned}
$$

It is defined for every $f\in L_p(w)(I)$ and $t>0$. Note that $g\in AC^{r-1}_{loc}(I)$ means that the infimum in (1.1) is taken on the largest possible subspace of $L_p(w)(I)$. If $p=\infty$, then in (1.1) the space $L_\infty(w)(I)$ can be replaced by $C(w)(I)=\{g\,:\,wg\in C(I)\}$, where $C(I)$ is the space of all continuous *bounded* on I functions. The case of $C(w)(I)$ is considered in the last section, while in the previous sections the space $L_\infty(w)(I)$ is understood when $p=\infty$.

The class of functions f for which we can calculate exactly the infimum in (1.1) for any $t\in(0,t_0]$ is quite narrow. That is why it is useful to have other function characteristics – moduli of smoothness – which can be calculated for a wider class of functions and are equivalent to the K-functional. Up to now several definitions of moduli of smoothness have been introduced to treat K-functionals acting on weighted L_p-spaces: Ivanov [9, 10], Ditzian and Totik [2], Ky [11], etc. The ideas in these papers are not suitable to treat the case $\lambda_a<0$, $\gamma_a\ne 0$ or the case $0\le\lambda_a<1$, $\gamma_a<0$. In all of these approaches the definitions of the moduli of smoothness are modified in order to fit the weights in (1.1).

In this article we present a characterization of K-functionals (1.1) by the classical (unweighted, fixed step) moduli of smoothness as the latter are taken not on the function f itself but on certain modifications of it. This approach can cover the K-functionals (1.1) *for all real values of* γ*'s and* λ*'s.* These characterizations will be valid not only for the weights w and φ listed in Table 1 but for any other weights $\bar{w}$ and $\bar{\varphi}$ equivalent to them on I. For treatment of weights with more general asymptotic at the end-points of the domain I see [3, Section 6]. Examples of applications of weighted K-functionals to some areas of the approximation theory, as characterization of the rate of convergence, are given in [2, 3, 6].

In [3] we have applied the following approach in order to get characterizations of K-functionals (1.1) for certain values of the parameters. First, we find a linear operator $\mathcal{A}$ which provides the equivalence

$$K(f,t^r;L_p(w)(I),AC_{loc}^{r-1},\varphi^r D^r) \sim K(\mathcal{A}f,t^r;L_p(\tilde{w})(\tilde{I}),AC_{loc}^{r-1},\tilde{\varphi}^r D^r) \tag{1.2}$$

with $\tilde{w}=1,\tilde{\varphi}=1$ for a proper interval $\tilde{I}$ of the types listed above. Next, since the second K-functional in (1.2) is equivalent to the classical r-th modulus of smoothness ω_r we get

$$K(f,t^r;L_p(w)(I),AC_{loc}^{r-1},\varphi^r D^r) \sim \omega_r(\mathcal{A}f,t)_{p(\tilde{I})}. \tag{1.3}$$

As usual $\psi_1(F,t)\sim\psi_2(F,t)$ means that the ratio of the functions ψ_1 and ψ_2 is bounded between two positive numbers independent of F and t. Relation (1.2) for general w, φ, $\tilde{w}$, $\tilde{\varphi}$ is of independent interest. For more details see Subsection 6.5.

The values of the parameters for which (1.3) was proved in [3] can be summarized as: $I=(a,b)$; $\lambda_a,\lambda_b<1$; $\gamma_a,\gamma_b>-1/p$; $1\le p<\infty$. (For $p=\infty$ only the case $\gamma_a,\gamma_b=0$ was solved.) In this article we extend the assumptions for the validity of (1.3) in the following directions:

(a) The restriction on the powers of the weight φ is $\lambda\neq 1$. Here λ stands for $\lambda_a,\lambda_b,\lambda_{\pm\infty}$. (The cases when some of the λ's are 1 need special treatment – see [4, 6].)

(b) I is an interval of the types listed above. On the one hand K-functionals between functional spaces on unbounded intervals are naturally connected with several sequences of operators, e.g. Sasz-Mirakyan's, Baskakov's, Post-Widder's, etc. On the other hand, let us emphasize that when treating the case when there is a $\lambda>1$ we arrive at the necessity to consider unbounded intervals. Indeed, in (1.3) if we have $I=(a,b)$ and $\lambda_a<1$, $\lambda_b>1$ then $\tilde{I}=(a,\infty)$ and if we have $I=(a,\infty)$ and $\lambda_a<1$, $\lambda_\infty>1$ then $\tilde{I}=(a,b)$.

(c) The restriction on the powers of the weight w for $1\le p<\infty$ is relaxed from $\gamma>-1/p$ to $\gamma\neq 1-r-1/p,2-r-1/p,\dots,-1/p$ at both ends of the interval. (Here and below γ stands for $\gamma_a,\gamma_b,\gamma_{\pm\infty}$.) This might be considered as the main achievement of the article because the treatment of such weights is impeded by difficulties, some of which are not only of technical nature (cf. the "finite overlapping condition", which is essential for Ditzian-Totik moduli [2, p. 8]). In order to get such results we generalize [3, Proposition 2.1] to Proposition 2.1 below by replacing the inverse operator by another one which we call "quasi-inverse". Let us note that the inverse of the operators $\mathcal{A}$ (constructed, for example, in Theorems 6.2, 6.5) are bounded only if the γ's are bigger than $-1/p$. For the other values of γ discussed in the article we construct different quasi-invertible continuous operators depending on the range to which γ belongs. The values $\gamma = 1-r-1/p, 2-r-1/p, \ldots, -1/p$ are *exceptional* in the sense that the constructed here $\mathcal{A}$ has no quasi-inverse bounded operators built by the operators from Sections 3 and 4. These values are not treated in this paper except $\gamma = 0$ for $p = \infty$. The construction of a proper operator $\mathcal{A}$ such that (1.3) is fulfilled for the exceptional values of γ demands new elements and will be given in [7].

The construction of operators $\mathcal{A}$ is explicit. It involves linear and power changes of the variable, multiplication by power functions, including algebraic polynomials, and antiderivative. In particular, fractional integrals are extensively used. Let us note that the computation of the classical unweighted moduli of the function $\mathcal{A}f$ is of the same degree of difficulty as of the function f itself.

The cases when (1.2) holds are summarized in Theorems 6.12, 6.13 and 6.14, while (1.3) is true under the conditions described in Theorem 6.15. Similarly to [3] the results concerning the validity of (1.3) are mainly in the case $1 \le p < \infty$ (with restrictions $\gamma_a = 0, \gamma_b = 0, \gamma_\infty = 0$ for $p = \infty$), while the validity of (1.2) is established under the condition $1 \le p \le \infty$. The reasons for such discrepancy are discussed in [7], where the case $p = \infty$ is studied in detail.

Finally, let us mention one tool of technical nature, which simplifies the problem for characterizing the K-functionals (1.1) in terms of moduli of smoothness. This is Lemma 7.1, which allows us to separate the singularities of the weights w and φ by "splitting" the interval I beforehand. After the usage of Lemma 7.1 we get two K-functionals whose weights w and φ have the initial power-type behaviour at one of the ends (finite or infinite) of the interval while at the other *finite* end of the interval the weights w and φ are equivalent to 1, i.e. $\gamma = \lambda = 0$ there. After applying Lemma 7.1 characterization (1.3) is modified to

$$(1.4) \qquad K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}_1 f, t)_{p(\tilde{I}_1)} + \omega_r(\mathcal{A}_2 f, t)_{p(\tilde{I}_2)}.$$

Note that Lemma 7.1 allows the unification of two moduli (or K-functionals) into a single one only if the underlying functions *coincide* on the intersection of the domains. As far as this is a rare case, one disadvantage of the application of Lemma 7.1 is the increase of the number of moduli used on the right-hand side of (1.4). So, we apply Lemma 7.1 only at the end of our study, having also in mind that the validity of (1.2) is of independent importance. The cases when we cannot avoid the separation of the singularities are listed at the end of Subsection 6.5.

The paper is organized as follows. The next section contains a variety of auxiliary results. In Sections 3 and 4 we construct and study two kinds of operators – type A and type B respectively – used as ingredients in the solutions of (1.2), (1.3) and (1.4). The type A operators change only the weight w, while the type B operators change simultaneously both weights w and φ. Some algebraic properties of these operators are listed in Section 5. Operators that are solutions of (1.2) and (1.3) under a variety of conditions on I, w and φ are constructed in Section 6, while the solutions of (1.4) are given in Section 7. Results about continuous functions are sketched in Section 8.

2. Preliminaries

2.1. Notations. Throughout the paper we shall use the following notations.

The order of the derivative in the K-functional is always denoted by $r \in \mathbb{N}$. For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ let Π_n be the set of all algebraic polynomials of degree at most n. For $\xi \in \mathbb{R}$ set $\chi_\xi(x) = |x - \xi|$.

The restrictions on the parameters γ are described by:

$$
\begin{aligned}
&\Gamma_+(p) = (-1/p, \infty) \text{ for } 1 \le p < \infty \text{ and } \Gamma_+(\infty) = [0, \infty);\\
&\Gamma_0(p) = (-1/p, \infty);\\
&\Gamma_i(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 1, \dots, r-1;\\
&\Gamma_r(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1/p\};\\
&\Gamma_1^*(p) = (-1 - 1/p, \infty);\\
&\Gamma_i^*(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 2, \dots, r-1;\\
&\Gamma_r^*(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}^*(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1 - 1/p\}.
\end{aligned}
$$

Intervals $\Gamma_i(p)$ and $\Gamma_i^*(p)$ coincide for $i = 2, \dots, r$, but we prefer using the two sets of notations to be able to state the results more shortly.

In few cases the standard convention $\zeta < \eta$ in the notation of the interval (ζ, η) may be violated. Then (ζ, η) has to be understood as the interval $(\min\{\zeta, \eta\}, \max\{\zeta, \eta\})$.

We denote by c positive numbers independent of the functions f and the parameter t of the K-functionals. The numbers c may differ at each occurrence. All constants denoted by c can be explicitly evaluated using algebraic expressions and the constants in the Hardy-type inequalities (which are known).

2.2. Quasi-invertible continuous maps. In order to give a general approach in establishing an equivalence like (1.2) we define the Peetre K-functional between abstract spaces by

$$K(f,t) = K(f,t;X,Y,\mathcal{D}) = \inf\big\{\|f-g\|_X + t\|\mathcal{D}g\|_X \ : \ g \in Y \cap \mathcal{D}^{-1}(X)\big\},$$

where X is a Banach space, $\mathcal{D}$ is a differential operator and $\mathcal{D}^{-1}(X) = \{g \in X : \mathcal{D}g \in X\}$. Usually $Y \cap \mathcal{D}^{-1}(X)$ is a dense subspace of X. In the notations of (1.1) $X = L_p(w)(I)$, $Y = AC_{loc}^{r-1}$ and $\mathcal{D} = \varphi^r D^r$. We call $(X, Y, \mathcal{D})$ a triplet.

For a differential operator $\mathcal{D}$, acting on a subspace of the Banach space X, we set $\ker \mathcal{D} = \{g \in \mathcal{D}^{-1}(X) : \mathcal{D}g = 0\}$. Note that $\ker \mathcal{D} \subset \mathcal{D}^{-1}(X) \subset X$.

We shall need a certain generalization of Proposition 2.1 in [3] in order to extend the results there to some of the functional classes discussed in this paper. First, we introduce the following

Definition 2.1. *We say that the linear operator $\mathcal{A}$ is **a quasi-invertible continuous map of the triplet** $(X_1, Y_1, \mathcal{D}_1)$ **onto the triplet** $(X_2, Y_2, \mathcal{D}_2)$ if and only if there exists a linear operator $\mathcal{B} : X_2 \to X_1$, related to $\mathcal{A} : X_1 \to X_2$, which we call **a quasi-inverse operator to** $\mathcal{A}$, and both operators satisfy the conditions:*

(a) $\|\mathcal{A}f\|_{X_2} \le c\|f\|_{X_1}$ *for any* $f \in X_1$*;*

(b) $\|\mathcal{D}_2\mathcal{A}f\|_{X_2} \le c\|\mathcal{D}_1 f\|_{X_1}$ *for any* $f \in Y_1 \cap \mathcal{D}_1^{-1}(X_1)$*;*

(c) $\|\mathcal{B}F\|_{X_1} \le c\|F\|_{X_2}$ *for any* $F \in X_2$*;*

(d) $\|\mathcal{D}_1\mathcal{B}F\|_{X_1} \le c\|\mathcal{D}_2 F\|_{X_2}$ *for any* $F \in Y_2 \cap \mathcal{D}_2^{-1}(X_2)$*;*

(e) $\mathcal{A}(Y_1 \cap \mathcal{D}_1^{-1}(X_1)) \subseteq Y_2 \cap \mathcal{D}_2^{-1}(X_2)$*;*

(f) $\mathcal{B}(Y_2 \cap \mathcal{D}_2^{-1}(X_2)) \subseteq Y_1 \cap \mathcal{D}_1^{-1}(X_1)$*;*

(g) $f - \mathcal{B}\mathcal{A}f \in Y_1 \cap \ker \mathcal{D}_1$ *for any* $f \in X_1$*;*

(h) $F - \mathcal{A}\mathcal{B}F \in Y_2 \cap \ker \mathcal{D}_2$ *for any* $F \in X_2$*.*

If $\mathcal{A}$ *is a quasi-invertible continuous map of* $(X_1, Y_1, \mathcal{D}_1)$ *onto* $(X_2, Y_2, \mathcal{D}_2)$ *and* $\mathcal{B}$ *is a quasi-inverse operator to* $\mathcal{A}$*, we write*

$$\mathcal{A} : (X_1, Y_1, \mathcal{D}_1) \rightleftharpoons (X_2, Y_2, \mathcal{D}_2) : \mathcal{B}.$$

Remark 2.1. When the operator $\mathcal{A} : X_1 \to X_2$ is invertible and its inverse $\mathcal{A}^{-1}$ is bounded and satisfies conditions (d) and (f) (in the place of $\mathcal{B}$), then $\mathcal{A}^{-1}$ is a quasi-inverse operator to $\mathcal{A}$. This case was considered in [3].

Remark 2.2. Note that if $\mathcal{A} : (X_1, Y_1, \mathcal{D}_1) \rightleftharpoons (X_2, Y_2, \mathcal{D}_2) : \mathcal{B}$, then $\mathcal{A}$ is a quasi-inverse operator to $\mathcal{B}$ and $\mathcal{B} : (X_2, Y_2, \mathcal{D}_2) \rightleftharpoons (X_1, Y_1, \mathcal{D}_1) : \mathcal{A}$. We use a notation, which points out a quasi-inverse operator because we need relations between the triplets $(X_1, Y_1, \mathcal{D}_1)$ and $(X_2, Y_2, \mathcal{D}_2)$ in both directions. Let us also note that the quasi-inverse operator may not be unique, which is the case considered in this article.

Sometimes we call "initial" the triplet $(X_1, Y_1, \mathcal{D}_1)$ or the weights in its functional spaces, while we call "target" the triplet $(X_2, Y_2, \mathcal{D}_2)$. Such terminology, which has no strict mathematical meaning, reflects the process of construction of the operator $\mathcal{A}$, which goes through several intermediate triplets between $(X_1, Y_1, \mathcal{D}_1)$ and $(X_2, Y_2, \mathcal{D}_2)$. Also the roles of $(X_1, Y_1, \mathcal{D}_1)$ and $(X_2, Y_2, \mathcal{D}_2)$ in the current investigation are slightly different – from the point of view of (1.3) the target triplet in (1.2) is described with the specific weights $\tilde{w} = 1$ and $\tilde{\varphi} = 1$.

A connection between quasi-invertible continuous maps and equivalence of K-functionals is given in:

Proposition 2.1. *Let the linear operator* $\mathcal{A}$ *be a quasi-invertible continuous map of* $(X_1, Y_1, \mathcal{D}_1)$ *onto* $(X_2, Y_2, \mathcal{D}_2)$ *and* $\mathcal{B}$ *be quasi-inverse to* $\mathcal{A}$*. Then for any* $f \in X_1$ *and* $t > 0$ *we have*

$$K(f, t; X_1, Y_1, \mathcal{D}_1) \sim K(\mathcal{A}f, t; X_2, Y_2, \mathcal{D}_2) \tag{2.1}$$

and for any $F \in X_2$ *and* $t > 0$ *we have*

$$K(F, t; X_2, Y_2, \mathcal{D}_2) \sim K(\mathcal{B}F, t; X_1, Y_1, \mathcal{D}_1). \tag{2.2}$$

Proof. Let $g \in Y_1 \cap \mathcal{D}_1^{-1}(X_1)$ be arbitrary. Then (e) from Definition 2.1 gives $\mathcal{A}g \in Y_2 \cap \mathcal{D}_2^{-1}(X_2)$. Applying (a) and (b) we get

$$\inf_{G \in Y_2 \cap \mathcal{D}_2^{-1}(X_2)} \{\|\mathcal{A}f - G\|_{X_2} + t\|\mathcal{D}_2 G\|_{X_2}\} \le \|\mathcal{A}f - \mathcal{A}g\|_{X_2} + t\|\mathcal{D}_2 \mathcal{A}g\|_{X_2}$$
$$\le c(\|f - g\|_{X_1} + t\|\mathcal{D}_1 g\|_{X_1}).$$

Now, taking an infimum over $g \in Y_1 \cap \mathcal{D}_1^{-1}(X_1)$, we get

$$K(\mathcal{A}f, t; X_2, Y_2, \mathcal{D}_2) \le cK(f, t; X_1, Y_1, \mathcal{D}_1). \tag{2.3}$$

Just similarly, using (c), (d) and (f) from Definition 2.1, we show that

$$K(\mathcal{B}F, t; X_1, Y_1, \mathcal{D}_1) \le cK(F, t; X_2, Y_2, \mathcal{D}_2). \tag{2.4}$$

Next, (g) implies $K(f, t; X_1, Y_1, \mathcal{D}_1) = K(\mathcal{B}\mathcal{A}f, t; X_1, Y_1, \mathcal{D}_1)$. Then inequality (2.4) with $F = \mathcal{A}f$ implies

$$K(f, t; X_1, Y_1, \mathcal{D}_1) \le cK(\mathcal{A}f, t; X_2, Y_2, \mathcal{D}_2),$$

which together with (2.3) gives (2.1). Similarly, (h), (2.3) and (2.4) yield (2.2) and complete the proof. □

We shall use extensively Proposition 2.1 for obtaining equivalences like (1.2). In the next proposition we give a sufficient condition for an operator to be a quasi-invertible continuous map in the context of the functional spaces treated in this article.

Proposition 2.2. *Let* $(X_1, Y_1, \mathcal{D}_1) = (L_p(w)(I), AC_{loc}^{r-1}(I), \varphi^r D^r)$ *and* $(X_2, Y_2, \mathcal{D}_2) = (L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}(\tilde{I}), \tilde{\varphi}^r D^r)$, *where* $I, \tilde{I}$ *are real intervals and* w, φ *and* $\tilde{w}, \tilde{\varphi}$ *are non-negative measurable functions defined respectively on* I *and* $\tilde{I}$. *Let the operators* $\mathcal{A} : X_1 \to X_2$ *and* $\mathcal{B} : X_2 \to X_1$ *satisfy conditions* (a), (b), (c) *and* (d) *of Definition* 2.1. *Let* $\bar{X}_1$, $\bar{X}_2$ *be two functional spaces such that* $\bar{X}_1 \supset L_p(w)(I)$, $\bar{X}_2 \supset L_p(\tilde{w})(\tilde{I})$. *If there exists an* ***invertible*** *linear operator* $\bar{\mathcal{A}} : \bar{X}_1 \to \bar{X}_2$ *such that* $\bar{\mathcal{A}}(\Pi_{r-1}) \subset \Pi_{r-1}$, $\bar{\mathcal{A}}^{-1}(\Pi_{r-1}) \subset \Pi_{r-1}$ *and*

- (e′) $\bar{\mathcal{A}}(X_1 \cap Y_1) \subseteq Y_2$;
- (f′) $\bar{\mathcal{A}}^{-1}(X_2 \cap Y_2) \subseteq Y_1$;
- (g′) $\mathcal{A}f - \bar{\mathcal{A}}f \in \Pi_{r-1}$ *for any* $f \in X_1$;
- (h′) $\mathcal{B}F - \bar{\mathcal{A}}^{-1}F \in \Pi_{r-1}$ *for any* $F \in X_2$,

then $\mathcal{A}$ *is a quasi-invertible continuous map of* $(X_1, Y_1, \mathcal{D}_1)$ *onto* $(X_2, Y_2, \mathcal{D}_2)$ *and* $\mathcal{B}$ *is a quasi-inverse operator to* $\mathcal{A}$.

P r o o f. We shall establish conditions (e), (f), (g) and (h) from Definition 2.1. We have

$$\mathcal{D}_1^{-1}(X_1) = \{f \in X_1 \cap Y_1 \ : \ D^r f \in L_p(w\varphi^r)(I)\},$$
$$\mathcal{D}_2^{-1}(X_2) = \{F \in X_2 \cap Y_2 \ : \ D^r F \in L_p(\tilde{w}\tilde{\varphi}^r)(\tilde{I})\}.$$

Hence, $Y_1 \cap \mathcal{D}_1^{-1}(X_1) = \mathcal{D}_1^{-1}(X_1)$ and $Y_2 \cap \mathcal{D}_2^{-1}(X_2) = \mathcal{D}_2^{-1}(X_2)$. Now, from conditions (a), (e′) and (g′) we get $\mathcal{A}(AC_{loc}^{r-1}(I) \cap L_p(w)(I)) \subseteq AC_{loc}^{r-1}(\tilde{I}) \cap L_p(\tilde{w})(\tilde{I})$, which together with condition (b) implies condition (e). Moreover, from conditions (b), (f′) and (h′) we get $\mathcal{B}(AC_{loc}^{r-1}(\tilde{I}) \cap L_p(\tilde{w})(\tilde{I})) \subseteq AC_{loc}^{r-1}(I) \cap L_p(w)(I)$, which together with condition (d) implies condition (f).

From the definitions we have $\ker \mathcal{D}_1 = \Pi_{r-1} \cap L_p(w)(I)$ and $\ker \mathcal{D}_2 = \Pi_{r-1} \cap L_p(\tilde{w})(\tilde{I})$.

Now we shall establish (g). Let $f \in L_p(w)(I)$. Set $Q_1 = \bar{\mathcal{A}}f - \mathcal{A}f$, $F = \mathcal{A}f \in L_p(\tilde{w})(\tilde{I})$ and $Q_2 = \bar{\mathcal{A}}^{-1}F - \mathcal{B}F$. Then

$$f - \mathcal{B}\mathcal{A}f = f - \mathcal{B}F = f - \bar{\mathcal{A}}^{-1}F + Q_2 = f - \bar{\mathcal{A}}^{-1}(\bar{\mathcal{A}}f - Q_1) + Q_2 = \bar{\mathcal{A}}^{-1}Q_1 + Q_2.$$

Now (g′) implies $Q_1 \in \Pi_{r-1}$, (h′) implies $Q_2 \in \Pi_{r-1}$ and hence $f - \mathcal{B}\mathcal{A}f \in \Pi_{r-1}$. Moreover $f \in X_1$ implies $\mathcal{A}f \in X_2$ and $f - \mathcal{B}\mathcal{A}f \in X_1$. Hence $f - \mathcal{B}\mathcal{A}f \in \Pi_{r-1} \cap L_p(w)(I) = Y_1 \cap \ker \mathcal{D}_1$. This proves (g).

In a similar way we get for any $F \in L_p(\tilde{w})(\tilde{I})$ that $F - \mathcal{A}\mathcal{B}F \in \Pi_{r-1} \cap L_p(\tilde{w})(\tilde{I}) = Y_2 \cap \ker \mathcal{D}_2$, which is (h). This completes the proof. $\square$

In this article we use $\bar{X}_1 = L_{1,loc}(I)$, $\bar{X}_2 = L_{1,loc}(\tilde{I})$. The operator $\bar{\mathcal{A}}$ will be explicitly given with (3.1), (3.2) or with (4.1), (4.2). It is invertible, preserves Π_{r-1} and satisfies (e′), (f′), (g′) and (h′). Note that no boundedness conditions are required from $\bar{\mathcal{A}}$ for the validity of the above proposition.

2.3. Hardy's inequalities. Muckenhoupt generalized in [12] Hardy's inequalities (see [8, p. 245]). A partial case of [12] are the following Hardy's inequalities, which will be used extensively.

Proposition 2.3. *Let $\zeta < \eta$ and let F be a measurable function on $[\zeta, \eta]$.*

a) *If $1 \le p \le \infty$, $\beta > 0$, $\gamma \le \beta$ or $p = 1$, $\beta = 0$, $\gamma < 0$, then*

$$\left(\int_\zeta^\eta \left|(x-\zeta)^{-\gamma-\frac{1}{p}} \int_\zeta^x F(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le c\left(\int_\zeta^\eta |(x-\zeta)^{-\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

b) *If $1 \le p \le \infty$, $\beta \le \gamma$, $\gamma > 0$ or $p = \infty$, $\beta < 0$, $\gamma = 0$, then*

$$\left(\int_\zeta^\eta \left|(x-\zeta)^{\gamma-\frac{1}{p}} \int_x^\eta F(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le c\left(\int_\zeta^\eta |(x-\zeta)^{\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Proposition 2.4. *Let $\eta > 0$ and let F be a measurable function on $[\eta, \infty)$.*

a) *If* $1 \le p \le \infty$, $\beta \le \gamma$, $\gamma > 0$ *or* $p = \infty$, $\beta < 0$, $\gamma = 0$, *then*

$$\left(\int_\eta^\infty \left| x^{-\gamma-\frac{1}{p}} \int_\eta^x F(y)\,dy \right|^p dx\right)^{\frac{1}{p}} \le c\left(\int_\eta^\infty |x^{-\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

b) *If* $1 \le p \le \infty$, $\beta \ge \gamma$, $\beta > 0$ *or* $p = 1$, $\beta = 0$, $\gamma < 0$, *then*

$$\left(\int_\eta^\infty \left| x^{\gamma-\frac{1}{p}} \int_x^\infty F(y)\,dy \right|^p dx\right)^{\frac{1}{p}} \le c\left(\int_\eta^\infty |x^{\beta+1-\frac{1}{p}} F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

2.4. Translations and dilations. Let us denote by $\mathcal{T}(u)$ the translation operator, i.e. $\mathcal{T}(u)f(x) = f(x+u)$, and by $\mathcal{S}(u)$ the dilation operator, i.e. $\mathcal{S}(u)f(x) = f(ux)$. Their inverse operators are $\mathcal{T}^{-1}(u) = \mathcal{T}(-u)$ and $\mathcal{S}^{-1}(u) = \mathcal{S}(u^{-1})$ $(u \ne 0)$. It is obvious that if $(X_1, Y_1, \mathcal{D}_1) = (L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ and $(X_2, Y_2, \mathcal{D}_2)$ is properly defined, then $\mathcal{T}(u)$ and $\mathcal{S}(u)$ $(u \ne 0)$ are quasi-invertible continuous maps and, hence, they satisfy the assumptions of Proposition 2.1. Thus, we can apply any linear change of the variables to the spaces involved in the definition of the K-functionals (1.1). Hence it is sufficient to consider only K-functionals of functions defined on the intervals $(0,1)$, $(0,\infty)$ or $\mathbb{R}$. Having constructed an operator $\mathcal{A}$, satisfying (1.2) for $I = (0,1)$ or $I = (0,\infty)$, then the operators $\mathcal{T}(-a)\,\mathcal{S}((b-a)^{-1})\,\mathcal{A}\,\mathcal{S}(b-a)\,\mathcal{T}(a)$ and $\mathcal{T}(-a)\,\mathcal{A}\,\mathcal{T}(a)$ satisfy (1.2) for $I = (a,b)$ or $I = (a,\infty)$. The same is true if (1.3) is in the place of (1.2). This follows from the propositions

Proposition 2.5. *We have*

$$\mathcal{S}(b-a)\,\mathcal{T}(a) : (L_p(\chi_a^{\gamma_a}\chi_b^{\gamma_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b} D^r) \rightleftharpoons (L_p(\chi_0^{\gamma_a}\chi_1^{\gamma_b})(0,1), AC_{loc}^{r-1}, \chi_0^{r\lambda_a}\chi_1^{r\lambda_b} D^r) : \mathcal{T}(-a)\,\mathcal{S}((b-a)^{-1}).$$

Proposition 2.6. *We have*

$$\mathcal{T}(a) : (L_p(\chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)} D^r) \rightleftharpoons (L_p(\chi_0^{\gamma_a}\chi_{-1}^{\gamma_\infty-\gamma_a})(0,\infty), AC_{loc}^{r-1}, \chi_0^{r\lambda_a}\chi_{-1}^{r(\lambda_\infty-\lambda_a)} D^r) : \mathcal{T}(-a).$$

The case of semi-infinite interval $(-\infty, -a)$ is reduced to (a,∞) by

Proposition 2.7.

$$\mathcal{S}(-1) : (L_p(\chi_{-a}^{\gamma_{-a}}\chi_{-a+1}^{\gamma_{-\infty}-\gamma_{-a}})(-\infty,-a), AC_{loc}^{r-1}, \chi_{-a}^{r\lambda_{-a}}\chi_{-a+1}^{r(\lambda_{-\infty}-\lambda_{-a})} D^r) \rightleftharpoons (L_p(\chi_a^{\gamma_{-a}}\chi_{a-1}^{\gamma_{-\infty}-\gamma_{-a}})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_{-a}}\chi_{a-1}^{r(\lambda_{-\infty}-\lambda_{-a})} D^r) : \mathcal{S}(-1).$$

Using "mirror" change of the variable we can interchange simultaneously the exponents of the weights w and φ at the two finite ends of the domain. Namely,

Proposition 2.8. *We have*

$$\mathcal{S}(-1)\,\mathcal{T}(a+b) : (L_p(\chi_a^{\gamma_a}\chi_b^{\gamma_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r) \rightleftharpoons \\ (L_p(\chi_a^{\gamma_b}\chi_b^{\gamma_a})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_b}\chi_b^{r\lambda_a}D^r) : \mathcal{T}(-a-b)\,\mathcal{S}(-1).$$

Note that $\mathcal{S}(-1)\,\mathcal{T}(a+b) = \mathcal{T}(-a-b)\,\mathcal{S}(-1)$.

3. Operators that change only the weight w

3.1. Basic operators of type A. Let $r \in \mathbb{N}$, $J \subseteq (0,\infty)$ be an open interval and $\xi \in J$ be fixed. For $\rho \in \mathbb{R}$ in [3] we defined the linear operator $A(\rho;\xi) : L_{1,loc}(J) \to L_{1,loc}(J)$ by

$$(A(\rho)f)(x) = (A(\rho;\xi)f)(x) = x^\rho f(x) + \sum_{k=1}^{r} \alpha_{r,k}(\rho) x^{k-1} \int_\xi^x y^{-k+\rho} f(y)\,dy, \tag{3.1}$$

where

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!}\binom{r-1}{k-1}\prod_{\nu=0}^{r-1}(\rho+r-k-\nu), \quad k=1,2,\dots,r. \tag{3.2}$$

In the cited paper we proved for $\rho, \sigma \in \mathbb{R}$

$$A(\rho;\xi)A(\sigma;\xi) = A(\rho+\sigma;\xi), \tag{3.3}$$

hence, in view of $\alpha_{r,k}(0) = 0$ for $k = 1,2,\dots,r$, $A(\rho)$ is invertible and

$$A^{-1}(\rho;\xi) = A(-\rho;\xi). \tag{3.4}$$

We also showed that $A(\rho)$ preserves the local smoothness of the function as

$$(A(\rho;\xi)f)^{(r)}(x) = x^\rho f^{(r)}(x) \quad \text{a.e. in } J \quad \forall f \in AC_{loc}^{r-1}(J). \tag{3.5}$$

Hence $A(\rho)$ maps the set of all algebraic polynomials of degree $r-1$ into itself.

In [3] we used these operators to treat the singularity of the weight w at 0 if the exponent γ_0 is greater than $-1/p$. Now we shall modify $A(\rho)$ to relax this restriction to $\gamma_0 \neq 1-r-1/p, 2-r-1/p, \dots, -1/p$.

Definition 3.1. *Let $\rho \in \mathbb{R}$, $\xi \in (0,\infty)$ and $i,j \in \mathbb{N}_0$ as $i \le j \le r$. For $x \in (0,\infty)$ and functions $f \in L_{1,loc}(0,\infty)$, satisfying the additional requirements $\chi_0^{-i+\rho} f \in L_1(0,1)$ if $i > 0$ and $\chi_0^{-j-1+\rho} f \in L_1(1,\infty)$ if $j < r$, we set*

$$
\begin{aligned}
(A_{i,j}(\rho;\xi)f)(x) = x^\rho f(x) &+ \sum_{k=1}^{i} \alpha_{r,k}(\rho) x^{k-1} \int_0^x y^{-k+\rho} f(y)\,dy \\
&+ \sum_{k=i+1}^{j} \alpha_{r,k}(\rho) x^{k-1} \int_\xi^x y^{-k+\rho} f(y)\,dy \\
&- \sum_{k=j+1}^{r} \alpha_{r,k}(\rho) x^{k-1} \int_x^\infty y^{-k+\rho} f(y)\,dy.
\end{aligned} \tag{3.6}
$$

As usually we assume that a sum is 0 if the upper bound is smaller than the lower one. The integral terms for $k = 1,\dots,i$ and $k = j+1,\dots,r$ are well defined under the assumptions made on f.

The following assertion holds true.

Proposition 3.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi > 0$, $i,j = 0,1,\dots,r$, $i \le j$ and $w = \chi_0^{\gamma_0} \chi_{-1}^{\gamma_\infty - \gamma_0}$, where $\gamma_0 \in \Gamma_i(p)$ and $\gamma_\infty \in \Gamma_j(p)$. Then for every $f \in L_p(\chi_0^\rho w)(0,\infty)$ we have*

$$
\|w A_{i,j}(\rho;\xi) f\|_{p(0,\infty)} \le c \|w \chi_0^\rho f\|_{p(0,\infty)}.
$$

Also for every non-negative measurable on $(0,\infty)$ function ϕ and every $g \in AC_{loc}^{r-1}(0,\infty)$ we have

$$
\|w\phi (A_{i,j}(\rho;\xi) g)^{(r)}\|_{p(0,\infty)} = \|w \chi_0^\rho \phi g^{(r)}\|_{p(0,\infty)}.
$$

Proof. The second assertion follows directly from (3.5) taking into consideration that

$$
\begin{aligned}
(A_{i,j}(\rho;\xi)f)(x) - (A(\rho;1)f)(x) &= \sum_{k=1}^{i} \alpha_{r,k}(\rho) x^{k-1} \int_0^1 y^{-k+\rho} f(y)\,dy \\
+ \sum_{k=i+1}^{j} \alpha_{r,k}(\rho) x^{k-1} \int_\xi^1 y^{-k+\rho} f(y)\,dy &- \sum_{k=j+1}^{r} \alpha_{r,k}(\rho) x^{k-1} \int_1^\infty y^{-k+\rho} f(y)\,dy,
\end{aligned} \tag{3.7}
$$

i.e. $A_{i,j}(\rho,\xi)f$ and $A(\rho,1)f$ differ with a polynomial of degree at most $r-1$.

Let us set

$$\psi_{k,\zeta}(x) = x^{k-1}\int_\zeta^x y^{-k+\rho} f(y)\,dy, \quad k = 1,\ldots,r, \quad \zeta \in [0,\infty].$$

Assume we have established the inequalities

$$\text{(3.8)} \quad \|w\psi_{k,0}\|_{p(0,\infty)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}, \quad k = 1,\ldots,i, \\ \text{if } \gamma_0 < 1-i-1/p,\ \gamma_\infty < 1-i-1/p;$$

$$\text{(3.9)} \quad \|w\psi_{k,\xi}\|_{p(0,\infty)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}, \quad k = i+1,\ldots,j, \\ \text{if } \gamma_0 > -i-1/p,\ \gamma_\infty < 1-j-1/p;$$

$$\text{(3.10)} \quad \|w\psi_{k,\infty}\|_{p(0,\infty)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}, \quad k = j+1,\ldots,r, \\ \text{if } \gamma_0 > -j-1/p,\ \gamma_\infty > -j-1/p;.$$

Then the first assertion of the proposition will follow if we multiply (3.6) by w, take L_p norm and apply Minkowski's inequality according to the terms on the right-hand side of (3.6). The first norm is $\|w\chi_0^\rho f\|_{p(0,\infty)}$ and the other norms are estimated with the same quantity in view of (3.8), (3.9) and (3.10).

In order to establish (3.8) we estimate the norm separately on the intervals $(0,\xi)$ and (ξ,∞). For $k = 1,\ldots,i$ (if any), we get by Proposition 2.3, a)

$$\text{(3.11)} \qquad \|w\psi_{k,0}\|_{p(0,\xi)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}, \quad k = 1,\ldots,i,$$

since $w \sim \chi_0^{\gamma_0}$ on $(0,\xi)$ and $\gamma_0 + k - 1 < 1 - i - 1/p + i - 1 = -1/p$. On (ξ,∞) we have $w \sim \chi_0^{\gamma_\infty}$ and $\gamma_\infty + k - 1 < -1/p$, hence by Proposition 2.4, a) we get

$$\text{(3.12)} \qquad \|w\psi_{k,\xi}\|_{p(\xi,\infty)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}.$$

Besides that by Hölder's inequality we have

$$\text{(3.13)} \quad \left\{\int_\xi^\infty \left| x^{\gamma_\infty+k-1}\int_0^\xi y^{-k+\rho} f(y)\,dy\right|^p dx\right\}^{\frac{1}{p}} \le c\,\|\chi_0^{-k+\rho} f\|_{1(0,\xi)} \\ \le c\,\|\chi_0^{\gamma_0+\rho} f\|_{p(0,\xi)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}.$$

Now, (3.12) – (3.13) imply

$$\|w\psi_{k,0}\|_{p(\xi,\infty)} \le c\,\|w\chi_0^\rho f\|_{p(0,\infty)}, \quad k = 1,\ldots,i,$$

which together with (3.11) proves (3.8).

In a similar way Proposition 2.3, b) and Proposition 2.4, a) imply (3.9), while Proposition 2.3, b), Proposition 2.4, b) and Hölder's inequality imply (3.10). This completes the proof. □

Remark 3.1. The intersections of the assumptions on γ_0 and γ_∞, under which inequalities (3.8)–(3.10) follow, are respectively $\Gamma_i(p)$ and $\Gamma_j(p)$ provided $i \le j$. In the case $i > j$ these intersections are empty. Thus, the assumption $i \le j$ is necessary in the construction (3.6) of $A_{i,j}(\rho, \xi)$.

Proposition 2.2 and Proposition 3.1 imply

Proposition 3.2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi, \eta > 0$ and $w = \chi_0^{\gamma_0}\chi_{-1}^{\gamma_\infty-\gamma_0}$ with $\gamma_0, \gamma_\infty, \gamma_0+\rho, \gamma_\infty+\rho \notin \Gamma_{exc}(p)$. Assume that $i \le j$ and $i' \le j'$, where i, j, i', j' are determined by $\Gamma_i(p) \ni \gamma_0$, $\Gamma_j(p) \ni \gamma_\infty$, $\Gamma_{i'}(p) \ni \gamma_0+\rho$ and $\Gamma_{j'}(p) \ni \gamma_\infty+\rho$. Finally, let ϕ be measurable and non-negative on $(0,\infty)$. Then we have*

$$A_{i,j}(\rho;\xi) : (L_p(w\chi_0^\rho)(0,\infty), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_p(w)(0,\infty), AC_{loc}^{r-1}, \phi D^r) : A_{i',j'}(-\rho;\eta).$$

P r o o f. We apply Proposition 2.2 with $I = \widetilde{I} = (0,\infty)$, $\varphi = \widetilde{\varphi} = \phi$, $\chi_0^\rho w$ and w in place of w and $\widetilde{w}$ respectively, $\mathcal{A} = A_{i,j}(\rho;\xi)$ and $\mathcal{B} = A_{i',j'}(-\rho;\eta)$, $\bar{X}_1 = L_{1,loc}(0,\infty)$, $\bar{X}_2 = L_{1,loc}(0,\infty)$ and $\bar{\mathcal{A}} = A(\rho, 1)$. Proposition 3.1 implies that $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions (a)–(d) of Definition 2.1. The invertibility of $\bar{\mathcal{A}}$ is given in (3.4), its action on Π_{r-1} and (e′) are implied by (3.5), (f′) follows from (3.4) and (3.5), while the validity of conditions (g′) and (h′) follows from (3.7). In view of Proposition 2.2 $A_{i,j}(\rho;\xi)$ is a quasi-invertible continuous map of $(L_p(w\chi_0^\rho)(0,\infty), AC_{loc}^{r-1}, \phi D^r)$ onto $(L_p(w)(0,\infty), AC_{loc}^{r-1}, \phi D^r)$ and $A_{i',j'}(-\rho;\eta)$ is a quasi-inverse to it. □

Let us observe that the operators $A_{i,j}(\rho;\xi)$ change the behaviour of the w-weight at both ends of the interval $(0,\infty)$. But we can consider their action on functions defined on subintervals as $(1,\infty)$ and $(0,1)$. In the domain $(1,\infty)$ it is natural to set $i = 0$ and, similarly, $j = r$ if $(0,1)$ is treated. When we consider $(A_{0,j}(\rho;\xi)f)(x)$ with $\xi > 1$ for $f \in L_{1,loc}(1,\infty)$, satisfying the additional requirement $\chi_0^{-j-1+\rho} f \in L_1(2,\infty)$ if $j < r$, and $x \in (1,\infty)$ we get an operator that treats only the singularity at infinity. This follows by the equivalence $\chi_0^\rho \sim 1$ in a neighborhood of 1.

Proposition 3.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi > 1$, $j = 0, 1, \dots, r$ and $w = \chi_1^{\gamma_1} \chi_0^{\gamma_\infty - \gamma_1}$, where $\gamma_1 \in \Gamma_+(p)$ and $\gamma_\infty \in \Gamma_j(p)$. Then we have*

$$\|wA_{0,j}(\rho;\xi)f\|_{p(1,\infty)} \le c\|w\chi_0^\rho f\|_{p(1,\infty)}.$$

Also for every non-negative measurable on $(1,\infty)$ function ϕ and every $g \in AC_{loc}^{r-1}(1,\infty)$ we have

$$\|w\phi(A_{0,j}(\rho;\xi)g)^{(r)}\|_{p(1,\infty)} = \|w\chi_0^\rho \phi g^{(r)}\|_{p(1,\infty)}.$$

P r o o f. Proposition 3.1, but when estimating $\|w\psi_{k,\xi}\|_{p(1,\xi)}$ by $\|w\chi_0^\rho f\|_{p(1,\xi)}$ we use Proposition 2.3, b) with $\gamma = \gamma_1 + 1/p$ and $\beta = \gamma_1 - 1 + 1/p$. $\square$

Hence as above from Propositions 2.2 and 3.3 we get

Proposition 3.4. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi, \eta > 1$ and $w = \chi_1^{\gamma_1} \chi_0^{\gamma_\infty - \gamma_1}$ with $\gamma_1 \in \Gamma_+(p)$ and $\gamma_\infty, \gamma_\infty + \rho \notin \Gamma_{exc}(p)$. Let j, j' be determined by $\Gamma_j(p) \ni \gamma_\infty$ and $\Gamma_{j'}(p) \ni \gamma_\infty + \rho$. Finally, let ϕ be measurable and non-negative on $(1,\infty)$. Then we have*

$$A_{0,j}(\rho;\xi) : (L_p(w\chi_0^\rho)(1,\infty), AC_{loc}^{r-1}, \phi D^r) \rightleftharpoons (L_p(w)(1,\infty), AC_{loc}^{r-1}, \phi D^r) : A_{0,j'}(-\rho;\eta).$$

Remark 3.2. Formally Proposition 3.2 and Proposition 3.4 look very similar – only $(0,\infty)$ is replaced by $(1,\infty)$. If we would like to have an operator in $(1,\infty)$, whose action is similar to that of $A_{i,j}(\rho;\xi)$, we could take $\mathcal{T}(-1)\, A_{i,j}(\rho;\xi)\, \mathcal{T}(1)$ (cf. Proposition 2.6) which differs from $A_{0,j}(\rho;\xi)$. This indicates why $A_{0,j}(\rho;\xi)$ preserves the asymptotic $(x-1)^{\gamma_1}$ of the weight w at the end-point 1, while $A_{i,j}(\rho;\xi)$ $(0 \le i \le j \le r)$ will change the asymptotic of the weight at the end-point 0 from $x^{\gamma_0+\rho}$ to x^{γ_0}. Therefore, the assumption imposed on $\gamma_0 + \rho$ in Proposition 3.2 is omitted in Proposition 3.4. The replacement of $\Gamma_0(p)$ with $\Gamma_+(p)$ is of importance when the case $p = \infty$ is treated. In this article we use it in respect to the operators defined in the next section.

When we consider $(A_{i,r}(\rho;\xi)f)(x)$ with $\xi \in (0,1)$ for $x \in (0,1)$ and $f \in L_{1,loc}(0,1)$, satisfying the additional requirement $\chi_0^{-i+\rho} f \in L_1(0,1/2)$ if $i > 0$, we get an operator that treats only the singularity at 0. Like in the case of a semi-infinite domain we establish the assertions:

Proposition 3.5. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi \in (0,1)$, $i = 0, 1, \dots, r$ and $w = \chi_0^{\gamma_0} \chi_1^{\gamma_1}$, where $\gamma_0 \in \Gamma_i(p)$ and $\gamma_1 \in \Gamma_+(p)$. Then for every $f \in L_p(\chi_0^\rho w)(0,1)$ we have*

$$\|wA_{i,r}(\rho;\xi)f\|_{p(0,1)} \le c\|w\chi_0^\rho f\|_{p(0,1)}.$$

Also for every non-negative measurable on $(0,1)$ *function* ϕ *and every* $g \in AC_{loc}^{r-1}(0,1)$ *we have*

$$\|w\phi(A_{i,r}(\rho;\xi)g)^{(r)}\|_{p(0,1)} = \|w\chi_0^{\rho}\phi g^{(r)}\|_{p(0,1)}.$$

Proposition 3.6. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi,\eta \in (0,1)$ *and* $w = \chi_0^{\gamma_0}\chi_1^{\gamma_1}$ *with* $\gamma_0, \gamma_0+\rho \notin \Gamma_{exc}(p)$ *and* $\gamma_1 \in \Gamma_+(p)$. *Let* i, i' *be determined by* $\Gamma_i(p) \ni \gamma_0$ *and* $\Gamma_{i'}(p) \ni \gamma_0+\rho$. *Finally, let* ϕ *be measurable and non-negative on* $(0,1)$. *Then we have*

$$\begin{aligned}A_{i,r}(\rho;\xi) : (L_p(w\chi_0^{\rho})(0,1), AC_{loc}^{r-1}, \phi D^r)&\\ \rightleftharpoons (L_p(w)(0,1), AC_{loc}^{r-1}, \phi D^r) &: A_{i',r}(-\rho;\eta).\end{aligned}$$

3.2. Transformed operators of type *A*. So far we have constructed operators through which we can treat K-functionals of functions defined on the intervals $(0,\infty)$, $(1,\infty)$ and $(0,1)$. On their basis, in view of Propositions 2.5 and 2.6, we can get their analogues, which act on functions defined on (a,∞) and (a,b).

Definition 3.2. *Let* $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $i,j \in \mathbb{N}_0$ *as* $i \le j \le r$, *and* $\xi \in (a,\infty)$. *For* $x \in (a,\infty)$ *and* $f \in L_{1,loc}(a,\infty)$, *satisfying the additional requirements* $\chi_a^{-i+\rho}f \in L_1(a,a+1)$ *if* $i>0$ *and* $\chi_a^{-j-1+\rho}f \in L_1(a+1,\infty)$ *if* $j<r$, *we set*

$$(A_{i,j}(\rho;a,\infty;\xi)f)(x) = (\mathcal{T}(-a)\,A_{i,j}(\rho;\xi-a)\,\mathcal{T}(a)f)(x),$$

that is,

$$\begin{aligned}(A_{i,j}(\rho;a,\infty;\xi)f)(x) &= (x-a)^{\rho}f(x)\\ &+\sum_{k=1}^{i}\alpha_{r,k}(\rho)(x-a)^{k-1}\int_a^x (y-a)^{-k+\rho}f(y)\,dy\\ &+\sum_{k=i+1}^{j}\alpha_{r,k}(\rho)(x-a)^{k-1}\int_\xi^x (y-a)^{-k+\rho}f(y)\,dy\\ &-\sum_{k=j+1}^{r}\alpha_{r,k}(\rho)(x-a)^{k-1}\int_x^\infty (y-a)^{-k+\rho}f(y)\,dy,\end{aligned} \tag{3.14}$$

where $\alpha_{r,k}(\rho)$ *are defined in* (3.2).

From Propositions 3.2 and 2.6 we get

Proposition 3.7. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi, \eta > a$ and $w = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ with $\gamma_a, \gamma_\infty, \gamma_a + \rho, \gamma_\infty + \rho \notin \Gamma_{exc}(p)$. Assume that $i \le j$ and $i' \le j'$, where i, j, i', j' are determined by $\Gamma_i(p) \ni \gamma_a$, $\Gamma_j(p) \ni \gamma_\infty$, $\Gamma_{i'}(p) \ni \gamma_a + \rho$ and $\Gamma_{j'}(p) \ni \gamma_\infty + \rho$. Finally, let ϕ be measurable and non-negative on (a, ∞). Then we have*

$$\begin{aligned} A_{i,j}(\rho; a, \infty; \xi) : (L_p(w\chi_a^\rho)(a,\infty), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_p(w)(a,\infty), AC_{loc}^{r-1}, \phi D^r) : A_{i',j'}(-\rho; a, \infty; \eta). \end{aligned}$$

Definition 3.3. *Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $j \in \mathbb{N}_0$ as $j \le r$, and $\xi \in (a, \infty)$. For $x \in (a, \infty)$ and $f \in L_{1,loc}(a, \infty)$, satisfying the additional requirement $\chi_a^{-j-1+\rho} f \in L_1(a+1, \infty)$ if $j < r$, we set*

$$(A_j(\rho; \infty, a; \xi) f)(x) = (\mathcal{T}(1-a)\, A_{0,j}(\rho; \xi - a + 1)\, \mathcal{T}(a-1) f)(x),$$

that is,

$$\begin{aligned} (A_j(\rho;\infty, a; \xi) f)(x) &= (x - a + 1)^\rho f(x) \\ &+ \sum_{k=1}^{j} \alpha_{r,k}(\rho)(x-a+1)^{k-1} \int_\xi^x (y-a+1)^{-k+\rho} f(y)\, dy \\ &- \sum_{k=j+1}^{r} \alpha_{r,k}(\rho)(x-a+1)^{k-1} \int_x^\infty (y-a+1)^{-k+\rho} f(y)\, dy, \end{aligned} \tag{3.15}$$

where $\alpha_{r,k}(\rho)$ are defined in (3.2).

From Propositions 3.4 and 2.6 we get

Proposition 3.8. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi, \eta > a$ and $w = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ with $\gamma_a \in \Gamma_+(p)$ and $\gamma_\infty, \gamma_\infty + \rho \notin \Gamma_{exc}(p)$. Let j, j' be determined by $\Gamma_j(p) \ni \gamma_\infty$ and $\Gamma_{j'}(p) \ni \gamma_\infty + \rho$. Finally, let ϕ be measurable and non-negative on (a, ∞). Then we have*

$$\begin{aligned} A_j(\rho; \infty, a; \xi) : (L_p(w\chi_{a-1}^\rho)(a,\infty), AC_{loc}^{r-1}, \phi D^r) \\ \rightleftharpoons (L_p(w)(a,\infty), AC_{loc}^{r-1}, \phi D^r) : A_{j'}(-\rho; \infty, a; \eta). \end{aligned}$$

Definition 3.4. *Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$, $i \in \mathbb{N}_0$ as $i \le r$, and $\xi \in (a, b)$. Let s be one of the ends of the finite interval (a, b) and e – the other. For $x \in (a, b)$ and*

$f \in L_{1,loc}(a,b)$, satisfying the additional requirement $\chi_s^{-i+\rho} f \in L_1(s,(s+e)/2)$ if $i > 0$, we set

$$(A_i(\rho;s,e;\xi)f)(x) = (\mathcal{T}(-s)\mathcal{S}((e-s)^{-1})\, A_{i,r}(\rho;(\xi-s)/(e-s))\, \mathcal{S}(e-s)\mathcal{T}(s)f)(x),$$

that is,

$$\begin{aligned}(A_i(\rho;s,e;\xi)f)(x) = {} & \left(\frac{x-s}{e-s}\right)^{\rho} f(x) \\ & + \frac{1}{e-s}\sum_{k=1}^{i} \alpha_{r,k}(\rho)\left(\frac{x-s}{e-s}\right)^{k-1} \int_s^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\,dy \\ & + \frac{1}{e-s}\sum_{k=i+1}^{r} \alpha_{r,k}(\rho)\left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\,dy,\end{aligned}$$

where $\alpha_{r,k}(\rho)$ are defined in (3.2).

This operator treats the singularity of the w-weight at the end s and does not affect its behaviour at the other end e. More precisely, from Propositions 3.6 and 2.5 we get

Proposition 3.9. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\xi, \eta \in (a,b)$, s be one of the points a, b and e be the other one, $w = \chi_s^{\gamma_s}\chi_e^{\gamma_e}$ with $\gamma_s, \gamma_s + \rho \notin \Gamma_{exc}(p)$ and $\gamma_e \in \Gamma_+(p)$. Let i, i' be determined by $\Gamma_i(p) \ni \gamma_s$ and $\Gamma_{i'}(p) \ni \gamma_s + \rho$. Finally, let ϕ be measurable and non-negative on (a,b). Then we have*

$$\begin{aligned} A_i(\rho;s,e;\xi) : {} & (L_p(w\chi_s^{\rho})(a,b), AC_{loc}^{r-1}, \phi D^r) \\ & \rightleftharpoons (L_p(w)(a,b), AC_{loc}^{r-1}, \phi D^r) : A_{i'}(-\rho;s,e;\eta).\end{aligned}$$

Let us note that the last proposition generalizes [3, Proposition 5.4].

4. Operators that change both weights w and φ

4.1. Basic operators of type B. Let $r \in \mathbb{N}$ and $\xi > 0$ be fixed. For $\sigma \in \mathbb{R}\backslash\{0\}$ we defined in [3] the linear operator $B(\sigma;\xi) : L_{1,loc}(0,\infty) \to L_{1,loc}(0,\infty)$ by

$$(4.1)\quad (B(\sigma)f)(x) = (B(\sigma;\xi)f)(x) = f(x^\sigma) + \sum_{k=2}^{r} \beta_{r,k}(\sigma)\, x^{k-1} \int_\xi^x y^{-k} f(y^\sigma)\,dy,$$

where

$$(4.2)\qquad \beta_{r,k}(\sigma)=\frac{(-1)^{r-k}}{(r-2)!}\binom{r-2}{k-2}\prod_{i=1}^{r-1}(k-1-i\sigma),\quad k=2,3,\dots,r.$$

Obviously, operators of type (4.1) preserve the local smoothness of the functions. Moreover, it is proved in the paper cited above that

$$(4.3)\quad (B(\sigma;\xi)f)^{(r)}(x)=\sigma^r x^{r(\sigma-1)}f^{(r)}(x^\sigma)\text{ a.e. in }(0,\infty)\ \forall f\in AC_{loc}^{r-1}(0,\infty).$$

Hence $B(\sigma)$ maps the set of all algebraic polynomials of degree $r-1$ into itself.

The following basic algebraic property of the operators $B(\sigma)$ holds true.

Theorem 4.1. *Let $\rho,\sigma\in\mathbb{R}\backslash\{0\}$ and $\xi>0$. Then*

$$(4.4)\qquad B(\sigma;\xi)B(\rho;\xi^\sigma)=B(\sigma\rho;\xi)$$

and hence $B(\sigma)$ is invertible as

$$(4.5)\qquad B(\sigma;\xi)^{-1}=B(\sigma^{-1};\xi^\sigma).$$

P r o o f. We follow the proof of the partial case $\xi=1$ given in [3, Theorem 4.2]. Applying twice (4.1) we get for every $f\in L_{1,loc}(0,\infty)$

$$\begin{aligned}
&(B(\sigma;\xi)B(\rho;\xi^\sigma)f)(x)\\
&=(B(\rho;\xi^\sigma)f)(x^\sigma)+\sum_{k=2}^{r}\beta_{r,k}(\sigma)\,x^{k-1}\int_\xi^x y^{-k}(B(\rho;\xi^\sigma)f)(y^\sigma)\,dy\\
&=f(x^{\sigma\rho})+\sum_{\ell=2}^{r}\beta_{r,\ell}(\rho)\,x^{\sigma(\ell-1)}\int_{\xi^\sigma}^{x^\sigma}y^{-\ell}f(y^\rho)\,dy\\
&+\sum_{k=2}^{r}\beta_{r,k}(\sigma)\,x^{k-1}\int_\xi^x y^{-k}\left(f(y^{\sigma\rho})+\sum_{\ell=2}^{r}\beta_{r,\ell}(\rho)\,y^{\sigma(\ell-1)}\int_{\xi^\sigma}^{y^\sigma}u^{-\ell}f(u^\rho)\,du\right)dy\\
&=f(x^{\sigma\rho})+\sum_{\ell=2}^{r}\sigma\beta_{r,\ell}(\rho)\,x^{\sigma(\ell-1)}\int_\xi^x u^{-\sigma(\ell-1)-1}f(u^{\sigma\rho})\,du\\
&+\sum_{k=2}^{r}\beta_{r,k}(\sigma)\,x^{k-1}\int_\xi^x y^{-k}f(y^{\sigma\rho})\,dy\\
&+\sum_{k=2}^{r}\sum_{\ell=2}^{r}\sigma\beta_{r,k}(\sigma)\beta_{r,\ell}(\rho)\,x^{k-1}\int_\xi^x y^{\sigma(\ell-1)-k}\left(\int_\xi^y u^{-\sigma(\ell-1)-1}f(u^{\sigma\rho})\,du\right)dy.
\end{aligned}$$

If $\sigma(\ell-1)=k-1$, then $\beta_{r,k}(\sigma)=0$, and if $\sigma(\ell-1)\neq k-1$, then

$$x^{k-1}\int_\xi^x y^{\sigma(\ell-1)-k}\Big(\int_\xi^y u^{-\sigma(\ell-1)-1}f(u^{\sigma\rho})\,du\Big)\,dy=\frac{1}{\sigma(\ell-1)-(k-1)} \times\Big(x^{\sigma(\ell-1)}\int_\xi^x u^{-\sigma(\ell-1)-1}f(u^{\sigma\rho})\,du-x^{k-1}\int_\xi^x u^{-k}f(u^{\sigma\rho})\,du\Big).$$

Hence

(4.6)

$$\begin{aligned}(B(\sigma;\xi)B(\rho;\xi^\sigma)f)(x)&=f(x^{\sigma\rho})\\ &+\sigma\sum_{\ell=2}^r\beta_{r,\ell}(\rho)\Big(1+\sum_{k=2}^r\frac{\beta_{r,k}(\sigma)}{\sigma(\ell-1)-(k-1)}\Big)x^{\sigma(\ell-1)}\int_\xi^x u^{-\sigma(\ell-1)-1}f(u^{\sigma\rho})\,du\\ &+\sum_{k=2}^r\Big(\beta_{r,k}(\sigma)-\sigma\sum_{\ell=2}^r\frac{\beta_{r,k}(\sigma)\beta_{r,\ell}(\rho)}{\sigma(\ell-1)-(k-1)}\Big)x^{k-1}\int_\xi^x u^{-k}f(u^{\sigma\rho})\,du\end{aligned}$$

as the ratio $\beta_{r,k}(\sigma)/(\sigma(\ell-1)-(k-1))$ is defined by continuity for $\sigma=(k-1)/(\ell-1)$. In [3] (see (4.8) and (4.11)) we proved the combinatorial identities

$$\sum_{k=2}^r\frac{\beta_{r,k}(\sigma)}{k-1-\sigma(\ell-1)}=1,\quad \ell=2,3,\dots,r,$$

and

$$\beta_{r,k}(\sigma)-\sigma\sum_{\ell=2}^r\frac{\beta_{r,k}(\sigma)\beta_{r,\ell}(\rho)}{\sigma(\ell-1)-(k-1)}=\beta_{r,k}(\sigma\rho),\quad k=2,3,\dots,r.$$

Using these identities in (4.6) we get (4.4). □

It is worth noting that $A(\rho)$ and $B(\sigma)$ change places in the following way:

$$B(\sigma;\xi)A(\rho;\xi^\sigma)=A(\rho\sigma;\xi)B(\sigma;\xi),\quad \sigma\neq 0. \tag{4.7}$$

In [3, Proposition 4.1] the partial case $\xi=1$ of the relation above was established. The proof in the general case is based on the approach used in the proof of Theorem 4.1 as this time we need to use the combinatorial identities [3, (3.10), (4.8), (4.14) and (4.15)]. Identity (4.7) is extended in Section 5 to properties xiii)–xvii).

In [3] we used operator $B(\sigma)$ to variate the behaviour of the φ-weight (with $\lambda<1$, which implies $\sigma>0$) in the second term of the K-functional of functions with a finite domain. Now we shall modify $B(\sigma)$ like we did with $A(\rho)$ in Section 3 in order to apply it under weaker restrictions on the w-weight. The following arguments imply that there is higher number of possible modifications of $B(\sigma)$ compared to $A(\rho)$.

Consider $0 \le \zeta < \eta \le \infty$ and $\xi \in (\zeta, \eta)$. If $\sigma > 0$, then operator $B(\sigma;\xi)$ relates functions, defined on $(\zeta^\sigma, \eta^\sigma)$, to functions, defined on (ζ, η); and if $\sigma < 0$, then operator $B(\sigma;\xi)$ relates functions, defined on $(\eta^\sigma, \zeta^\sigma)$, to functions, defined on (ζ, η). Above we assume that $\zeta^\sigma = \infty$ if $\zeta = 0$ and $\sigma < 0$; $\eta^\sigma = \infty$ if $\eta = \infty$ and $\sigma > 0$; and $\eta^\sigma = 0$ if $\eta = \infty$ and $\sigma < 0$. Thus, in particular, we have:

i) $B(\sigma;\xi) : L_{1,loc}(0,\infty) \to L_{1,loc}(0,\infty)$ for $\sigma \ne 0$ and $\xi \in (0,\infty)$;

ii) $B(\sigma;\xi) : L_{1,loc}(0,1) \to L_{1,loc}(0,1)$ for $\sigma > 0$ and $\xi \in (0,1)$;

iii) $B(\sigma;\xi) : L_{1,loc}(1,\infty) \to L_{1,loc}(1,\infty)$ for $\sigma > 0$ and $\xi \in (1,\infty)$;

iv) $B(\sigma;\xi) : L_{1,loc}(0,1) \to L_{1,loc}(1,\infty)$ for $\sigma < 0$ and $\xi \in (1,\infty)$;

v) $B(\sigma;\xi) : L_{1,loc}(1,\infty) \to L_{1,loc}(0,1)$ for $\sigma < 0$ and $\xi \in (0,1)$.

Definition 4.1. *Let $\sigma > 0$, $\xi \in (0,\infty)$, $i,j \in \mathbb{N}$ as $i \le j \le r$. For $x \in (0,\infty)$ and functions $f \in L_{1,loc}(0,\infty)$, satisfying the additional requirements $\chi_0^{(1-i)/\sigma-1} f \in L_1(0,1)$ if $i > 1$ and $\chi_0^{-j/\sigma-1} f \in L_1(1,\infty)$ if $j < r$, we set*

$$\begin{aligned}(B_{i,j}(\sigma;\xi)f)(x) = f(x^\sigma) &+ \sum_{k=2}^{i} \beta_{r,k}(\sigma)\, x^{k-1} \int_0^x y^{-k} f(y^\sigma)\, dy \\ &+ \sum_{k=i+1}^{j} \beta_{r,k}(\sigma)\, x^{k-1} \int_\xi^x y^{-k} f(y^\sigma)\, dy \\ &- \sum_{k=j+1}^{r} \beta_{r,k}(\sigma)\, x^{k-1} \int_x^\infty y^{-k} f(y^\sigma)\, dy,\end{aligned} \tag{4.8}$$

where the coefficients $\beta_{r,k}(\sigma)$ are given in (4.2).

Analogously to the results in Section 3 we have the following boundedness property.

Proposition 4.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi > 0$, $i,j = 1,2,\ldots,r$, $i \le j$, $w = \chi_0^{\gamma_0} \chi_{-1}^{\gamma_\infty - \gamma_0}$, where $\gamma_0 \in \Gamma_i^*(p)$ and $\gamma_\infty \in \Gamma_j^*(p)$, and $\lambda = 1 - 1/\sigma$. Then for every $f \in L_p(w\chi_0^{-(\gamma_0+1/p)\lambda} \chi_{-1}^{-(\gamma_\infty-\gamma_0)\lambda})(0,\infty)$ we have*

$$\|wB_{i,j}(\sigma;\xi)f\|_{p(0,\infty)} \le c\|w\chi_0^{-(\gamma_0+1/p)\lambda} \chi_{-1}^{-(\gamma_\infty-\gamma_0)\lambda} f\|_{p(0,\infty)}.$$

Also for every $\tau_0, \tau_\infty \in \mathbb{R}$, $\phi = \chi_0^{\tau_0}\chi_{-1}^{\tau_\infty-\tau_0}$ *and* $g \in AC_{loc}^{r-1}(0,\infty)$ *we have*

$$\|w\phi(B_{i,j}(\sigma;\xi)g)^{(r)}\|_{p(0,\infty)} \sim \|w\chi_0^{-(\gamma_0+1/p)\lambda}\chi_{-1}^{-(\gamma_\infty-\gamma_0)\lambda}\phi\chi_0^{(r-\tau_0)\lambda}\chi_{-1}^{-(\tau_\infty-\tau_0)\lambda}g^{(r)}\|_{p(0,\infty)}.$$

Proof. From (4.1) and (4.8) we have

$$B_{i,j}(\sigma;\xi)f - B(\sigma;1)f \in \Pi_{r-1}. \tag{4.9}$$

Repeating the arguments in the proof of (3.8), (3.9) and (3.10) (with $\rho = 0$) in Proposition 3.1 we get

$$\|wB_{i,j}(\sigma;\xi)f\|_{p(0,\infty)} \le c\left\{\int_0^\infty \left|x^{\gamma_0}(x+1)^{\gamma_\infty-\gamma_0}f(x^\sigma)\right|^p dx\right\}^{1/p}.$$

Making the change of the variable $x^\sigma = y$ in the integral on the right side and taking into account that $y^{1/\sigma}+1 \sim (y+1)^{1/\sigma}$ for $0 < y < \infty$ we get the first statement of the proposition.

The second statement follows from (4.3) and (4.9) by applying the same change of the variable. □

Remark 4.1. Note that the conditions on the γ's are a little bit different than in Section 3. This is due to the fact that the integral summand for $k = 1$ is missing in the definition of $B_{i,j}(\sigma;\xi)$ unlike in the one of $A_{i,j}(\rho;\xi)$. This allows us to replace the sets $\Gamma_1(p) = (-1-1/p, -1/p)$ and $\Gamma_0(p) = (-1/p, \infty)$, used for $A_{i,j}(\rho)$, by $\Gamma_1^*(p) = (-1-1/p, \infty) = \Gamma_1(p) \cup \Gamma_0(p) \cup \{-1/p\}$.

Following the lines of the proof of Proposition 3.2 now Propositions 2.2 and 4.1, identities (4.3) and (4.5) and the fact that $B(\sigma)(\Pi_{r-1}) \subseteq \Pi_{r-1}$ imply

Proposition 4.2. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi, \eta > 0$, *and* $w = \chi_0^{\gamma_0}\chi_{-1}^{\gamma_\infty-\gamma_0}$ *with* $\gamma_0, \gamma_\infty, (\gamma_0+1/p)/\sigma - 1/p, (\gamma_\infty+1/p)/\sigma - 1/p \notin \Gamma_{exc}^*(p)$. *Assume that* $i \le j$ *and* $i' \le j'$, *where* i, j, i', j' *are determined by* $\Gamma_i^*(p) \ni \gamma_0$, $\Gamma_j^*(p) \ni \gamma_\infty$, $\Gamma_{i'}^*(p) \ni (\gamma_0+1/p)/\sigma - 1/p$ *and* $\Gamma_{j'}^*(p) \ni (\gamma_\infty+1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_0^{\tau_0}\chi_{-1}^{\tau_\infty-\tau_0}$, $\tau_0, \tau_\infty \in \mathbb{R}$. *Then*

$$B_{i,j}(\sigma;\xi): (L_p(w\chi_0^{-(\gamma_0+1/p)\lambda}\chi_{-1}^{-(\gamma_\infty-\gamma_0)\lambda})(0,\infty), AC_{loc}^{r-1}, \phi\chi_0^{(r-\tau_0)\lambda}\chi_{-1}^{-(\tau_\infty-\tau_0)\lambda}D^r) \rightleftharpoons (L_p(w)(0,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{i',j'}(\sigma^{-1};\eta),$$

where $\lambda = 1 - 1/\sigma$*.*

Proposition 4.2 shows that the linear operator $B_{i,j}(\sigma;\xi)$ clears the multiplier $\chi_0^{(r-\tau_0)\lambda}\chi_{-1}^{-(\tau_\infty-\tau_0)\lambda}$ from the second term of the K-functional, but also clears $\chi_0^{-(\gamma_0+1/p)\lambda}\chi_{-1}^{-(\gamma_\infty-\gamma_0)\lambda}$ as an additional weight in both terms. The exponents of the weight in the first term of the K-functional are restricted by Hardy's inequality and there are practically no restrictions on the second term weight. The applying of that operator affects the power of the Jacobean-type weights in both terms of the K-functional at both ends of the domain.

If we consider $(B_{1,j}(\sigma;\xi)f)(x)$ with $\xi > 1$ for $x \in (1,\infty)$ and functions $f \in L_{1,loc}(1,\infty)$ such that $\chi_0^{-j/\sigma-1}f \in L_1(2,\infty)$ if $j < r$, we get an operator with similar properties which affects the powers of the Jacobean-type weights in both terms of the K-functional but only at infinity.

Proposition 4.3. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi > 1$, $j = 1,2,\dots,r$, $w = \chi_1^{\gamma_1}\chi_0^{\gamma_\infty-\gamma_1}$*, where* $\gamma_1 \in \Gamma_+(p)$ *and* $\gamma_\infty \in \Gamma_j^*(p)$*, and* $\lambda = 1 - 1/\sigma$*. Then for every* $f \in L_p(w\chi_0^{-(\gamma_\infty+1/p)\lambda})(1,\infty)$ *we have*

$$\|wB_{1,j}(\sigma;\xi)f\|_{p(1,\infty)} \le c\|w\chi_0^{-(\gamma_\infty+1/p)\lambda}f\|_{p(1,\infty)}.$$

Also for every $\tau_1,\tau_\infty \in \mathbb{R}$, $\phi = \chi_1^{\tau_1}\chi_0^{\tau_\infty-\tau_1}$ *and* $g \in AC_{loc}^{r-1}(1,\infty)$ *we have*

$$\|w\phi(B_{1,j}(\sigma;\xi)g)^{(r)}\|_{p(1,\infty)} \sim \|w\chi_0^{-(\gamma_\infty+1/p)\lambda}\phi\chi_0^{(r-\tau_\infty)\lambda}g^{(r)}\|_{p(1,\infty)}.$$

P r o o f. We proceed as in the proof of Proposition 4.1 but take into account that $y^{1/\sigma} - 1 \sim (y-1)y^{-\lambda}$ for $1 < y < \infty$. Also let us note that when estimating the L_p-norm on the interval $(1,\xi)$ of the integral summands of $B_{1,j}(\sigma;\xi)f$ we use Proposition 2.3, b) with $\gamma = \gamma_1 + 1/p$ and $\beta = \gamma_1 - 1 + 1/p$ as in the proof of Proposition 3.3 and then make the change of the variable $x^\sigma = y$. $\square$

As above from Propositions 2.2 and 4.3 we get

Proposition 4.4. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi,\eta > 1$ *and* $w = \chi_1^{\gamma_1}\chi_0^{\gamma_\infty-\gamma_1}$ *with* $\gamma_1 \in \Gamma_+(p)$ *and* $\gamma_\infty, (\gamma_\infty+1/p)/\sigma - 1/p \notin \Gamma_{exc}^*(p)$*. Let* j, j' *be determined by* $\Gamma_j^*(p) \ni \gamma_\infty$ *and* $\Gamma_{j'}^*(p) \ni (\gamma_\infty+1/p)/\sigma - 1/p$*. Finally, let* $\phi = \chi_1^{\tau_1}\chi_0^{\tau_\infty-\tau_1}$, $\tau_1,\tau_\infty \in \mathbb{R}$*. Then*

$$\begin{aligned} B_{1,j}(\sigma;\xi) : (L_p(w\chi_0^{-(\gamma_\infty+1/p)\lambda})(1,\infty), AC_{loc}^{r-1}, \phi\chi_0^{(r-\tau_\infty)\lambda}D^r) \\ \rightleftharpoons (L_p(w)(1,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{1,j'}(\sigma^{-1};\eta), \end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Proposition 4.4 shows that $B_{1,j}(\sigma;\xi)$ clears the multiplier $\chi_0^{(r-\tau_\infty)\lambda}$ from the second term of the K-functional, but also clears $\chi_0^{-(\gamma_\infty+1/p)\lambda}$ as an additional weight in both terms. Once again the exponents of the weight in the first term are restricted by Hardy's inequality and there are practically no restrictions on the second term weight. The applying of that operator affects the power of the Jacobean-type weights in both terms of the K-functional only at infinity.

Let $\sigma > 0$ and $\xi \in (0,1)$ be fixed. Then the operator $B_{i,r}(\sigma;\xi)$ is well defined for $f \in L_{1,loc}(0,1)$ such that $\chi_0^{(1-i)/\sigma-1} f \in L_1(0,1/2)$ if $i > 1$. Like in the case of a semi-infinite domain we establish the assertions:

Proposition 4.5. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi \in (0,1)$, $i = 1,2,\ldots,r$, $w = \chi_0^{\gamma_0}\chi_1^{\gamma_1}$, *where* $\gamma_0 \in \Gamma_i^*(p)$ *and* $\gamma_1 \in \Gamma_+(p)$, *and* $\lambda = 1 - 1/\sigma$. *Then for every* $f \in L_p(w\chi_0^{-(\gamma_0+1/p)\lambda})(0,1)$ *we have*

$$\|wB_{i,r}(\sigma;\xi)f\|_{p(0,1)} \le c\|w\chi_0^{-(\gamma_0+1/p)\lambda} f\|_{p(0,1)}.$$

Also for every $\tau_0,\tau_1 \in \mathbb{R}$, $\phi = \chi_0^{\tau_0}\chi_1^{\tau_1}$ *and* $g \in AC_{loc}^{r-1}(0,1)$ *we have*

$$\|w\phi(B_{i,r}(\sigma;\xi)g)^{(r)}\|_{p(0,1)} \sim \|w\chi_0^{-(\gamma_0+1/p)\lambda}\phi\chi_0^{(r-\tau_0)\lambda} g^{(r)}\|_{p(0,1)}.$$

In the proof of the above proposition we take into account that $1-y^{1/\sigma} \sim 1-y$ for $0 < y < 1$.

Proposition 4.6. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi,\eta \in (0,1)$ *and* $w = \chi_0^{\gamma_0}\chi_1^{\gamma_1}$ *with* $\gamma_0, (\gamma_0+1/p)/\sigma - 1/p \notin \Gamma_{exc}^*(p)$ *and* $\gamma_1 \in \Gamma_+(p)$. *Let* i, i' *be determined by* $\Gamma_i^*(p) \ni \gamma_0$ *and* $\Gamma_{i'}^*(p) \ni (\gamma_0+1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_0^{\tau_0}\chi_1^{\tau_1}$, $\tau_0,\tau_1 \in \mathbb{R}$. *Then*

$$B_{i,r}(\sigma;\xi) : (L_p(w\chi_0^{-(\gamma_0+1/p)\lambda})(0,1), AC_{loc}^{r-1}, \phi\chi_0^{(r-\tau_0)\lambda}D^r) \rightleftharpoons (L_p(w)(0,1), AC_{loc}^{r-1}, \phi D^r) : B_{i',r}(\sigma^{-1};\eta),$$

where $\lambda = 1 - 1/\sigma$.

Now we shall investigate the behaviour of operators of the type of $B(\sigma)$ for $\sigma < 0$. First, let us observe that $B_{i,j}(\sigma;\xi)$, defined in (4.8), can also be considered for a negative σ, $x \in (0,\infty)$ and $f \in L_{1,loc}(0,\infty)$ such that $\chi_0^{(1-i)/\sigma-1} f \in L_1(1,\infty)$ if $i > 1$ and $\chi_0^{-j/\sigma-1} f \in L_1(0,1)$ if $j < r$.

Proposition 4.7. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi > 0$, $i, j = 1, 2, \ldots, r$, $i \le j$, $w = \chi_0^{\gamma_0}\chi_{-1}^{\gamma_\infty-\gamma_0}$ and $\bar{w} = \chi_0^{\gamma_\infty}\chi_{-1}^{\gamma_0-\gamma_\infty}$, where $\gamma_0 \in \Gamma_i^*(p)$ and $\gamma_\infty \in \Gamma_j^*(p)$, and $\lambda = 1 - 1/\sigma$. Then for every $f \in L_p(\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda}\chi_{-1}^{-(\gamma_0-\gamma_\infty)\lambda})(0,\infty)$ we have*

$$\|wB_{i,j}(\sigma;\xi)f\|_{p(0,\infty)} \le c\|\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda}\chi_{-1}^{-(\gamma_0-\gamma_\infty)\lambda}f\|_{p(0,\infty)}.$$

Also for every $\tau_0, \tau_\infty \in \mathbb{R}$, $\phi = \chi_0^{\tau_0}\chi_{-1}^{\tau_\infty-\tau_0}$, $\bar{\phi} = \chi_0^{\tau_\infty}\chi_{-1}^{\tau_0-\tau_\infty}$ and $g \in AC_{loc}^{r-1}(0,\infty)$ we have

$$\|w\phi(B_{i,j}(\sigma;\xi)g)^{(r)}\|_{p(0,\infty)} \sim \|\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda}\chi_{-1}^{-(\gamma_0-\gamma_\infty)\lambda}\bar{\phi}\chi_0^{(r-\tau_\infty)\lambda}\chi_{-1}^{-(\tau_0-\tau_\infty)\lambda}g^{(r)}\|_{p(0,\infty)}.$$

Proof. The proof is similar to that of Proposition 4.1 – it is based on the same Hardy's inequalities and change of the variable $x^\sigma = y$, but now, since $\sigma < 0$, we have $y^{1/\sigma} + 1 \sim y^{1/\sigma}(y+1)^{-1/\sigma}$ for $0 < y < \infty$. □

Proposition 4.8. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi, \eta > 0$, $w = \chi_0^{\gamma_0}\chi_{-1}^{\gamma_\infty-\gamma_0}$ and $\bar{w} = \chi_0^{\gamma_\infty}\chi_{-1}^{\gamma_0-\gamma_\infty}$ with $\gamma_0, \gamma_\infty, (\gamma_0+1/p)/\sigma - 1/p, (\gamma_\infty+1/p)/\sigma - 1/p \notin \Gamma_{exc}^*(p)$. Assume that $i \le j$ and $i' \le j'$, where i, j, i', j' are determined by $\Gamma_i^*(p) \ni \gamma_0$, $\Gamma_j^*(p) \ni \gamma_\infty$, $\Gamma_{i'}^*(p) \ni (\gamma_\infty+1/p)/\sigma - 1/p$ and $\Gamma_{j'}^*(p) \ni (\gamma_0+1/p)/\sigma - 1/p$. Finally, let $\phi = \chi_0^{\tau_0}\chi_{-1}^{\tau_\infty-\tau_0}$ and $\bar{\phi} = \chi_0^{\tau_\infty}\chi_{-1}^{\tau_0-\tau_\infty}$ with $\tau_0, \tau_\infty \in \mathbb{R}$. Then*

$$B_{i,j}(\sigma;\xi): (L_p(\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda}\chi_{-1}^{-(\gamma_0-\gamma_\infty)\lambda})(0,\infty), AC_{loc}^{r-1}, \bar{\phi}\chi_0^{(r-\tau_\infty)\lambda}\chi_{-1}^{-(\tau_0-\tau_\infty)\lambda}D^r) \rightleftharpoons (L_p(w)(0,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{i',j'}(\sigma^{-1};\eta),$$

where $\lambda = 1 - 1/\sigma$.

We notice that $B_{i,j}(\sigma;\xi)$ with $\sigma < 0$ changes the weights like $B_{i,j}(\sigma;\xi)$ with $\sigma > 0$ as it interchanges the behaviour of the weights at 0 and at infinity, i.e. it changes the places of the exponents γ_0 and γ_∞.

$(B_{1,j}(\sigma;\xi)f)(x)$ with $\sigma < 0$ and $\xi > 1$ is well defined for $x \in (1,\infty)$ and $f \in L_{1,loc}(0,1)$ such that $\chi_0^{-j/\sigma-1}f \in L_1(0,1/2)$ if $j < r$. As above we establish the assertion:

Proposition 4.9. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi > 1$, $j = 1, 2, \ldots, r$, $w = \chi_1^{\gamma_1}\chi_0^{\gamma_\infty-\gamma_1}$ and $\bar{w} = \chi_0^{\gamma_\infty}\chi_1^{\gamma_1}$, where $\gamma_1 \in \Gamma_+(p)$ and $\gamma_\infty \in \Gamma_j^*(p)$, and $\lambda = 1 - 1/\sigma$. Then for every $f \in L_p(\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda})(0,1)$ we have*

$$\|wB_{1,j}(\sigma;\xi)f\|_{p(1,\infty)} \le c\|\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda}f\|_{p(0,1)}.$$

Also for every $\tau_1, \tau_\infty \in \mathbb{R}$, $\phi = \chi_1^{\tau_1}\chi_0^{\tau_\infty - \tau_1}$, $\bar\phi = \chi_0^{\tau_\infty}\chi_1^{\tau_1}$ *and* $g \in AC_{loc}^{r-1}(0,1)$ *we have*

$$\|w\phi(B_{1,j}(\sigma;\xi)g)^{(r)}\|_{p(1,\infty)} \sim \|\bar w\chi_0^{-(\gamma_\infty+1/p)\lambda}\bar\phi\chi_0^{(r-\tau_\infty)\lambda}g^{(r)}\|_{p(0,1)}.$$

P r o o f. We proceed as in the proof of Proposition 4.3. Since $\sigma < 0$ the change of the variable $y = x^\sigma$ maps the interval $(1,\infty)$ onto $(0,1)$ and we have $y^{1/\sigma} - 1 \sim y^{1/\sigma}(1-y)$ for $0 < y < 1$. $\square$

Next, we observe that $(B_{i,r}(\sigma;\eta)f)(x)$ with $\sigma < 0$ and $\eta \in (0,1)$ is well defined for $x \in (0,1)$ and $f \in L_{1,loc}(1,\infty)$ such that $\chi_0^{(1-i)/\sigma-1}f \in L_1(2,\infty)$ if $i > 1$. Now, we have

Proposition 4.10. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\eta \in (0,1)$, $i = 1,2,\dots,r$, $w = \chi_0^{\gamma_0}\chi_1^{\gamma_1}$ *and* $\bar w = \chi_1^{\gamma_1}\chi_0^{\gamma_0-\gamma_1}$, *where* $\gamma_0 \in \Gamma_i^*(p)$ *and* $\gamma_1 \in \Gamma_+(p)$, *and* $\lambda = 1 - 1/\sigma$. *Then for every* $f \in L_p(\bar w\chi_0^{-(\gamma_0+1/p)\lambda})(1,\infty)$ *we have*

$$\|wB_{i,r}(\sigma;\eta)f\|_{p(0,1)} \le c\|\bar w\chi_0^{-(\gamma_0+1/p)\lambda}f\|_{p(1,\infty)}.$$

Also for every $\tau_0, \tau_1 \in \mathbb{R}$, $\phi = \chi_0^{\tau_0}\chi_1^{\tau_1}$, $\bar\phi = \chi_1^{\tau_1}\chi_0^{\tau_0-\tau_1}$ *and* $g \in AC_{loc}^{r-1}(1,\infty)$ *we have*

$$\|w\phi(B_{i,r}(\sigma;\eta)g)^{(r)}\|_{p(0,1)} \sim \|\bar w\chi_0^{-(\gamma_0+1/p)\lambda}\bar\phi\chi_0^{(r-\tau_0)\lambda}g^{(r)}\|_{p(1,\infty)}.$$

In the proof of the above proposition we take into account that $1 - y^{1/\sigma} \sim (y-1)y^{-1}$ for $1 < y < \infty$.

Unlike the previous operators, now the quasi-inverse operators of $B_{1,j}(\sigma;\xi)$ with $\sigma < 0$ are among $B_{i,r}(\sigma^{-1};\eta)$ and vice versa. Propositions 2.2, 4.9, 4.10 and property (4.5) imply

Proposition 4.11. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi \in (1,\infty)$, $\eta \in (0,1)$, $w = \chi_1^{\gamma_1}\chi_0^{\gamma_\infty-\gamma_1}$ *and* $\bar w = \chi_0^{\gamma_\infty}\chi_1^{\gamma_1}$ *with* $\gamma_1 \in \Gamma_+(p)$ *and* $\gamma_\infty, (\gamma_\infty + 1/p)/\sigma - 1/p \notin \Gamma_{exc}^*(p)$. *Let* j, i' *be determined by* $\Gamma_j^*(p) \ni \gamma_\infty$ *and* $\Gamma_{i'}^*(p) \ni (\gamma_\infty + 1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_1^{\tau_1}\chi_0^{\tau_\infty-\tau_1}$, $\bar\phi = \chi_0^{\tau_\infty}\chi_1^{\tau_1}$ *with* $\tau_1, \tau_\infty \in \mathbb{R}$. *Then*

$$\begin{aligned}(4.10)\quad B_{1,j}(\sigma;\xi) : (L_p(\bar w\chi_0^{-(\gamma_\infty+1/p)\lambda})(0,1), AC_{loc}^{r-1}, \bar\phi\chi_0^{(r-\tau_\infty)\lambda}D^r)\\ \rightleftharpoons (L_p(w)(1,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{i',r}(\sigma^{-1};\eta),\end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Proof. Proposition 4.9 and Proposition 4.10 imply that the operators $\mathcal{A} = B_{1,j}(\sigma;\xi)$ and $\mathcal{B} = B_{i',r}(\sigma^{-1};\eta)$ satisfy conditions (a) – (d) of Definition 2.1 with $(X_1, Y_1, \mathcal{D}_1) = (L_p(\bar{w}\chi_0^{-(\gamma_\infty+1/p)\lambda})(0,1), AC_{loc}^{r-1}, \bar{\phi}\chi_0^{(r-\tau_\infty)\lambda}D^r)$ and $(X_2, Y_2, \mathcal{D}_2) = (L_p(w)(1,\infty), AC_{loc}^{r-1}, \phi D^r)$. Next, we observe that if $\xi > 1$, then $\xi^\sigma \in (0,1)$. Consequently, (4.3) and (4.9) imply the remaining assumptions of Proposition 2.2 with $\bar{X}_1 = L_{1,loc}(0,1)$, $\bar{X}_2 = L_{1,loc}(1,\infty)$ and $\bar{\mathcal{A}} = B(\sigma;\xi)$ (hence $\bar{\mathcal{A}}^{-1} = B(\sigma^{-1};\xi^\sigma)$, see (4.5)), which proves (4.10). $\square$

For the sake of completeness we also give explicitly relation (4.10) in reverse order.

Proposition 4.12. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\eta \in (0,1)$, $\xi \in (1,\infty)$, $w = \chi_0^{\gamma_0}\chi_1^{\gamma_1}$ and $\bar{w} = \chi_1^{\gamma_1}\chi_0^{\gamma_0-\gamma_1}$ with $\gamma_0, (\gamma_0+1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$ and $\gamma_1 \in \Gamma_+(p)$. Let i, j' be determined by $\Gamma^*_i(p) \ni \gamma_0$ and $\Gamma^*_{j'}(p) \ni (\gamma_0+1/p)/\sigma - 1/p$. Finally, let $\phi = \chi_0^{\tau_0}\chi_1^{\tau_1}$, $\bar{\phi} = \chi_1^{\tau_1}\chi_0^{\tau_0-\tau_1}$ with $\tau_0, \tau_1 \in \mathbb{R}$. Then*

$$B_{i,r}(\sigma;\eta) : (L_p(\bar{w}\chi_0^{-(\gamma_0+1/p)\lambda})(1,\infty), AC_{loc}^{r-1}, \bar{\phi}\chi_0^{(r-\tau_0)\lambda}D^r) \rightleftharpoons (L_p(w)(0,1), AC_{loc}^{r-1}, \phi D^r) : B_{1,j'}(\sigma^{-1};\xi),$$

where $\lambda = 1 - 1/\sigma$.

Note that in Propositions 4.9 or 4.11 the exponent of the initial weight $\bar{w}\chi_0^{-(\gamma_0+1/p)\lambda}$ at 0 is $(\gamma_\infty+1/p)/\sigma - 1/p$. It is changed by the operator $B_{1,j}(\sigma,\xi)$ to exponent of the target weight w at ∞ equal to γ_∞. This value is returned to $(\gamma_\infty+1/p)/\sigma - 1/p$ by the operator $B_{i,r}(\sigma^{-1},\eta)$ in Propositions 4.10 or 4.12. In the four propositions the exponents of the weights of type w at 1 remain equal to γ_1, i.e. unchanged. These operators have similar action on the weights of type ϕ.

4.2. Transformed operators of type *B*. Now, using Propositions 2.5 and 2.6, we give the explicit form of the results from the previous subsection for functions defined on (a,∞) and (a,b).

Definiton 4.2. *Let $r \in \mathbb{N}$, $\sigma > 0$, $i, j \in \mathbb{N}$ as $i \le j \le r$, and $\xi \in (a,\infty)$. For $x \in (a,\infty)$ and $f \in L_{1,loc}(a,\infty)$, satisfying the additional requirements $\chi_a^{(1-i)/\sigma-1} f \in L_1(a,a+1)$ if $i > 1$ and $\chi_a^{-j/\sigma-1} f \in L_1(a+1,\infty)$ if $j < r$, we set*

$$(B_{i,j}(\sigma;a,\infty;\xi)f)(x) = (\mathcal{T}(-a)\, B_{i,j}(\sigma,\xi-a)\, \mathcal{T}(a)f)(x), \tag{4.11}$$

that is,

$$
\begin{aligned}
(B_{i,j}(\sigma;a,\infty;\xi)f)(x) &= f(a+(x-a)^\sigma)\\
&+\sum_{k=2}^{i}\beta_{r,k}(\sigma)(x-a)^{k-1}\int_a^x (y-a)^{-k} f(a+(y-a)^\sigma)\,dy\\
&+\sum_{k=i+1}^{j}\beta_{r,k}(\sigma)(x-a)^{k-1}\int_\xi^x (y-a)^{-k} f(a+(y-a)^\sigma)\,dy\\
&-\sum_{k=j+1}^{r}\beta_{r,k}(\sigma)(x-a)^{k-1}\int_x^\infty (y-a)^{-k} f(a+(y-a)^\sigma)\,dy,
\end{aligned}
$$

where $\beta_{r,k}(\sigma)$ *are defined in* (4.2).

Proposition 4.2 generalizes to

Proposition 4.13. *Let* $r\in\mathbb{N}$, $1\le p\le\infty$, $\sigma>0$, $\xi,\eta>a$, *and* $w=\chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ *with* $\gamma_a,\gamma_\infty,(\gamma_a+1/p)/\sigma-1/p,(\gamma_\infty+1/p)/\sigma-1/p\notin\Gamma^*_{exc}(p)$. *Assume that* $i\le j$ *and* $i'\le j'$, *where* i,j,i',j' *are determined by* $\Gamma^*_i(p)\ni\gamma_a$, $\Gamma^*_j(p)\ni\gamma_\infty$, $\Gamma^*_{i'}(p)\ni(\gamma_a+1/p)/\sigma-1/p$ *and* $\Gamma^*_{j'}(p)\ni(\gamma_\infty+1/p)/\sigma-1/p$. *Finally, let* $\phi=\chi_a^{\tau_a}\chi_{a-1}^{\tau_\infty-\tau_a}$, $\tau_a,\tau_\infty\in\mathbb{R}$. *Then*

$$
\begin{aligned}
&B_{i,j}(\sigma;a,\infty;\xi):\\
&\quad(L_p(w\chi_a^{-(\gamma_a+1/p)\lambda}\chi_{a-1}^{-(\gamma_\infty-\gamma_a)\lambda})(a,\infty),AC^{r-1}_{loc},\phi\chi_a^{(r-\tau_a)\lambda}\chi_{a-1}^{-(\tau_\infty-\tau_a)\lambda}D^r)\\
&\qquad\rightleftharpoons(L_p(w)(a,\infty),AC^{r-1}_{loc},\phi D^r):B_{i',j'}(\sigma^{-1};a,\infty;\eta),
\end{aligned}
$$

where $\lambda=1-1/\sigma$.

Definition 4.3. *Let* $r\in\mathbb{N}$, $\sigma>0$, $j\in\mathbb{N}$ *as* $j\le r$, *and* $\xi\in(a,\infty)$. *For* $x\in(a,\infty)$ *and* $f\in L_{1,loc}(a,\infty)$, *satisfying the additional requirement* $\chi_a^{-j/\sigma-1}f\in L_1(a+1,\infty)$ *if* $j<r$, *we set*

$$(B_j(\sigma;\infty,a;\xi)f)(x)=(\mathcal{T}(1-a)\,B_{1,j}(\sigma;\xi-a+1)\,\mathcal{T}(a-1)f)(x),$$

that is,

$$
\begin{aligned}
(B_j(\sigma;\infty,a;\xi)f)(x) &= f(a-1+(x-a+1)^\sigma)\\
&+\sum_{k=2}^{j}\beta_{r,k}(\sigma)(x-a+1)^{k-1}\int_\xi^x (y-a+1)^{-k} f(a-1+(y-a+1)^\sigma)\,dy\\
&-\sum_{k=j+1}^{r}\beta_{r,k}(\sigma)(x-a+1)^{k-1}\int_x^\infty (y-a+1)^{-k} f(a-1+(y-a+1)^\sigma)\,dy,
\end{aligned}
$$

where $\beta_{r,k}(\sigma)$ *are defined in* (4.2).

Proposition 4.4 generalizes to

Proposition 4.14. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi, \eta > a$ *and* $w = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ *with* $\gamma_a \in \Gamma_+(p)$ *and* $\gamma_\infty, (\gamma_\infty + 1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$. *Let* j, j' *be determined by* $\Gamma^*_j(p) \ni \gamma_\infty$ *and* $\Gamma^*_{j'}(p) \ni (\gamma_\infty + 1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_a^{\tau_a} \chi_{a-1}^{\tau_\infty - \tau_a}$, $\tau_a, \tau_\infty \in \mathbb{R}$. *Then*

$$\begin{aligned} B_j(\sigma; \infty, a; \xi) : (L_p(w \chi_{a-1}^{-(\gamma_\infty + 1/p)\lambda})(a, \infty), AC^{r-1}_{loc}, \phi \chi_{a-1}^{(r-\tau_\infty)\lambda} D^r) \\ \rightleftharpoons (L_p(w)(a,\infty), AC^{r-1}_{loc}, \phi D^r) : B_{j'}(\sigma^{-1}; \infty, a; \eta), \end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Definition 4.4. *Let* $r \in \mathbb{N}$, $\sigma > 0$, $i \in \mathbb{N}$ *as* $i \le r$, *and* $\xi \in (a, b)$. *Let* s *be one of the ends of the interval* (a, b) *and* e *– the other. For* $x \in (a, b)$ *and* $f \in L_{1,loc}(a, b)$, *satisfying the additional requirement* $\chi_s^{(1-i)/\sigma - 1} f \in L_1(s, (s+e)/2)$ *if* $i > 1$, *we set*

$$\begin{aligned} &(B_i(\sigma; s, e; \xi) f)(x) \\ &\qquad = (\mathcal{T}(-s)\, \mathcal{S}((e-s)^{-1})\, B_{i,r}(\sigma; (\xi - s)/(e-s))\, \mathcal{S}(e-s)\, \mathcal{T}(s) f)(x), \end{aligned}$$

that is,

$$\begin{aligned} &(B_i(\sigma; s, e; \xi) f)(x) = f\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^\sigma\right) \\ &+ \frac{1}{e-s} \sum_{k=2}^{i} \beta_{r,k}(\sigma) \left(\frac{x-s}{e-s}\right)^{k-1} \int_s^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s + (e-s)\left(\frac{y-s}{e-s}\right)^\sigma\right) dy, \\ &+ \frac{1}{e-s} \sum_{k=i+1}^{r} \beta_{r,k}(\sigma) \left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s + (e-s)\left(\frac{y-s}{e-s}\right)^\sigma\right) dy, \end{aligned}$$

where $\beta_{r,k}(\sigma)$ *are defined in* (4.2).

Proposition 4.6 generalizes to

Proposition 4.15. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\xi, \eta \in (a, b)$ *and* $w = \chi_s^{\gamma_s} \chi_e^{\gamma_e}$ *with* $\gamma_s, (\gamma_s + 1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$ *and* $\gamma_e \in \Gamma_+(p)$. *Let* i, i' *be determined by* $\Gamma^*_i(p) \ni \gamma_s$ *and* $\Gamma^*_{i'}(p) \ni (\gamma_s + 1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_s^{\tau_s} \chi_e^{\tau_e}$, $\tau_s, \tau_e \in \mathbb{R}$. *Then*

$$\begin{aligned} B_i(\sigma; s, e; \xi) : (L_p(w \chi_s^{-(\gamma_s + 1/p)\lambda})(a, b), AC^{r-1}_{loc}, \phi \chi_s^{(r-\tau_s)\lambda} D^r) \\ \rightleftharpoons (L_p(w)(a, b), AC^{r-1}_{loc}, \phi D^r) : B_{i'}(\sigma^{-1}; s, e; \eta), \end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Let us note that the last proposition generalizes [3, Proposition 5.5].

Definition 4.5. *Let* $r \in \mathbb{N}$, $\sigma < 0$, $i, j \in \mathbb{N}$ *as* $i \le j \le r$, *and* $\xi \in (a, \infty)$. *For* $x \in (a, \infty)$ *and* $f \in L_{1,loc}(a, \infty)$, *satisfying the additional requirements* $\chi_a^{(1-i)/\sigma-1} f \in L_1(a+1, \infty)$ *if* $i > 1$ *and* $\chi_a^{-j/\sigma-1} f \in L_1(a, a+1)$ *if* $j < r$, *we define* $(B_{i,j}(\sigma; a, \infty; \xi) f)(x)$ *by* (4.11).

From Proposition 4.8 we derive the analogue of Proposition 4.13 for a negative σ.

Proposition 4.16. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi, \eta > a$, $w = \chi_a^{\gamma_a} \chi_{a-1}^{\gamma_\infty - \gamma_a}$ *and* $\bar{w} = \chi_a^{\gamma_\infty} \chi_{a-1}^{\gamma_a - \gamma_\infty}$ *with* $\gamma_a, \gamma_\infty, (\gamma_a + 1/p)/\sigma - 1/p, (\gamma_\infty + 1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$. *Assume that* $i \le j$ *and* $i' \le j'$, *where* i, j, i', j' *are determined by* $\Gamma^*_i(p) \ni \gamma_a$, $\Gamma^*_j(p) \ni \gamma_\infty$, $\Gamma^*_{i'}(p) \ni (\gamma_\infty + 1/p)/\sigma - 1/p$ *and* $\Gamma^*_{j'}(p) \ni (\gamma_a + 1/p)/\sigma - 1/p$. *Finally, let* $\phi = \chi_a^{\tau_a} \chi_{a-1}^{\tau_\infty - \tau_a}$ *and* $\bar{\phi} = \chi_a^{\tau_\infty} \chi_{a-1}^{\tau_a - \tau_\infty}$ *with* $\tau_a, \tau_\infty \in \mathbb{R}$. *Then*

$$
\begin{aligned}
&B_{i,j}(\sigma; a, \infty; \xi) : \\
&\quad (L_p(\bar{w} \chi_a^{-(\gamma_\infty + 1/p)\lambda} \chi_{a-1}^{-(\gamma_a - \gamma_\infty)\lambda})(a, \infty), AC_{loc}^{r-1}, \bar{\phi} \chi_a^{(r-\tau_\infty)\lambda} \chi_{a-1}^{-(\tau_a - \tau_\infty)\lambda} D^r) \\
&\qquad\qquad \rightleftharpoons (L_p(w)(a, \infty), AC_{loc}^{r-1}, \phi D^r) : B_{i',j'}(\sigma^{-1}; a, \infty; \eta),
\end{aligned}
$$

where $\lambda = 1 - 1/\sigma$.

Definition 4.6. *Let* $r \in \mathbb{N}$, $\sigma < 0$, $j \in \mathbb{N}$ *as* $j \le r$ *and* (a, b) *be an interval. Let* s *be one of the ends of the interval* (a, b), e *– the other and* $\xi \in (e, \infty)$. *For* $x \in (e, \infty)$ *and* $f \in L_{1,loc}(a, b)$, *satisfying the additional requirement* $\chi_s^{-j/\sigma-1} f \in L_1(s, (s+e)/2)$ *if* $j < r$, *we set*

$$
B_j(\sigma; s, e; \infty, e; \xi) f)(x) = (\mathcal{T}(1-e)\, B_{1,j}(\sigma; \xi - e + 1) \mathcal{S}(e-s)\, \mathcal{T}(s) f)(x),
$$

that is,

$$
\begin{aligned}
B_j(\sigma; &s, e; \infty, e; \xi) f)(x) = f(s + (e-s)(x-e+1)^\sigma) \\
&+ \sum_{k=2}^{j} \beta_{r,k}(\sigma)(x-e+1)^{k-1} \int_\xi^x y^{-k} f(s + (e-s)(y-e+1)^\sigma)\, dy \\
&- \sum_{k=j+1}^{r} \beta_{r,k}(\sigma)(x-e+1)^{k-1} \int_x^\infty y^{-k} f(s + (e-s)(y-e+1)^\sigma)\, dy,
\end{aligned}
$$

where $\beta_{r,k}(\sigma)$ are defined in (4.2).

Definition 4.7. *Let $r \in \mathbb{N}$, $\sigma < 0$, $i \in \mathbb{N}$ as $i \le r$ and (a,b) be an interval. Let s be one of the ends of the interval (a,b), e – the other and $\eta \in (a,b)$. For $x \in (a,b)$ and $f \in L_{1,loc}(e,\infty)$, satisfying the additional requirement $\chi_e^{(1-i)/\sigma-1} f \in L_1(e+1,\infty)$ if $i > 1$, we set*

$$\begin{aligned}(B_i(\sigma;\infty,e;s,e;\eta)f)(x) \\ = (\mathcal{T}(-s)\,\mathcal{S}((e-s)^{-1})\,B_{i,r}(\sigma;(\eta-s)/(e-s))\,\mathcal{T}(e-1)f)(x),\end{aligned}$$

that is,

$$\begin{aligned}(B_i(\sigma;\infty,e;s,e;\eta)f)(x) &= f\left(\left(\frac{x-s}{e-s}\right)^{\sigma}+e-1\right)\\ &+\frac{1}{e-s}\sum_{k=2}^{i}\beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_s^x\left(\frac{y-s}{e-s}\right)^{-k} f\left(\left(\frac{y-s}{e-s}\right)^{\sigma}+e-1\right)dy\\ &+\frac{1}{e-s}\sum_{k=i+1}^{r}\beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_\eta^x\left(\frac{y-s}{e-s}\right)^{-k} f\left(\left(\frac{y-s}{e-s}\right)^{\sigma}+e-1\right)dy,\end{aligned}$$

where $\beta_{r,k}(\sigma)$ are defined in (4.2).

Proposition 4.11 and Proposition 4.12 generalize to

Proposition 4.17. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\xi \in (e,\infty)$, $\eta \in (a,b)$, $w = \chi_e^{\gamma_e}\chi_{e-1}^{\gamma_\infty-\gamma_e}$ and $\bar{w} = \chi_s^{\gamma_\infty}\chi_e^{\gamma_e}$ with $\gamma_e \in \Gamma_+(p)$ and $\gamma_\infty, (\gamma_\infty+1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$. Let j, i' be determined by $\Gamma^*_j(p) \ni \gamma_\infty$ and $\Gamma^*_{i'}(p) \ni (\gamma_\infty+1/p)/\sigma - 1/p$. Finally, let $\phi = \chi_e^{\tau_e}\chi_{e-1}^{\tau_\infty-\tau_e}$, $\bar{\phi} = \chi_s^{\tau_\infty}\chi_e^{\tau_e}$ with $\tau_e, \tau_\infty \in \mathbb{R}$. Then*

$$\begin{aligned}B_j(\sigma;s,e;\infty,e;\xi) : (L_p(\bar{w}\chi_s^{-(\gamma_\infty+1/p)\lambda})(a,b), AC_{loc}^{r-1}, \bar{\phi}\chi_s^{(r-\tau_\infty)\lambda}D^r) \\ \rightleftharpoons (L_p(w)(e,\infty), AC_{loc}^{r-1}, \phi D^r) : B_{i'}(\sigma^{-1};\infty,e;s,e;\eta),\end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Proposition 4.18. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma < 0$, $\eta \in (a,b)$, $\xi \in (e,\infty)$, $w = \chi_s^{\gamma_s}\chi_e^{\gamma_e}$ and $\bar{w} = \chi_e^{\gamma_e}\chi_{e-1}^{\gamma_s-\gamma_e}$ with $\gamma_s, (\gamma_s+1/p)/\sigma - 1/p \notin \Gamma^*_{exc}(p)$ and $\gamma_e \in \Gamma_+(p)$. Let i, j' be determined by $\Gamma^*_i(p) \ni \gamma_s$ and $\Gamma^*_{j'}(p) \ni (\gamma_s+1/p)/\sigma - 1/p$. Finally, let $\phi = \chi_s^{\tau_s}\chi_e^{\tau_e}$, $\bar{\phi} = \chi_e^{\tau_e}\chi_{e-1}^{\tau_s-\tau_e}$ with $\tau_s, \tau_e \in \mathbb{R}$. Then*

$$\begin{aligned}B_i(\sigma;\infty,e;s,e;\eta) : (L_p(\bar{w}\chi_{e-1}^{-(\gamma_s+1/p)\lambda})(e,\infty), AC_{loc}^{r-1}, \bar{\phi}\chi_{e-1}^{(r-\tau_s)\lambda}D^r) \\ \rightleftharpoons (L_p(w)(a,b), AC_{loc}^{r-1}, \phi D^r) : B_{j'}(\sigma^{-1};s,e;\infty,e;\xi),\end{aligned}$$

where $\lambda = 1 - 1/\sigma$.

Remark 4.2. In the case $r = 1$ all results in Section 4 are valid without any restrictions on the weights $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ and $\bar{w} = \chi_a^{\gamma_a}\chi_b^{\gamma_b}$.

4.3. Overview of the transformed operators. We summarize the notations for the operators of type A and B in Table 2. All of them act from a subspace of $L_{1,loc}(\zeta,\eta)$ to a subspace of $L_{1,loc}(\zeta',\eta')$. We denote by s one of the ends of the interval (ζ,η) at which the operator modify a weight singularity and by e the other end. Both $s < e$ and $s > e$ are possible. When $e = \infty$ the operator modifies singularities at both ends s and e and has two indexes. When $e < \infty$ the operator modifies a singularity only at s and has one index.

In order to simplify a little bit the notations we try to consider when possible only the case $(\zeta,\eta) = (\zeta',\eta')$ and to omit ζ',η' from the notation. This is so in all types of operators but for $B_\ell(\sigma)$ with $\sigma < 0$.

Table 2. Overview of the operators

Operators	s, e	Description
$A_{i,j}(\rho; s, e; \xi)$, $\rho \in \mathbb{R}$	$e = \infty$	modify the singularities of w at both ends s and e, $0 \le i \le j \le r$.
$A_\ell(\rho; s, e; \xi)$, $\rho \in \mathbb{R}$	$e < \infty$ $s = \infty$	modify a singularity of w only at s, $0 \le \ell \le r$, ℓ stays for j while $i = 0$.
$A_\ell(\rho; s, e; \xi)$, $\rho \in \mathbb{R}$	$e < \infty$ $s < \infty$	modify a singularity of w only at s, $0 \le \ell \le r$, ℓ stays for i while $j = r$.
$B_{i,j}(\sigma; s, e; \xi)$, $\sigma > 0$	$e = \infty$	modify singularities of w and φ at both ends s and e, $1 \le i \le j \le r$.
$B_\ell(\sigma; s, e; \xi)$, $\sigma > 0$	$e < \infty$ $s = \infty$	modify singularities of w and φ only at s, $1 \le \ell \le r$, ℓ stays for j while $i = 1$.
$B_\ell(\sigma; s, e; \xi)$, $\sigma > 0$	$e < \infty$ $s < \infty$	modify singularities of w and φ only at s, $1 \le \ell \le r$, ℓ stays for i while $j = r$.
$B_{i,j}(\sigma; s, e; \xi)$, $\sigma < 0$	$e = \infty$	modify singularities of w and φ at both ends s and e while interchanging them, $1 \le i \le j \le r$.
$B_\ell(\sigma; s, e; s', e'; \xi)$, $\sigma < 0$, $e = e'$	$e < \infty$ $s < \infty$ $s' = \infty$	modify singularities of w and φ at s and s', preserve singularities at e and e', $1 \le \ell \le r$, ℓ stays for j while $i = 1$.
$B_\ell(\sigma; s, e; s', e'; \xi)$, $\sigma < 0$, $e = e'$	$e < \infty$ $s = \infty$ $s' < \infty$	modify singularities of w and φ at s and s', preserve singularities at e and e', $1 \le \ell \le r$, ℓ stays for i while $j = r$.

The indexes i and j are connected with the behaviour of the image function at a finite end of (ζ', η') and at ∞ respectively. We always require $f \in L_{1,loc}(\zeta, \eta)$. When $i > 0$ in $A(\rho)$ or $i > 1$ in $B(\sigma)$, or when $j < r$ we impose on f additional requirements, which are given in the definitions in Sections 3 and 4.

In all cases ξ belongs to the domain where the images are defined, i.e. $\xi \in (\zeta', \eta')$. The operators do not depend on the value of ξ provided $i = j$ in the description part of the table. In this case we may replace ξ by $*$.

5. Algebraic properties of the operators. Relations (3.3), (4.4) and (4.7) can be extended to the operators we considered in the previous two sections if the fixed integral limit is one and the same in all integral summands in the definition of the operator. Under these assumptions with the notations

$$\eta_{\xi,\sigma} = s + (e-s)\left(\frac{\xi - s}{e - s}\right)^{\sigma}, \quad \tilde{\eta}_{\xi,\sigma} = e - 1 + \left(\frac{\xi - s}{e - s}\right)^{\sigma}$$

the following relations hold provided that all operators involved are defined:

i) $A_{i,j}(\rho; a, \infty; \xi) A_{i,j}(\sigma; a, \infty; \xi) = A_{i,j}(\rho + \sigma; a, \infty; \xi)$ for either $i = j = 0$, or $i = 0, j = r$, or $i = j = r$;

ii) $A_j(\rho; \infty, a; \xi) A_j(\sigma; \infty, a; \xi) = A_j(\rho + \sigma; \infty, a; \xi)$ for either $j = 0$, or $j = r$;

iii) $A_i(\rho; s, e; \xi) A_i(\sigma; s, e; \xi) = A_i(\rho + \sigma; s, e; \xi)$ for either $i = 0$, or $i = r$;

iv) $B_{i,j}(\sigma; a, \infty; \xi) B_{i,j}(\rho; a, \infty; a + (\xi - a)^{\sigma}) = B_{i,j}(\sigma\rho; a, \infty; \xi)$ for either $i = j = 1$, or $i = 1, j = r$, or $i = j = r$;

v) $B_j(\sigma; \infty, a; \xi) B_j(\rho; \infty, a; a - 1 + (\xi - a + 1)^{\sigma}) = B_j(\sigma\rho; \infty, a; \xi)$ for either $j = 0$, or $j = r$;

vi) $B_i(\sigma; s, e; \xi) B_i(\rho; s, e; \eta_{\xi,\sigma}) = B_i(\sigma\rho; s, e; \xi)$ for either $i = 0$, or $i = r$;

vii) $B_j(\sigma; s, e; \infty, e; \xi) B_i(\rho; s, e; s + (e - s)(\xi - e + 1)^{\sigma}) = B_j(\sigma\rho; s, e; \infty, e; \xi)$ for either $i = 1, j = r$, or $i = r, j = 1$;

viii) $B_j(\sigma; \infty, e; \xi) B_j(\rho; s, e; \infty, e, e - 1 + (\xi - e + 1)^{\sigma}) = B_j(\sigma\rho; s, e; \infty, e; \xi)$ for either $j = 0$, or $j = r$;

ix) $B_i(\sigma; \infty, e; s, e; \xi) B_j(\rho; s, e; \infty, e; \tilde{\eta}_{\xi,\sigma}) = B_i(\sigma\rho; s, e; \xi)$ for either $i = 1, j = r$, or $i = r, j = 1$;

x) $B_i(\sigma;\infty,e;s,e;\xi)B_j(\rho;\infty,e;\tilde\eta_{\xi,\sigma}) = B_i(\sigma\rho;\infty,e;s,e;\xi)$ for either $i=1$, $j=r$, or $i=r, j=1$;

xi) $B_i(\sigma;s,e;\xi)B_i(\rho;\infty,e;s,e;\eta_{\xi,\sigma}) = B_i(\sigma\rho;\infty,e;s,e;\xi)$ for either $i=0$, or $i=r$;

xii) $B_j(\sigma;s,e;\infty,e;\xi)B_i(\rho;\infty,e;s,e;s+(e-s)(\xi-e+1)^\sigma) = B_j(\sigma\rho;\infty,e;\xi)$ for either $i=1, j=r$, or $i=r, j=1$;

xiii) $B_{i,j}(\sigma;a,\infty;\xi)A_{i',j'}(\rho;a,\infty;a+(\xi-a)^\sigma) = A_{i',j'}(\rho\sigma;a,\infty;\xi)B_{i,j}(\sigma;a,\infty;\xi)$ for either $i=j=1, i'=j'=0$, or $i=1, i'=0, j=j'=r$, or $i=i'=j=j'=r$;

xiv) $B_j(\sigma;\infty,a;\xi)A_{j'}(\rho;\infty,a;a-1+(\xi-a+1)^\sigma) = A_{j'}(\rho\sigma;\infty,a;\xi)B_j(\sigma;\infty,a;\xi)$ for either $j=1, j'=0$, or $j=j'=r$;

xv) $B_i(\sigma;s,e;\xi)A_{i'}(\rho;s,e;\eta_{\xi,\sigma}) = A_{i'}(\rho\sigma;s,e;\xi)B_i(\sigma;s,e;\xi)$ for either $i=1$, $i'=0$, or $i=i'=r$;

xvi) $B_j(\sigma;s,e;\infty,e;\xi)A_i(\rho;s,e;s+(e-s)(\xi-e+1)^\sigma)$
$= A_{j'}(\rho\sigma;\infty,e;\xi)B_j(\sigma;s,e;\infty,e;\xi)$ for either $i=0, j=j'=r$, or $i=r, j=1, j'=0$;

xvii) $B_i(\sigma;\infty,e;s,e;\xi)A_j(\rho;\infty,e;\tilde\eta_{\xi,\sigma}) = A_{i'}(\rho\sigma;s,e;\xi)B_i(\sigma;\infty,e;s,e;\xi)$ for either $i=1, i'=0, j=r$, or $i=i'=r, j=0$.

All above properties concern operators which treat one and the same singular point (or its image if $B(\sigma)$ with $\sigma<0$ is involved). For the treatment of different singular points we established in [3, Proposition 5.1] that the operators $A_0(\rho;a,b;\xi)$ and $A_0(\sigma;b,a;\xi)$ commute. We can extend this property (provided that all operators involved are defined) to

xviii) $A_i(\rho;a,b;\xi)A_i(\sigma;b,a;\xi) = A_i(\sigma;b,a;\xi)A_i(\rho;a,b;\xi)$ for either $i=0$, or $i=r$;

xix) $A_{0,j}(\rho;a,\infty;\xi)A_j(\sigma;\infty,a;\xi) = A_j(\sigma;\infty,a;\xi)A_{0,j}(\rho;a,\infty;\xi)$ for either $j=0$, or $j=r$.

As it was demonstrated in [3, Remark 5.1] the commutativity is not intrinsic when two consecutive operators, one of which is of type B, are used for treating singularities at opposite ends of the domain.

6. Equivalence between K-functionals. Characterization of K-functionals by one modulus. The results in Sections 3 and 4 enable us to deduce a number of equivalences between different K-functionals and hence their characterization by appropriately defined moduli of smoothness.

Throughout the section we shall use the notations given in Table 3, where the κ's, λ's, μ's and ν's are real numbers.

Table 3. Initial and target weights

$I,\ \tilde{I}$	w	φ	$\tilde{w}$	$\tilde{\varphi}$
(a,b)	$\chi_a^{\kappa_a}\chi_b^{\kappa_b}$	$\chi_a^{\lambda_a}\chi_b^{\lambda_b}$	$\chi_a^{\mu_a}\chi_b^{\mu_b}$	$\chi_a^{\nu_a}\chi_b^{\nu_b}$
(a,∞)	$\chi_a^{\kappa_a}\chi_{a-1}^{\kappa_\infty-\kappa_a}$	$\chi_a^{\lambda_a}\chi_{a-1}^{\lambda_\infty-\lambda_a}$	$\chi_a^{\mu_a}\chi_{a-1}^{\mu_\infty-\mu_a}$	$\chi_a^{\nu_a}\chi_{a-1}^{\nu_\infty-\nu_a}$

We recall that Proposition 2.1 implies that if the linear operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the relation

$$\mathcal{A} : (L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}, \tag{6.1}$$

then we have the equivalences:

$$K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{A}f, t^r; L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$$

and

$$K(F, t^r; L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) \sim K(\mathcal{B}F, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r).$$

We shall construct operators $\mathcal{A}$ and $\mathcal{B}$ satisfying (6.1) as combinations of operators of type A and B, studied in Sections 3 and 4. In some cases we shall need to index the A and B operators by subscripts of the type i_k, j_k, i'_k and j'_k in order to emphasize their place and role. In forming the subscripts, we follow the rules:

- Subscripts i_k, j_k are used in the definition of $\mathcal{A}$ and primed subscripts i'_k, j'_k – in the definition of $\mathcal{B}$;
- The use of i or j is in conformity with the definitions of the A and B operators (see Subsection 4.3);

- k corresponds to the position of the operator from left to right in the definition of $\mathcal{A}$ and from right to left in the definition of $\mathcal{B}$. Thus the operators with equal values of k in the definition of $\mathcal{A}$ and $\mathcal{B}$ are quasi-inverse to one another.

We arrange the results into several subsections according to the relations between the exponents of the weights φ and $\tilde{\varphi}$ in (1.2). Examining all operators in Section 4 we see that only the two operators $B_\ell(\sigma; s, e; s', e'; \xi)$ (with $\sigma < 0$ and one of s and s' being a finite number and the other – infinity) change the sign of $(1-\lambda_s)(1-\lambda_e)$, but simultaneously they change the type of the interval from finite to semi-infinite or vice versa. All other operators preserve both the sign of $(1-\lambda_s)(1-\lambda_e)$ and the type of the interval (either finite or semi-infinite). So, the cases when $(1-\lambda_s)(1-\lambda_e)(1-\nu_{s'})(1-\nu_{e'}) > 0$ are considered in the first three subsections – the finite interval is treated in Subsection 6.1, while the semi-infinite interval is treated in Subsections 6.2 and 6.3. The cases when $(1-\lambda_s)(1-\lambda_e)(1-\nu_{s'})(1-\nu_{e'}) < 0$ together with a change of the type of the interval (three of the points s, e, s', e' are finite and one is infinite) are considered in Subsection 6.4.

6.1. The case $(1-\lambda_a)(1-\lambda_b)(1-\nu_a)(1-\nu_b) > 0$. We start with the case $(1-\lambda_a)(1-\nu_a) > 0, (1-\lambda_b)(1-\nu_b) > 0$ on the finite interval (a, b). By means of Propositions 3.9 and 4.15 we first extend the result in [3, Theorem 5.3] by weakening the condition $\kappa_a > -1/p$ to $\kappa_a \notin \Gamma_{exc}(p)$.

Proposition 6.1. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_b, \mu_a, \mu_b, \lambda_a, \lambda_b, \nu_a, \nu_b$ and the integer i' satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_a) > 0, \quad (1-\lambda_b)(1-\nu_b) > 0,$$

$$\kappa_a \in \Gamma_{i'}(p), \quad \kappa_b, \mu_a, \mu_b > -1/p.$$

Set

$$\mathcal{A} = B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_b; b, a; \xi) A_0(\rho_a; a, b; \xi),$$

$$\mathcal{B} = A_{i'}(-\rho_a; a, b; \eta) A_0(-\rho_b; b, a; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_b^{-1}; b, a; \eta),$$

where $\xi, \eta \in (a, b)$ and

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\nu_b}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a + 1/p}{\sigma_a}, \quad \rho_b = \kappa_b + \frac{1}{p} - \frac{\mu_b + 1/p}{\sigma_b}.$$

Then

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. The assertion of the theorem follows from the relations:

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{\mu_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\nu_b}D^r)}$$

Step 1 $\qquad B_1(\sigma_b; b, a; \xi) \ \upharpoonleft\downharpoonright \ B_1(\sigma_b^{-1}; b, a; \eta)$

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{(\mu_b+1/p)/\sigma_b-1/p})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\lambda_b}D^r)}$$

Step 2 $\qquad B_1(\sigma_a; a, b; \xi) \ \upharpoonleft\downharpoonright \ B_1(\sigma_a^{-1}; a, b; \eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_b^{(\mu_b+1/p)/\sigma_b-1/p})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}$$

Step 3 $\qquad A_0(\rho_b; b, a; \xi) \ \upharpoonleft\downharpoonright \ A_0(-\rho_b; b, a; \eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}$$

Step 4 $\qquad A_0(\rho_a; a, b; \xi) \ \upharpoonleft\downharpoonright \ A_{i'}(-\rho_a; a, b; \eta)$

$$\boxed{(L_p(\chi_a^{\kappa_a}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}.$$

In this scheme we consider the operators in the definition of $\mathcal{A}$ from left to right because that simplifies the application of Proposition 3.9 and Proposition 4.15.

Step 1. We use Proposition 4.15 with $s = b$, $e = a$, $i = i' = 1$, $\sigma = \sigma_b > 0$, $\gamma_e = \mu_a \in \Gamma_+(p)$, $\gamma_s = \mu_b \in \Gamma_1^*(p)$, $\tau_s = r\nu_s$ and $\tau_e = r\nu_e$ as we take into consideration that $(\mu_b + 1/p)/\sigma_b - 1/p \in \Gamma_1^*(p)$ for $\mu_b > -1/p$ and $\sigma_b > 0$, and also $\tau_b + (r - \tau_b)(1 - 1/\sigma_b) = r\nu_b + r - r\nu_b - r(1 - \nu_b)/\sigma_b = r\lambda_b$.

Step 2. We apply again Proposition 4.15 but now in respect to the weight singularity at the other end of the interval, i.e. for $s = a$, $e = b$. So we put in Proposition 4.15 $i = i' = 1$, $\sigma = \sigma_a > 0$, $\gamma_s = \mu_a \in \Gamma_1^*(p)$, $\gamma_e = (\mu_b + 1/p)/\sigma_b - 1/p \in \Gamma_+(p)$, $\tau_s = r\nu_s$ and $\tau_e = r\lambda_e$. As above we also have $(\mu_a + 1/p)/\sigma_a - 1/p \in \Gamma_1^*(p)$ and $\tau_a + (r - \tau_a)(1 - 1/\sigma_a) = r\lambda_a$.

Step 3. We apply Proposition 3.9 with $s = b$, $e = a$, $i = i' = 0$, $\rho = \rho_b$, $\gamma_e = (\mu_a + 1/p)/\sigma_a - 1/p \in \Gamma_+(p)$ (since $\mu_a > -1/p$, $\sigma_a > 0$), $\gamma_s = (\mu_b + 1/p)/\sigma_b - 1/p \in \Gamma_0(p)$ (since $\mu_b > -1/p$, $\sigma_b > 0$) and $\phi = \chi_a^{r\lambda_a} \chi_b^{r\lambda_b}$. Let us observe that $\gamma_s + \rho = \kappa_b \in \Gamma_0(p)$.

Step 4. We use Proposition 3.9 with $s = a$, $e = b$, $i = 0$, $\rho = \rho_a$, $\gamma_s = (\mu_a + 1/p)/\sigma_a - 1/p \in \Gamma_0(p)$, $\gamma_e = \kappa_b \in \Gamma_+(p)$ and $\phi = \chi_a^{r\lambda_a} \chi_b^{r\lambda_b}$ as $\gamma_s + \rho = \kappa_a \in \Gamma_{i'}(p)$. $\square$

Remark 6.1. It is not necessary for the point $\xi \in (a, b)$ to be one and the same in all the components of the definition of the operator $\mathcal{A}$. We use one and the same point ξ in order to simplify the notations. The same is true for the point η and the operator $\mathcal{B}$.

If we combine operators of type A and B in another way, then we get equivalence between the K-functionals under weaker assumptions than in Proposition 6.1 – the restriction $\mu_b > -1/p$ is replaced by $\mu_b \notin \Gamma_{exc}(p)$. Note that the $\mathcal{A}$ and $\mathcal{B}$ operators in the two propositions are different as in Proposition 6.2 below λ_b and ν_b are interchanged in σ_b as well as κ_b and μ_b in ρ_b, comparing to σ_a and ρ_a respectively (cf. Sec. 5, xv).

Proposition 6.2. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_b, \mu_a, \mu_b, \lambda_a, \lambda_b, \nu_a, \nu_b$ and the integers i_1, i_4' satisfy the conditions:*

$$(1 - \lambda_a)(1 - \nu_a) > 0, \quad (1 - \lambda_b)(1 - \nu_b) > 0,$$

$$\kappa_a \in \Gamma_{i_4'}(p), \quad \kappa_b, \mu_a > -1/p, \quad \mu_b \in \Gamma_{i_1}(p).$$

Set

$$\mathcal{A} = A_{i_1}(-\rho_b; b, a; \xi) B_1(\sigma_b^{-1}; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_a; a, b; \xi),$$

$$\mathcal{B} = A_{i_4'}(-\rho_a; a, b; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_b; b, a; \eta) A_0(\rho_b; b, a; \eta),$$

where $\xi, \eta \in (a, b)$ and

$$\sigma_a = \frac{1 - \nu_a}{1 - \lambda_a}, \ \sigma_b = \frac{1 - \lambda_b}{1 - \nu_b},$$

$$\rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a + 1/p}{\sigma_a}, \ \rho_b = \mu_b + \frac{1}{p} - \frac{\kappa_b + 1/p}{\sigma_b}.$$

Then

$$\mathcal{A} : (L_p(w)(a, b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a, b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. As above the assertion of the theorem follows from the relations:

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{\mu_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\nu_b}D^r)}$$

Step 1 $$A_{i_1}(-\rho_b; b, a; \xi) \quad \upharpoonleft\downharpoonright \quad A_0(\rho_b; b, a; \eta)$$

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{(\kappa_b+1/p)/\sigma_b-1/p})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\nu_b}D^r)}$$

Step 2 $$B_1(\sigma_b^{-1}; b, a; \xi) \quad \upharpoonleft\downharpoonright \quad B_1(\sigma_b; b, a; \eta)$$

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\lambda_b}D^r)}$$

Step 3 $$B_1(\sigma_a; a, b; \xi) \quad \upharpoonleft\downharpoonright \quad B_1(\sigma_a^{-1}; a, b; \eta)$$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}$$

Step 4 $$A_0(\rho_a; a, b; \xi) \quad \upharpoonleft\downharpoonright \quad A_{i_4'}(-\rho_a; a, b; \eta)$$

$$\boxed{L_p(\chi_a^{\kappa_a}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}\,.$$

At steps 1 and 4 we use Proposition 3.9 and at steps 2 and 3 – Proposition 4.15. □

Finally, we generalize Proposition 6.1 and Proposition 6.2 by imposing two independent couples of restrictions – one on the κ's and another on the μ's.

Theorem 6.1. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_a) > 0$ *and* $(1-\lambda_b)(1-\nu_b) > 0$. *Let also* $\kappa_a, \kappa_b, \mu_a, \mu_b \notin \Gamma_{exc}(p)$ *as one of the* κ*'s and one of the* μ*'s are in* $\Gamma_0(p)$. *Then there exist linear operators* $\mathcal{A}$ *and* $\mathcal{B}$, *constructed as compositions of the operators in Sections* 3 *and* 4, *such that*

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

Remark 6.2. Explicit constructions of the operators $\mathcal{A}$ and $\mathcal{B}$, whose existence is stated in the theorem, are given in its proof.

P r o o f o f T h e o r e m 6.1. The restrictions on the exponents of the weights w and $\tilde{w}$ fall into at least one of the four combinations:

1) $\kappa_a, \mu_a \notin \Gamma_{exc}(p)$ and $\kappa_b, \mu_b > -1/p$;
2) $\kappa_a, \mu_b \notin \Gamma_{exc}(p)$ and $\kappa_b, \mu_a > -1/p$;
3) $\kappa_a, \mu_a > -1/p$ and $\kappa_b, \mu_b \notin \Gamma_{exc}(p)$; and
4) $\kappa_a, \mu_b > -1/p$ and $\kappa_b, \mu_a \notin \Gamma_{exc}(p)$.

Let us note that by interchanging a and b in 1) and 2), we get respectively 3) and 4). Consequently, if the operators $\mathcal{A}$ and $\mathcal{B}$ give a solution in case 1) (or 2)), then by interchanging a and b in their definition, we get operators, which solve case 3) (or 4)). Thus it is sufficient to consider only the cases 1) and 2). Case 2) is solved by Proposition 6.2.

It only remains to consider case 1). Let i_0 be such that $\Gamma_{i_0}(p) \ni \mu_a$. If $i_0 = 0$, then Proposition 6.1 (or Proposition 6.2) gives an appropriate definition of $\mathcal{A}$ and $\mathcal{B}$. If $i_0 \geq 1$, we fix $\mu_a^\# > -1/p$ and set $\tilde{w}^\# = \chi_a^{\mu_a^\#} \chi_b^{\mu_b}$. Let the parameters $\xi, \eta, \sigma_a, \sigma_b$, ρ_b and i' be defined in Proposition 6.1 and let $\rho_a^\# = \kappa_a + 1/p - (\mu_a^\# + 1/p)/\sigma_a$. Then Proposition 6.1 implies that the operators

$$\mathcal{A}^\# = B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_b; b, a; \xi) A_0(\rho_a^\#; a, b; \xi),$$
$$\mathcal{B}^\# = A_{i'}(-\rho_a^\#; a, b; \eta) A_0(-\rho_b; b, a; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_b^{-1}; b, a; \eta)$$

satisfy

$$\mathcal{A}^\# : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w}^\#)(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}^\#.$$

Next, by Proposition 3.9 we have

$$A_{i_0}(\mu_a^\# - \mu_a; a, b; \xi) : (L_p(\tilde{w}^\#)(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$$
$$\rightleftharpoons (L_p(\tilde{w})(a,b), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : A_0(\mu_a - \mu_a^\#; a, b; \eta).$$

Then $\mathcal{A} = A_{i_0}(\mu_a^\# - \mu_a; a, b; \xi)\mathcal{A}^\#$ and $\mathcal{B} = \mathcal{B}^\# A_0(\mu_a - \mu_a^\#; a, b; \eta)$ satisfy the assertion of the theorem in case 1) and complete the proof. □

As a corollary of Proposition 6.1 and [3, Theorems 5.4] in the partial case $\mu_a = \mu_b = 0$, $\nu_a = \nu_b = 0$, we get the following characterization of the weighted K-functional $K(f, t^r; L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r)$, which generalizes the result in [3, Corollary 5.2].

Theorem 6.2. *Let $r \in \mathbb{N}$, $1 \leq p \leq \infty$ and $\lambda_a, \lambda_b \in (-\infty, 1)$. For $p < \infty$ we assume that $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ as one of them is in $\Gamma_0(p)$, and for $p = \infty$ we assume that $\kappa_a = \kappa_b = 0$. Set*

$$\mathcal{A} = B_1(\sigma_b; b, a; \xi) B_1(\sigma_a; a, b; \xi) A_0(\rho_b; b, a; \xi) A_0(\rho_a; a, b; \xi),$$

where $\xi \in (a,b)$ and

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p}.$$

Then for $t > 0$ and $f \in L_p(w)(a,b)$ we have

$$K(f,t^r;L_p(w)(a,b),AC_{loc}^{r-1},\varphi^r D^r) \sim \omega_r(\mathcal{A}f,t)_{p(a,b)}.$$

P r o o f. The assertion for $p=\infty$ is contained in [3, Theorems 5.4] (or, equivalently, in the first two steps of the proof of Proposition 6.1). Note that $\rho_a=\rho_b=0$ and $\mathcal{A}$ is defined by only two operators of type B.

In the case $1 \le p < \infty$, if $\kappa_a \notin \Gamma_{exc}(p)$ and $\kappa_b > -1/p$, we set $\mu_a=\mu_b=0$ and $\nu_a=\nu_b=0$ in Proposition 6.1 and get

$$K(f,t^r;L_p(w)(a,b),AC_{loc}^{r-1},\varphi^r D^r) \sim K(\mathcal{A}f,t^r;L_p(a,b),AC_{loc}^{r-1},D^r). \tag{6.2}$$

In the opposite case when $\kappa_a > -1/p$ and $\kappa_b \notin \Gamma_{exc}(p)$ we follow the proof of Proposition 6.1 for $\mu_a=\mu_b=0$ and $\nu_a=\nu_b=0$. Steps 1 and 2 are the same whereas steps 3 and 4 are interchanged as the intermediate triplet is $(L_p(\chi_a^{\kappa_a}\chi_b^{(\mu_b+1/p)/\sigma_b-1/p})(a,b),AC_{loc}^{r-1},\chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)$. Thus we get

$$\mathcal{A}^{\#}:(L_p(w)(a,b),AC_{loc}^{r-1},\varphi^r D^r) \rightleftharpoons (L_p(a,b),AC_{loc}^{r-1},D^r):\mathcal{B}^{\#},$$

where

$$\begin{aligned}\mathcal{A}^{\#} &= B_1(\sigma_b;b,a;\xi)B_1(\sigma_a;a,b;\xi)A_0(\rho_a;a,b;\xi)A_0(\rho_b;b,a;\xi),\\ \mathcal{B}^{\#} &= A_{i_4'}(-\rho_b;b,a;\eta)A_0(-\rho_a;a,b;\eta)B_1(\sigma_a^{-1};a,b;\eta)B_1(\sigma_b^{-1};b,a;\eta)\end{aligned}$$

as i_4' is such that $\Gamma_{i_4'}(p) \ni \kappa_b$. Hence (6.2) is true with $\mathcal{A}^{\#}$ instead of $\mathcal{A}$. But it is established in property xviii) in Section 5 (or [3, Proposition 5.1]) that

$$A_0(\rho_a;a,b;\xi)A_0(\rho_b;b,a;\xi)=A_0(\rho_b;b,a;\xi)A_0(\rho_a;a,b;\xi),$$

hence $\mathcal{A}^{\#}=\mathcal{A}$ and (6.2) holds under the assumptions of the theorem. Since the unweighted K-functional on the right of (6.2) is equivalent to the unweighted fixed-step modulus of smoothness ω_r, we get the assertion of the theorem. □

Note that the operator $\mathcal{A}$ in Theorem 6.2 is one and the same for the restrictions $\kappa_a \notin \Gamma_{exc}(p), \kappa_b > -1/p$ and $\kappa_a > -1/p, \kappa_b \notin \Gamma_{exc}(p)$. The same is true for the operator $\mathcal{A}$ in Proposition 6.1, but is not true in general for the operators in Proposition 6.2 and the operator $\mathcal{B}$ in Proposition 6.1.

Let us now consider the case $(1-\lambda_a)(1-\nu_a)<0$, $(1-\lambda_b)(1-\nu_b)<0$. The sub-case $(1-\lambda_a)(1-\nu_b)<0$ has no solution (cf. classes C_1 and C_7 in

Subsection 6.5), while the other sub-case $(1-\lambda_a)(1-\nu_b)>0$ is covered by the next theorem, which easily follows from the results in this subsection.

Theorem 6.3. *Let $r\in\mathbb{N}$, $1\le p\le\infty$, $(1-\lambda_a)(1-\nu_b)>0$ and $(1-\lambda_b)(1-\nu_a)>0$. Let also $\kappa_a,\kappa_b,\mu_a,\mu_b\notin\Gamma_{exc}(p)$ as one of the κ's and one of the μ's are in $\Gamma_0(p)$. Set $\bar{w}=\chi_a^{\kappa_b}\chi_b^{\kappa_a}$ and $\bar{\varphi}=\chi_a^{\lambda_b}\chi_b^{\lambda_a}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the operators in Theorem 6.1, satisfying*

$$\mathcal{A}:(L_p(\bar{w})(a,b),AC_{loc}^{r-1},\bar{\varphi}^rD^r)\rightleftharpoons(L_p(\tilde{w})(a,b),AC_{loc}^{r-1},\tilde{\varphi}^rD^r):\mathcal{B}.$$

Then

$$\mathcal{A}\,\mathcal{S}(-1)\,\mathcal{T}(a+b):(L_p(w)(a,b),AC_{loc}^{r-1},\varphi^rD^r)\rightleftharpoons (L_p(\tilde{w})(a,b),AC_{loc}^{r-1},\tilde{\varphi}^rD^r):\mathcal{T}(-a-b)\,\mathcal{S}(-1)\,\mathcal{B}.$$

Proof. According to the assumptions of the theorem we can apply Theorem 6.1 to the triplets $(L_p(\bar{w})(a,b),AC_{loc}^{r-1},\bar{\varphi}^rD^r)$ and $(L_p(\tilde{w})(a,b),AC_{loc}^{r-1},\tilde{\varphi}^rD^r)$ and the result is modified by Proposition 2.8. □

Remark 6.3. The conditions on φ and $\tilde{\varphi}$ in Theorems 6.1 and 6.3 are different but not disjoint. Both theorems are applicable in the cases when the signs of $(1-\lambda_a)$, $(1-\lambda_b)$, $(1-\nu_a)$, $(1-\nu_b)$ are one and the same.

6.2. The case $(1-\lambda_a)(1-\nu_a)>0,(1-\lambda_\infty)(1-\nu_\infty)>0$. In the semi-finite interval (a,∞) we begin with the case $(1-\lambda_a)(1-\nu_a)>0,(1-\lambda_\infty)(1-\nu_\infty)>0$.

Proposition 6.3. *Let $r\in\mathbb{N}$ and $1\le p\le\infty$. Let the real numbers $\kappa_a,\kappa_\infty,\mu_a,\mu_\infty,\lambda_a,\lambda_\infty,\nu_a,\nu_\infty$ and the integers $j_1,j_2,j_3,j_4,i_4',j_4'$ satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_a)>0,\quad (1-\lambda_\infty)(1-\nu_\infty)>0,$$
$$\kappa_a\in\Gamma_{i_4'}(p),\quad \kappa_\infty\in\Gamma_{j_4'}(p),\quad i_4'\le j_4',\quad \mu_a>-1/p,\quad \mu_\infty\in\Gamma_{j_1}^*(p),$$
$$\left(\mu_\infty+\frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_a}-\frac{1}{p}\in\Gamma_{j_2}^*(p),\quad \left(\mu_\infty+\frac{1}{p}\right)\frac{1-\lambda_\infty}{1-\nu_\infty}-\frac{1}{p}\in\Gamma_{j_3}(p),$$
$$\kappa_\infty-\kappa_a+\left(\mu_a+\frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_a}-\frac{1}{p}\in\Gamma_{j_4}(p).$$

Set

$$\mathcal{A}=B_{1,j_1}(\sigma_a;a,\infty;\xi)B_{j_2}(\sigma_\infty;\infty,a;\xi)A_{j_3}(\rho_\infty;\infty,a;\xi)A_{0,j_4}(\rho_a;a,\infty;\xi),$$
$$\mathcal{B}=A_{i_4',j_4'}(-\rho_a;a,\infty;\eta)A_{j_3'}(-\rho_\infty;\infty,a;\eta)B_{j_2'}(\sigma_\infty^{-1};\infty,a;\eta)B_{1,j_1'}(\sigma_a^{-1};a,\infty;\eta),$$

where $\xi, \eta > a$, $j_1' = j_2$, $j_2' = \max\{1, j_3\}$, $j_3' = j_4$ *and*

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_\infty = \frac{1-\nu_\infty}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_a},$$
$$\rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a + 1/p}{\sigma_a}, \quad \rho_\infty = \kappa_\infty - \kappa_a - \frac{\mu_\infty + 1/p}{\sigma_a\sigma_\infty} + \frac{\mu_a + 1/p}{\sigma_a}.$$

Then

$$\mathcal{A} : (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

Proof. The proof follows the scheme:

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_{a-1}^{\mu_\infty-\mu_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_{a-1}^{r(\nu_\infty-\nu_a)}D^r)}$$

Step 1 $\quad B_{1,j_1}(\sigma_a; a,\infty;\xi) \quad \upharpoonleft\downharpoonright \quad B_{1,j_1'}(\sigma_a^{-1}; a,\infty;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_{a-1}^{(\mu_\infty-\mu_a)/\sigma_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\nu_\infty-\nu_a)/\sigma_a}D^r)}$$

Step 2 $\quad B_{j_2}(\sigma_\infty; \infty, a;\xi) \quad \upharpoonleft\downharpoonright \quad B_{j_2'}(\sigma_\infty^{-1}; \infty, a;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_{a-1}^{\vartheta})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)}$$

Step 3 $\quad A_{j_3}(\rho_\infty; \infty, a;\xi) \quad \upharpoonleft\downharpoonright \quad A_{j_3'}(-\rho_\infty; \infty, a;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_{a-1}^{\kappa_\infty-\kappa_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)}$$

Step 4 $\quad A_{0,j_4}(\rho_a; a,\infty;\xi) \quad \upharpoonleft\downharpoonright \quad A_{i_4',j_4'}(-\rho_a; a,\infty;\eta)$

$$\boxed{L_p(\chi_a^{\kappa_a}\chi_{a-1}^{\kappa_\infty-\kappa_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)},$$

where $\vartheta = (\mu_\infty + 1/p)/(\sigma_a\sigma_\infty) - (\mu_a + 1/p)/\sigma_a = \kappa_\infty - \kappa_a - \rho_\infty$. Step 1 is accomplished by Proposition 4.13, Step 2 – by Proposition 4.14, Step 3 – by Proposition 3.8, and Step 4 – by Proposition 3.7. □

Restrictions like $(\mu_\infty + 1/p)/\sigma_a - 1/p \notin \Gamma_{exc}(p)$ in Proposition 6.3 can be easily avoided by the use of additional operators of type A. By means of Proposition 6.3 and Proposition 3.7 we can construct operators $\mathcal{A}$ and $\mathcal{B}$ which satisfy the following property.

Theorem 6.4. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_a) > 0$ *and* $(1-\lambda_\infty)(1-\nu_\infty) > 0$. *Let also* $\kappa_a \in \Gamma_{i'}(p)$, $\kappa_\infty \in \Gamma_{j'}(p)$ *as* $i' \le j'$ *and* $\mu_a \in \Gamma_i(p)$,

$\mu_\infty \in \Gamma_j(p)$ as $i \le j$. Then there exist linear operators $\mathcal{A}$ and $\mathcal{B}$, constructed as compositions of the operators in Sections 3 and 4, such that

$$\mathcal{A} : (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. Let $\tilde{\rho}$ be so large that

$$\mu_a + \tilde{\rho},\ \mu_\infty + \tilde{\rho} > -1/p, \quad \kappa_\infty - \kappa_a + \left(\mu_a + \tilde{\rho} + \frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_a} > 0.$$

Then Proposition 6.3 implies that the operators

$$\begin{aligned}
\mathcal{A}^{\#} &= B_{1,1}(\sigma_a; a,\infty; *)B_1(\sigma_\infty; \infty, a; *)A_0(\rho_\infty^{\#}; \infty, a; *)A_{0,0}(\rho_a^{\#}; a,\infty; *),\\
\mathcal{B}^{\#} &= A_{i',j'}(-\rho_a^{\#}; a,\infty; \eta)A_0(-\rho_\infty^{\#}; \infty, a; *)B_1(\sigma_\infty^{-1}; \infty, a; *)B_{1,1}(\sigma_a^{-1}; a,\infty; *),
\end{aligned}$$

where $\eta, \sigma_a, \sigma_\infty$ are defined as in Proposition 6.3 and

$$\rho_a^{\#} = \kappa_a + \frac{1}{p} - \frac{\mu_a + \tilde{\rho} + 1/p}{\sigma_a}, \quad \rho_\infty^{\#} = \kappa_\infty - \kappa_a - \frac{\mu_\infty + \tilde{\rho} + 1/p}{\sigma_a\sigma_\infty} + \frac{\mu_a + \tilde{\rho} + 1/p}{\sigma_a},$$

satisfy the relation

$$\mathcal{A}^{\#} : (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w}\chi_a^{\tilde{\rho}})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}^{\#}.$$

By means of Proposition 3.7 we get

$$\begin{aligned}
A_{i,j}(\tilde{\rho}; a,\infty; \xi) : (L_p(\tilde{w}\chi_a^{\tilde{\rho}})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)&\\
\rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) &: A_{0,0}(-\tilde{\rho}; a,\infty; *),
\end{aligned}$$

where $\xi > a$. The operators $\mathcal{A} = A_{i,j}(\tilde{\rho}; a,\infty; \xi)\mathcal{A}^{\#}$ and $\mathcal{B} = \mathcal{B}^{\#}A_{0,0}(-\tilde{\rho}; a,\infty; *)$ satisfy the assertion of the theorem. $\Box$

If we use only operators of type B, we establish the relation

Proposition 6.4. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_\infty, \lambda_a, \lambda_\infty, \nu_a, \nu_\infty$ and the integers j_1, j_2, j_2' satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_a) > 0, \quad (1-\lambda_\infty)(1-\nu_\infty) > 0,$$
$$\kappa_a \in \Gamma_+(p), \quad \kappa_\infty \in \Gamma_{j_2'}^*(p), \quad \left(\kappa_\infty + \frac{1}{p}\right)\frac{1-\nu_\infty}{1-\lambda_\infty} - \frac{1}{p} \in \Gamma_{j_1}^*(p),$$
$$\left(\kappa_\infty + \frac{1}{p}\right)\frac{1-\nu_\infty}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_a} - \frac{1}{p} \in \Gamma_{j_2}^*(p).$$

Set

$$\mu_a = \left(\kappa_a + \frac{1}{p}\right)\sigma_a - \frac{1}{p}, \quad \mu_\infty = \left(\kappa_\infty + \frac{1}{p}\right)\sigma_a\sigma_\infty - \frac{1}{p}$$

and

$$\mathcal{A} = B_{1,j_1}(\sigma_a; a, \infty; \xi) B_{j_2}(\sigma_\infty; \infty, a; \xi),$$
$$\mathcal{B} = B_{j_2'}(\sigma_\infty^{-1}; \infty, a; \eta) B_{1,j_1'}(\sigma_a^{-1}; a, \infty; \eta),$$

where $\xi, \eta > a$, $j_1' = j_2$ *and*

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_\infty = \frac{1-\nu_\infty}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_a}.$$

Then

$$\mathcal{A} : (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. The first two steps in the proof of Proposition 6.3 verify this assertion. □

By Propositions 3.7, 6.3 and 6.4 we establish the following characterization.

Theorem 6.5. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$ *and* $\lambda_a, \lambda_\infty \in (-\infty, 1)$. *Let us set*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_\infty = \frac{1-\lambda_a}{1-\lambda_\infty}.$$

a) *For* $p < \infty$ *we assume that* $\kappa_a \in \Gamma_{i'}(p)$ *and* $\kappa_\infty \in \Gamma_{j'}(p)$ *as* $i' \le j'$ *and set*

$$\mathcal{A} = A_{0,0}(\tilde{\rho}; a, \infty; *) B_{1,1}(\sigma_a; a, \infty; *) B_1(\sigma_\infty; \infty, a; *) A_0(\rho_\infty; \infty, a; *) A_{0,0}(\rho_a; a, \infty; *),$$

where $\tilde{\rho} + 1/p > \max\{0, (\kappa_a - \kappa_\infty)\sigma_a\}$ *and*

$$\rho_a = \kappa_a - \tilde{\rho} + \left(\tilde{\rho} + \frac{1}{p}\right)\lambda_a, \quad \rho_\infty = \kappa_\infty - \kappa_a + \left(\tilde{\rho} + \frac{1}{p}\right)(\lambda_\infty - \lambda_a).$$

b) *For* $p = \infty$ *we assume that* $\kappa_a = \kappa_\infty = 0$ *and set*

$$\mathcal{A} = B_{1,1}(\sigma_a; a, \infty; *) B_1(\sigma_\infty; \infty, a; *).$$

Then for $t > 0$ and $f \in L_p(w)(a,\infty)$ we have

$$K(f,t^r;L_p(w)(a,\infty),AC_{loc}^{r-1},\varphi^r D^r) \sim \omega_r(\mathcal{A}f,t)_{p(a,\infty)}.$$

P r o o f. Assertion b) follows from Proposition 6.4 with $p=\infty$ and $\kappa_a = \kappa_\infty = 0$, $\nu_a = \nu_b = 0$. For $p < \infty$ we set in Proposition 6.3 $\mu_a = \mu_\infty = \tilde{\rho}$, $\nu_a = \nu_\infty = 0$ and get

$$\text{(6.3)} \quad \mathcal{A}^{\#} : (L_p(w)(a,\infty),AC_{loc}^{r-1},\varphi^r D^r) \rightleftharpoons (L_p(\chi_a^{\tilde{\rho}})(a,\infty),AC_{loc}^{r-1},D^r) : \mathcal{B}^{\#},$$

where

$$\mathcal{A}^{\#} = B_{1,1}(\sigma_a;a,\infty;*)B_1(\sigma_\infty;\infty,a;*)A_0(\rho_\infty;\infty,a;*)A_{0,0}(\rho_a;a,\infty;*),$$
$$\mathcal{B}^{\#} = A_{i',j'}(-\rho_a;a,\infty;\eta)A_0(-\rho_\infty;\infty,a;*)B_1(\sigma_\infty^{-1};\infty,a;*)B_{1,1}(\sigma_a^{-1};a,\infty;*)$$

and $\eta > a$. Next, by Proposition 3.7 we get

$$\text{(6.4)} \quad A_{0,0}(\tilde{\rho};a,\infty;*) : (L_p(\chi_a^{\tilde{\rho}})(a,\infty),AC_{loc}^{r-1},\varphi^r D^r) \rightleftharpoons (L_p(a,\infty),AC_{loc}^{r-1},\varphi^r D^r) : A_{0,0}(-\tilde{\rho};a,\infty;*).$$

Now, (6.3) and (6.4) yield assertion a). □

Let us note that in the case $p<\infty$ if $(\kappa_\infty - \kappa_a)\sigma_a > -1/p$, then we can fix $\tilde{\rho}=0$ and the operator $\mathcal{A}$ is defined by four operators of type A and B. Also, if $\kappa_a > -1/p$ and $\kappa_\infty \notin \Gamma_{exc}(p)$, then we can use in Theorem 6.5 the operator

$$\mathcal{A} = B_{1,1}(\sigma_a;a,\infty;*)A_{0,0}(\bar{\rho}_a;a,\infty;*)B_1(\sigma_\infty;\infty,a;*)A_0(\bar{\rho}_\infty;\infty,a;*),$$

where σ_a, σ_∞ are as in the theorem, $\bar{\rho}_a = \kappa_a + \lambda_a/p$ and $\bar{\rho}_\infty = \kappa_\infty + 1/p - (\kappa_a + 1/p)/\sigma_\infty$.

6.3. The case $(1-\lambda_a)(1-\nu_\infty) < 0, (1-\lambda_\infty)(1-\nu_a) < 0$. Let us now consider the other possible case for the semi-infinite interval (a,∞), namely $(1-\lambda_a)(1-\nu_a) < 0$, $(1-\lambda_\infty)(1-\nu_\infty) < 0$. Unlike the finite interval, here the sub-case $(1-\lambda_a)(1-\nu_\infty) > 0$ has no solution (cf. classes C_2 and C_8 in Subsection 6.5), while the other sub-case $(1-\lambda_a)(1-\nu_\infty) < 0$ permits quasi-invertible continuous mappings. The results do not follow easily from the results in Subsection 6.2 (no "mirror" operators for semi-infinite intervals) and require the use of operator $B_{i,j}(\sigma;a,\infty;\xi)$ with $\sigma < 0$.

The restrictions on the exponents of φ and $\tilde{\varphi}$ can be summarized as $(1-\lambda_a)(1-\nu_\infty)<0, (1-\lambda_\infty)(1-\nu_a)<0$. Note the specific restriction $\mu_\infty < -1/p$ in the following proposition.

Proposition 6.5. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_\infty, \mu_a, \mu_\infty, \lambda_a, \lambda_\infty, \nu_a, \nu_\infty$ and the integers $i_1, j_1, j_2, j_3, j_4, i'_4, j'_4$ satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_\infty)<0, \quad (1-\lambda_\infty)(1-\nu_a)<0,$$
$$\kappa_a \in \Gamma_{i'_4}(p), \quad \kappa_\infty \in \Gamma_{j'_4}(p),\ i'_4 \le j'_4, \quad \mu_a \in \Gamma^*_{i_1}(p), \quad \mu_\infty \in \Gamma_{j_1}(p),\ 1 \le i_1 \le j_1,$$
$$\left(\mu_a+\frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_\infty}-\frac{1}{p} \in \Gamma^*_{j_2}(p), \quad \left(\mu_a+\frac{1}{p}\right)\frac{1-\lambda_\infty}{1-\nu_a}-\frac{1}{p} \in \Gamma_{j_3}(p),$$
$$\kappa_\infty-\kappa_a+\left(\mu_\infty+\frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_\infty}-\frac{1}{p} \in \Gamma_{j_4}(p).$$

Set

$$\mathcal{A}=B_{i_1,j_1}(\sigma_{a,\infty};a,\infty;\xi)B_{j_2}(\sigma_{\infty,a};\infty,a;\xi)A_{j_3}(\rho_{\infty,a};\infty,a;\xi)A_{0,j_4}(\rho_{a,\infty};a,\infty;\xi),$$
$$\mathcal{B}=A_{i'_4,j'_4}(-\rho_{a,\infty};a,\infty;\eta)A_{j'_3}(-\rho_{\infty,a};\infty,a;\eta)$$
$$B_{j'_2}(\sigma^{-1}_{\infty,a};\infty,a;\eta)B_{1,j'_1}(\sigma^{-1}_{a,\infty};a,\infty;\eta),$$

where $\xi,\eta>a$, $j'_1=j_2$, $j'_2=\max\{1,j_3\}$, $j'_3=j_4$ and

$$\sigma_{a,\infty}=\frac{1-\nu_\infty}{1-\lambda_a}, \quad \sigma_{\infty,a}=\frac{1-\nu_a}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_\infty},$$
$$\rho_{a,\infty}=\kappa_a+\frac{1}{p}-\frac{\mu_\infty+1/p}{\sigma_{a,\infty}}, \quad \rho_{\infty,a}=\kappa_\infty-\kappa_a-\frac{\mu_a+1/p}{\sigma_{a,\infty}\sigma_{\infty,a}}+\frac{\mu_\infty+1/p}{\sigma_{a,\infty}}.$$

Then

$$\mathcal{A}:(L_p(w)(a,\infty),AC^{r-1}_{loc},\varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty),AC^{r-1}_{loc},\tilde{\varphi}^r D^r):\mathcal{B}.$$

P r o o f. The proof follows the scheme:

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_{a-1}^{\mu_\infty-\mu_a})(a,\infty),AC^{r-1}_{loc},\chi_a^{r\nu_a}\chi_{a-1}^{r(\nu_\infty-\nu_a)}D^r)}$$

Step 1 $\quad B_{i_1,j_1}(\sigma_{a,\infty};a,\infty;\xi) \quad \upharpoonleft\downharpoonright \quad B_{1,j'_1}(\sigma^{-1}_{a,\infty};a,\infty;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_\infty+1/p)/\sigma_{a,\infty}-1/p}\chi_{a-1}^{(\mu_a-\mu_\infty)/\sigma_{a,\infty}})(a,\infty),AC^{r-1}_{loc},\chi_a^{r\lambda_a}\chi_{a-1}^{r(\nu_a-\nu_\infty)/\sigma_{a,\infty}}D^r)}$$

Step 2 $\quad B_{j_2}(\sigma_{\infty,a};\infty,a;\xi) \quad \upharpoonleft\downharpoonright \quad B_{j'_2}(\sigma^{-1}_{\infty,a};\infty,a;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_\infty+1/p)/\sigma_{a,\infty}-1/p}\chi_{a-1}^{\vartheta})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)}$$

Step 3 $\qquad A_{j_3}(\rho_{\infty,a};\infty,a;\xi) \quad \upharpoonleft\downharpoonright \quad A_{j_3'}(-\rho_{\infty,a};\infty,a;\eta)$

$$\boxed{(L_p(\chi_a^{(\mu_\infty+1/p)/\sigma_{a,\infty}-1/p}\chi_{a-1}^{\kappa_\infty-\kappa_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)}$$

Step 4 $\qquad A_{0,j_4}(\rho_{a,\infty};a,\infty;\xi) \quad \upharpoonleft\downharpoonright \quad A_{i_4',j_4'}(-\rho_{a,\infty};a,\infty;\eta)$

$$\boxed{L_p(\chi_a^{\kappa_a}\chi_{a-1}^{\kappa_\infty-\kappa_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_{a-1}^{r(\lambda_\infty-\lambda_a)}D^r)},$$

where $\vartheta = (\mu_a + 1/p)/(\sigma_{a,\infty}\sigma_{\infty,a}) - (\mu_\infty + 1/p)/\sigma_{a,\infty} = \kappa_\infty - \kappa_a - \rho_{\infty,a}$. Step 1 is accomplished by Proposition 4.16, Step 2 – by Proposition 4.14, Step 3 – by Proposition 3.8, and Step 4 – by Proposition 3.7. $\square$

From Proposition 6.5 and Proposition 3.7 we get

Theorem 6.6. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_\infty) < 0$ and $(1-\lambda_\infty)(1-\nu_a) < 0$. Let also $\kappa_a \in \Gamma_{i'}(p)$, $\kappa_\infty \in \Gamma_{j'}(p)$ as $i' \le j'$ and $\mu_a \in \Gamma_i(p)$, $\mu_\infty \in \Gamma_j(p)$ as $i \le j$. Then there exist linear operators $\mathcal{A}$ and $\mathcal{B}$, constructed as compositions of the operators in Sections 3 and 4, such that*

$$\mathcal{A} : (L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. Since the proof is similar to the one of Theorem 6.4 we shall only sketch it. We fix $\tilde{\rho}$ such that

$$\mu_a + \tilde{\rho},\ \mu_\infty + \tilde{\rho} < 1 - r - 1/p, \quad \kappa_\infty - \kappa_a + \left(\mu_\infty + \tilde{\rho} + \frac{1}{p}\right)\frac{1-\lambda_a}{1-\nu_\infty} > 0.$$

Proposition 6.5 gives an operator which maps quasi-invertibly continuously the triplet $(L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r)$ onto $(L_p(\tilde{w}\chi_a^{\tilde{\rho}})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$. Now, Proposition 3.7 gives an operator of type A which maps quasi-invertibly continuously the latter onto the triplet $(L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$. $\square$

Remark 6.4. The conditions on φ and $\tilde{\varphi}$ in Theorems 6.4 and 6.6 are different but not disjoint. Both theorems are applicable in the cases when the signs of $(1-\lambda_a)$, $-(1-\lambda_\infty)$, $(1-\nu_a)$, $-(1-\nu_\infty)$ are one and the same (cf. Remark 6.3).

If we use only operators of type B, we establish the relation

Proposition 6.6. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_\infty, \lambda_a, \lambda_\infty, \nu_a, \nu_\infty$ and the integers i_1, j_1, j_2, j_2' satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_\infty) < 0, \quad (1-\lambda_\infty)(1-\nu_a) < 0,$$
$$\kappa_a \in \Gamma_+(p), \quad \kappa_\infty \in \Gamma^*_{j_2'}(p),$$
$$\left(\kappa_\infty + \frac{1}{p}\right)\frac{1-\nu_a}{1-\lambda_\infty} - \frac{1}{p} \in \Gamma^*_{i_1}(p), \quad \left(\kappa_a + \frac{1}{p}\right)\frac{1-\nu_\infty}{1-\lambda_a} - \frac{1}{p} \in \Gamma^*_{j_1}(p), \quad i_1 \le j_1,$$
$$\left(\kappa_\infty + \frac{1}{p}\right)\frac{1-\nu_a}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_\infty} - \frac{1}{p} \in \Gamma^*_{j_2}(p).$$

Set

$$\mu_a = \left(\kappa_\infty + \frac{1}{p}\right)\sigma_{a,\infty}\sigma_{\infty,a} - \frac{1}{p}, \quad \mu_\infty = \left(\kappa_a + \frac{1}{p}\right)\sigma_{a,\infty} - \frac{1}{p},$$

and

$$\mathcal{A} = B_{i_1,j_1}(\sigma_{a,\infty}; a, \infty; \xi) B_{j_2}(\sigma_{\infty,a}; \infty, a; \xi),$$
$$\mathcal{B} = B_{j_2'}(\sigma^{-1}_{\infty,a}; \infty, a; \eta) B_{1,j_1'}(\sigma^{-1}_{a,\infty}; a, \infty; \eta),$$

where $\xi, \eta > a$, $j_1' = j_2$ and

$$\sigma_{a,\infty} = \frac{1-\nu_\infty}{1-\lambda_a}, \quad \sigma_{\infty,a} = \frac{1-\nu_a}{1-\lambda_\infty}\frac{1-\lambda_a}{1-\nu_\infty}.$$

Then

$$\mathcal{A} : (L_p(w)(a,\infty), AC^{r-1}_{loc}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC^{r-1}_{loc}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. The first two steps in the proof of Proposition 6.5 verify this assertion. $\square$

Now we can establish (1.3) for K-functionals with $\lambda_a, \lambda_\infty > 1$.

Theorem 6.7. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$ and $\lambda_a, \lambda_\infty \in (1, \infty)$. Let us set*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_\infty = \frac{1-\lambda_a}{1-\lambda_\infty}.$$

a) *For $p < \infty$ we assume that $\kappa_a \in \Gamma_{i'}(p)$ and $\kappa_\infty \in \Gamma_{j'}(p)$ as $i' \le j'$ and set*

$$\mathcal{A} = A_{0,0}(\tilde{\rho}; a, \infty; *) B_{r,r}(\sigma_a; a, \infty; *) B_1(\sigma_\infty; \infty, a; *)$$
$$A_0(\rho_\infty; \infty, a; *) A_{0,0}(\rho_a; a, \infty; *),$$

where $\tilde{\rho} + 1/p < \min\{1 - r, (\kappa_a - \kappa_\infty)\sigma_a\}$ *and*

$$\rho_a = \kappa_a - \tilde{\rho} + \left(\tilde{\rho} + \frac{1}{p}\right)\lambda_a, \quad \rho_\infty = \kappa_\infty - \kappa_a + \left(\tilde{\rho} + \frac{1}{p}\right)(\lambda_\infty - \lambda_a).$$

b) *For* $p = \infty$ *we assume that* $\kappa_a = \kappa_\infty = 0$ *and set*

$$\mathcal{A} = B_{1,1}(\sigma_a; a, \infty; *) B_1(\sigma_\infty; \infty, a; *).$$

Then for $t > 0$ *and* $f \in L_p(w)(a, \infty)$ *we have*

$$K(f, t^r; L_p(w)(a, \infty), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p(a,\infty)}.$$

P r o o f. Assertion b) is contained in Proposition 6.6. To prove a) we proceed as in the proof of Theorem 6.5. We set

$$\begin{aligned}\mathcal{A}^{\#} &= B_{r,r}(\sigma_a; a, \infty; *) B_1(\sigma_\infty; \infty, a; *) A_0(\rho_\infty; \infty, a; *) A_{0,0}(\rho_a; a, \infty; *),\\ \mathcal{B}^{\#} &= A_{i',j'}(-\rho_a; a, \infty; \eta) A_0(-\rho_\infty; \infty, a; *) B_1(\sigma_\infty^{-1}; \infty, a; *) B_{1,1}(\sigma_a^{-1}; a, \infty; *),\end{aligned}$$

where $\eta > a$. Then Proposition 3.7 and Proposition 6.5 (with $\mu_a = \mu_\infty = \tilde{\rho}$ and $\nu_a = \nu_\infty = 0$) imply

$$\begin{array}{c} \boxed{(L_p(a, \infty), AC_{loc}^{r-1}, D^r)} \\ A_{0,0}(\tilde{\rho}; a, \infty; *) \quad \upharpoonleft\downharpoonright \quad A_{r,r}(-\tilde{\rho}; a, \infty; *) \\ \boxed{(L_p(\chi_a^{\tilde{\rho}})(a, \infty), AC_{loc}^{r-1}, D^r)} \\ \mathcal{A}^{\#} \quad \upharpoonleft\downharpoonright \quad \mathcal{B}^{\#} \\ \boxed{(L_p(w)(a, \infty), AC_{loc}^{r-1}, \varphi^r D^r)}, \end{array}$$

which verifies a). $\square$

6.4. Transfer between finite and semi-infinite intervals. In this subsection we consider the cases which require the type of the interval to be changed. The quasi-invertible maps work in both directions. So, without loss of generality, we may assume that the initial triplet is defined on a finite interval (a, b) and the target triplet is defined on (a, ∞). Hence, the requirement for λ's and ν's is $(1 - \lambda_a)(1 - \lambda_b)(1 - \nu_a)(1 - \nu_\infty) < 0$.

We start with the case $(1-\lambda_a)(1-\nu_a) > 0$, $(1-\lambda_b)(1-\nu_\infty) < 0$. Like in the previous subsection the specific restriction in the first proposition is $\mu_\infty < -1/p$.

Proposition 6.7. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_b, \mu_a, \mu_\infty, \lambda_a, \lambda_b, \nu_a, \nu_\infty$ and the integers j_1, i'_4 satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_a) > 0, \quad (1-\lambda_b)(1-\nu_\infty) < 0,$$
$$\kappa_a \in \Gamma_{i'_4}(p), \quad \kappa_b, \mu_a > -1/p, \quad \mu_\infty \in \Gamma_{j_1}(p), \quad j_1 > 0.$$

Set $\mathcal{A} = B_{j_1}(\sigma_{b,\infty}; b, a; \infty, a; \xi_1) B_1(\sigma_a; a, b; \xi_2) A_0(\rho_{b,\infty}; b, a; \xi_2) A_0(\rho_a; a, b; \xi_2)$,

$$\mathcal{B} = A_{i'_4}(-\rho_a; a, b; \eta) A_0(-\rho_{b,\infty}; b, a; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_{b,\infty}^{-1}; \infty, a; b, a; \eta),$$

where $\xi_1 \in (a, \infty), \xi_2, \eta \in (a, b)$ and

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_{b,\infty} = \frac{1-\nu_\infty}{1-\lambda_b},$$
$$\rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a + 1/p}{\sigma_a}, \quad \rho_{b,\infty} = \kappa_b + \frac{1}{p} - \frac{\mu_\infty + 1/p}{\sigma_{b,\infty}}.$$

Then

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. The proof follows the scheme (cf. the proof of Proposition 6.1 for steps 2–4):

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_{a-1}^{\mu_\infty-\mu_a})(a,\infty), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_{a-1}^{r(\nu_\infty-\nu_a)}D^r)}$$

Step 1 $\qquad B_{j_1}(\sigma_{b,\infty}; b, a; \infty, a; \xi_1) \ \upharpoonleft\downharpoonright\ B_1(\sigma_{b,\infty}^{-1}; \infty, a; b, a; \eta)$

$$\boxed{(L_p(\chi_a^{\mu_a}\chi_b^{(\mu_\infty+1/p)/\sigma_{b,\infty}-1/p})(a,b), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\lambda_b}D^r)}$$

Step 2 $\qquad B_1(\sigma_a; a, b; \xi_2) \ \upharpoonleft\downharpoonright\ B_1(\sigma_a^{-1}; a, b; \eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_b^{(\mu_\infty+1/p)/\sigma_{b,\infty}-1/p})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}$$

Step 3 $\qquad A_0(\rho_{b,\infty}; b, a; \xi_2) \ \upharpoonleft\downharpoonright\ A_0(-\rho_{b,\infty}; b, a; \eta)$

$$\boxed{(L_p(\chi_a^{(\mu_a+1/p)/\sigma_a-1/p}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}$$

Step 4 $\qquad A_0(\rho_a; a, b; \xi_2) \ \upharpoonleft\downharpoonright\ A_{i'_4}(-\rho_a; a, b; \eta)$

$$\boxed{(L_p(\chi_a^{\kappa_a}\chi_b^{\kappa_b})(a,b), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b}D^r)}\,.$$

Step 1 is accomplished by Proposition 4.17, Step 2 – by Proposition 4.15 and steps 3 and 4 – by Proposition 3.9. □

Remark 6.5. Let us note that by interchanging steps 3 and 4 in the proof of Proposition 6.7 we establish that if

$$\kappa_a, \mu_a > -1/p, \quad \kappa_b \in \Gamma_{i'_4}(p), \quad \mu_\infty \in \Gamma_{j_1}(p), \quad j_1 > 0,$$

then the operators

$$\begin{aligned} \mathcal{A} &= B_{j_1}(\sigma_{b,\infty}; b, a; \infty, a; \xi_1) B_1(\sigma_a; a, b; \xi_2) A_0(\rho_a; a, b; \xi_2) A_0(\rho_{b,\infty}; b, a; \xi_2), \\ \mathcal{B} &= A_{i'_4}(-\rho_{b,\infty}; b, a; \eta) A_0(-\rho_a; a, b; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_{b,\infty}^{-1}; \infty, a; b, a; \eta) \end{aligned}$$

will do the quasi-invertible continuous mapping of the proposition. Let us also recall that by property xviii) in Section 5 (or [3, Proposition 5.1])

$$A_0(\rho_a; a, b; \xi_2) A_0(\rho_{b,\infty}; b, a; \xi_2) = A_0(\rho_{b,\infty}; b, a; \xi_2) A_0(\rho_a; a, b; \xi_2),$$

i.e. in both cases operators $\mathcal{A}$ are one and the same.

As in the previous subsections we generalize the assertion of the last theorem by using additional operators of type A in the definition of $\mathcal{A}$ and $\mathcal{B}$.

Theorem 6.8. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_a) > 0$ and $(1-\lambda_b)(1-\nu_\infty) < 0$. Let also $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ as one of them is in $\Gamma_0(p)$, and $\mu_a \in \Gamma_i(p)$, $\mu_\infty \in \Gamma_j(p)$ as $i \le j$. Then there exist linear operators $\mathcal{A}$ and $\mathcal{B}$, constructed as compositions of the operators in Sections 3 and 4, such that*

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

P r o o f. Let $\kappa_a \in \Gamma_{i'_4}(p)$ and $\kappa_b > -1/p$. We choose $\tilde{\rho}_a$ and $\tilde{\rho}_\infty$ so that

$$\mu_a + \tilde{\rho}_a \in \Gamma_0(p), \quad \mu_\infty + \tilde{\rho}_a \in \Gamma_{j_0}(p), \quad \mu_\infty + \tilde{\rho}_\infty \in \Gamma_{j_1}(p) \text{ as } j_1 > 0. \tag{6.5}$$

Set $\mu_a^\# = \mu_a + \tilde{\rho}_a$, $\mu_\infty^\# = \mu_\infty + \tilde{\rho}_\infty$, $\tilde{w}^\# = \chi_a^{\mu_a^\#} \chi_{a-1}^{\mu_\infty^\# - \mu_a^\#}$, $j' = j_0$ and $j'_0 = j_1$. Define the operators

$$\begin{aligned} \mathcal{A}^\# &= B_{j_1}(\sigma_{b,\infty}; b, a; \infty, a; \xi) B_1(\sigma_a; a, b; \eta) A_0(\rho_{b,\infty}^\#; b, a; \eta) A_0(\rho_a^\#; a, b; \eta), \\ \mathcal{B}^\# &= A_{i'_4}(-\rho_a^\#; a, b; \eta) A_0(-\rho_{b,\infty}^\#; b, a; \eta) B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_{b,\infty}^{-1}; \infty, a; b, a; \eta), \end{aligned}$$

where the parameters $\eta, \sigma_a, \sigma_{b,\infty}$ are given in Proposition 6.7 and

$$\rho_a^{\#} = \kappa_a + \frac{1}{p} - \frac{\mu_a + \tilde{\rho}_a + 1/p}{\sigma_a}, \quad \rho_{b,\infty}^{\#} = \kappa_b + \frac{1}{p} - \frac{\mu_\infty + \tilde{\rho}_\infty + 1/p}{\sigma_{b,\infty}}.$$

We apply the scheme

$$\boxed{(L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)}$$

Step 1 $\quad A_{i,j}(\tilde{\rho}_a; a, \infty; \xi) \ \ \upharpoonleft\downharpoonright \ \ A_{0,j'}(-\tilde{\rho}_a; a, \infty; \xi)$

$$\boxed{(L_p(\chi_a^{\mu_a^{\#}} \chi_{a-1}^{\mu_\infty - \mu_a})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)}$$

Step 2 $\quad A_{j_0}(\tilde{\rho}_\infty - \tilde{\rho}_a; \infty, a; \xi) \ \ \upharpoonleft\downharpoonright \ \ A_{j_0'}(\tilde{\rho}_a - \tilde{\rho}_\infty; \infty, a; \xi)$

$$\boxed{(L_p(\tilde{w}^{\#})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)}$$

Step 3 $\quad \mathcal{A}^{\#} \ \ \upharpoonleft\downharpoonright \ \ \mathcal{B}^{\#}$

$$\boxed{(L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r)}\ .$$

Propositions 3.7 and 3.8 with $\xi > a$ are used respectively at Steps 1 and 2 taking into account that $\mu_a^{\#} + \mu_\infty - \mu_a \in \Gamma_{j_0}(p)$. At step 3 we use Proposition 6.7 with $\tilde{w}^{\#}$ instead of $\tilde{w}$. The above scheme implies that the operators

$$\begin{aligned} \mathcal{A} &= A_{i,j}(\tilde{\rho}_a; a, \infty; \xi) A_{j_0}(\tilde{\rho}_\infty - \tilde{\rho}_a; \infty, a; \xi)\mathcal{A}^{\#}, \\ \mathcal{B} &= \mathcal{B}^{\#} A_{j_0'}(\tilde{\rho}_a - \tilde{\rho}_\infty; \infty, a; \xi) A_{0,j'}(-\tilde{\rho}_a; a, \infty; \xi) \end{aligned}$$

satisfy the assertion of the theorem.

In the case $\kappa_a > -1/p$ and $\kappa_b \in \Gamma_{i_4'}(p)$ in view of Remark 6.5 it is enough to interchange the places of the operators of type A in the definition of $\mathcal{B}^{\#}$. $\square$

Remark 6.6. In both cases $\kappa_a \in \Gamma_i(p)$, $\kappa_b \in \Gamma_0(p)$ and $\kappa_a \in \Gamma_0(p)$, $\kappa_b \in \Gamma_{i'}(p)$ the operators $\mathcal{A}$ are one and the same. The operators $\mathcal{B}$ are one and the same for $i = i' = 0$ and different otherwise.

In general, the operators $\mathcal{A}$ and $\mathcal{B}$ consists of 6 operators of type A and B. But in the case $\mu_\infty < \mu_a$ (note that $\mu_\infty \geq \mu_a$ implies $i = j$) it is clear from (6.5) that we can choose $\tilde{\rho}_\infty = \tilde{\rho}_a$ and reduce the number of operators to 5. Similar reduction can be achieved for $\mu_a \in \Gamma_0(p)$ when we can choose $\tilde{\rho}_a = 0$.

The first two steps in the proof of Proposition 6.7 imply

Proposition 6.8. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the real numbers $\kappa_a, \kappa_b, \lambda_a, \lambda_b, \nu_a, \nu_\infty$ and the integer j satisfy the conditions:*

$$(1-\lambda_a)(1-\nu_a) > 0, \quad (1-\lambda_b)(1-\nu_\infty) < 0,$$
$$\kappa_a, \kappa_b \in \Gamma_+(p), \quad \left(\kappa_b + \frac{1}{p}\right)\frac{1-\nu_\infty}{1-\lambda_b} - \frac{1}{p} \in \Gamma_j^*(p).$$

Set

$$\mu_a = \left(\kappa_a + \frac{1}{p}\right)\sigma_a - \frac{1}{p}, \quad \mu_\infty = \left(\kappa_a + \frac{1}{p}\right)\sigma_{b,\infty} - \frac{1}{p}$$

and

$$\mathcal{A} = B_j(\sigma_{b,\infty}; b, a; \infty, a; \xi_1) B_1(\sigma_a; a, b; \xi_2),$$
$$\mathcal{B} = B_1(\sigma_a^{-1}; a, b; \eta) B_1(\sigma_{b,\infty}^{-1}; \infty, a; b, a; \eta),$$

where $\xi_1 \in (a, \infty)$, $\xi_2, \eta \in (a, b)$ and

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_{b,\infty} = \frac{1-\nu_\infty}{1-\lambda_b}.$$

Then

$$\mathcal{A} : (L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

We get the following two characterization theorems by means of the operators constructed in the proof of Theorem 6.8 and Remark 6.6 (for $p < \infty$) or by Proposition 6.8 (for $p = \infty$).

Theorem 6.9. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\xi \in (a, b)$, $\lambda_a \in (-\infty, 1)$ and $\lambda_b \in (1, \infty)$. Let us set*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}.$$

a) *For $p < \infty$ we assume that $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ as one of them is in $\Gamma_0(p)$, and set*

$$\mathcal{A} = A_0(\tilde{\rho}_\infty; \infty, a; *) B_{j_1}(\sigma_b; b, a; \infty, a; \xi_1) B_1(\sigma_a; a, b; \xi)$$
$$A_0(\rho_b; b, a; \xi) A_0(\rho_a; a, b; \xi),$$

where $\xi_1 \in (a, \infty)$, $\tilde{\rho}_\infty \in \Gamma_{j_1}(p)$, $j_1 > 0$ and

$$\rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p} - \tilde{\rho}_\infty(1-\lambda_b).$$

b) *For $p = \infty$ we assume that $\kappa_a = \kappa_\infty = 0$ and set*

$$\mathcal{A} = B_1(\sigma_b; b, a; \infty, a; *) B_1(\sigma_a; a, b; \xi).$$

Then for $t > 0$ and $f \in L_p(w)(a,b)$ we have

$$K(f, t^r; L_p(w)(a,b), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p(a,\infty)}.$$

P r o o f. Assertion a) follows from the considerations in the proof of Theorem 6.8 with $\mu_a = \mu_\infty = 0$, $\nu_a = \nu_\infty = 0$ and $\tilde{\rho}_a = 0$. Assertion b) follows from Proposition 6.8 with $p = \infty$, $\kappa_a = \kappa_b = 0$ and $\nu_a = \nu_\infty = 0$. □

Looking at the quasi-invertible continuous map in Theorem 6.8 with $\mathcal{B}$ being the direct operator and $\mathcal{A}$ its quasi-inverse we get

Theorem 6.10. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\eta \in (a,b)$, $\eta_1 \in (a,\infty)$, $\lambda_a \in (-\infty, 1)$ and $\lambda_\infty \in (1,\infty)$. Let us set*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_\infty = \frac{1}{1-\lambda_\infty}.$$

a) *For $p < \infty$ we assume that $\kappa_a \in \Gamma_i(p)$, $\kappa_\infty \in \Gamma_j(p)$ as $i \le j$ and set*

$$\mathcal{A} = A_0(\rho_a; a, b; \eta) A_0(\rho_\infty; b, a; \eta) B_1(\sigma_a; a, b; \eta) B_1(\sigma_\infty; \infty, a; b, a; \eta)$$
$$A_{j_1}(\tilde{\rho}_a - \tilde{\rho}_\infty; \infty, a; \eta_1) A_{0,j_0}(-\tilde{\rho}_a; a, \infty; \eta_1),$$

where $\tilde{\rho}_a$, $\tilde{\rho}_\infty$, j_0, j_1 are chosen so that

$$\kappa_a + \tilde{\rho}_a \in \Gamma_0(p), \quad \kappa_\infty + \tilde{\rho}_a \in \Gamma_{j_0}(p), \quad \kappa_\infty + \tilde{\rho}_\infty \in \Gamma_{j_1}(p), \quad j_1 > 0$$

and

$$\rho_a = \frac{\kappa_a + \tilde{\rho}_a + 1/p}{1-\lambda_a} - \frac{1}{p}, \quad \rho_\infty = \frac{\kappa_\infty + \tilde{\rho}_\infty + 1/p}{1-\lambda_\infty} - \frac{1}{p}.$$

b) *For $p = \infty$ we assume that $\mu_a = \mu_\infty = 0$ and set*

$$\mathcal{A} = B_1(\sigma_a; a, b; \eta) B_1(\sigma_\infty; \infty, a; b, a; \eta).$$

Then for $t > 0$ and $f \in L_p(w)(a,\infty)$ we have

$$K(f, t^r; L_p(w)(a,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p(a,b)}.$$

Proof. Assertion a) follows from the considerations in the proof of Theorem 6.8 as we set $\kappa_a = \kappa_b = 0$ and $\lambda_a = \lambda_b = 0$. Assertion b) follows from Proposition 6.8 with $p = \infty$, $\kappa_a = \kappa_b = 0$ and $\lambda_a = \lambda_b = 0$. In both cases we use the operator $\mathcal{B}$ and at the end replace the μ's by κ's, the ν's by λ's and finally denote operator $\mathcal{B}$ by $\mathcal{A}$. Note that the σ's here denote the reverse of the σ's in Theorem 6.8, while the ρ's are the opposite. □

In the case $p < \infty$ the number of operators of type A in Theorem 6.10 can be reduced if the κ's satisfy some additional restrictions. For example, if $\kappa_a > -1/p$, $\kappa_\infty \in \Gamma_j(p)$, $j > 0$, we can set $\tilde{\rho}_a = \tilde{\rho}_\infty = 0$ and then $\mathcal{A}$ is defined by four operators. If $\kappa_a, \kappa_\infty > -1/p$, then we can set $\tilde{\rho}_a = 0$ and get five operators. The same is true if $\kappa_\infty < \kappa_a$ because then we can choose $\tilde{\rho}_a = \tilde{\rho}_\infty$.

Let us now consider the other case $(1-\lambda_a)(1-\nu_a) < 0$, $(1-\lambda_b)(1-\nu_\infty) > 0$. The sub-case $(1-\lambda_a)(1-\lambda_b) > 0$ has no solution in the terms of the operators of type A and B (cf. classes C_1 for (a, b) and C_8 for (a,∞) or classes C_7 for (a, b) and C_2 for (a,∞) in Subsection 6.5).

The remaining sub-case $(1-\lambda_a)(1-\lambda_b) < 0$ is covered by the next remark, which easily follows from the previous results in this subsection. (Note that one cannot expect an analogue of Theorem 6.10 in this sub-case, because it would require target weight $\tilde{\varphi} = 1$ in a finite interval!)

Remark 6.7. The cases $\lambda_a > 1$, $\lambda_b, \nu_a, \nu_\infty < 1$ and $\lambda_a < 1$, $\lambda_b, \nu_a, \nu_\infty > 1$ can be solved by applying the "mirror" operator from Proposition 2.8 together with the operators from Propositions 6.7, 6.8 and Theorems 6.8, 6.9.

In order to demonstrate this approach we establish the following analogue of Theorem 6.8.

Theorem 6.11. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $(1-\lambda_a)(1-\nu_\infty) < 0$ and $(1-\lambda_b)(1-\nu_a) > 0$. Let also $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ as one of them is in $\Gamma_0(p)$ and $\mu_a \in \Gamma_i(p)$, $\mu_\infty \in \Gamma_j(p)$ as $i \le j$. Set $\bar{w} = \chi_a^{\kappa_b}\chi_b^{\kappa_a}$ and $\bar{\varphi} = \chi_a^{\lambda_b}\chi_b^{\lambda_a}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the operators in Theorem 6.8, satisfying*

$$\mathcal{A} : (L_p(\bar{w})(a, b), AC_{loc}^{r-1}, \bar{\varphi}^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{B}.$$

Then

$$\mathcal{A}\,\mathcal{S}(-1)\,\mathcal{T}(a+b) : (L_p(w)(a, b), AC_{loc}^{r-1}, \varphi^r D^r) \rightleftharpoons (L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r) : \mathcal{T}(-a-b)\,\mathcal{S}(-1)\,\mathcal{B}.$$

P r o o f. If we interchange λ_a and λ_b, then the hypotheses of Theorem 6.8 will be fulfilled and we can apply it to the triplets $(L_p(\bar{w})(a,b), AC_{loc}^{r-1}, \bar{\varphi}^r D^r)$ and $(L_p(\tilde{w})(a,\infty), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$. Then we apply Proposition 2.8. $\square$

If we would like to avoid using the "mirror" operator, then we can proceed as follows.

Remark 6.8. If we interchange a and b everywhere in Propositions 6.7, 6.8 and Theorem 6.8, we get their analogues in the case $(1-\lambda_a)(1-\nu_\infty) < 0$ and $(1-\lambda_b)(1-\nu_b) > 0$ and hence the analogue of Theorem 6.9 for $\lambda_a > 1$, $\lambda_b < 1$. Note that the interval of the target triplet is (b,∞) and not (a,∞) as in Remark 6.7. Of course, as explained in Subsection 2.1, relations like $\eta \in (b,a)$ are supposed to be understood as $\eta \in (a,b)$.

6.5. Solutions of (1.2) **and of** (1.3)**.** The results of the previous subsections related to (1.2) can be summarized in the following three theorems.

Theorem 6.12. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the interval and the weights of the triplet $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ be in one of the following classes:*

C_1) (a,b), $\lambda_a < 1$, $\lambda_b < 1$, $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as one of them is in* $\Gamma_0(p)$*;*

C_2) (a,∞), $\lambda_a < 1$, $\lambda_\infty > 1$, $\kappa_a \in \Gamma_i(p)$, $\kappa_\infty \in \Gamma_j(p)$ *as* $i \le j$*.*

Let the triplet $(L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$ satisfy the same condition. Then there is a linear operator $\mathcal{A}$, constructed as a composition of the operators in Sections 3 *and* 4*, such that* (1.2) *holds for every $f \in L_p(w)(I)$ and every $t \in (0,1]$.*

P r o o f. If both triplets $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ and $(L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$ are in the class C_1, then the conclusion follows from Theorem 6.1 and if they are in the class C_2, then the conclusion follows from Theorem 6.4. When one of the triplets is in the class C_1 and the other – in C_2 we apply Theorem 6.8. $\square$

Theorem 6.13. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. Let the interval and the weights of the triplet $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ be in one of the following classes:*

C_3) (a,∞), $\lambda_a < 1$, $\lambda_\infty < 1$, $\kappa_a \in \Gamma_i(p)$, $\kappa_\infty \in \Gamma_j(p)$ *as* $i \le j$*;*

C_4) (a,∞), $\lambda_a > 1$, $\lambda_\infty > 1$, $\kappa_a \in \Gamma_i(p)$, $\kappa_\infty \in \Gamma_j(p)$ *as* $i \le j$*;*

C_5) (a,b), $\lambda_a < 1$, $\lambda_b > 1$, $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as one of them is in* $\Gamma_0(p)$*;*

C_6) (a,b), $\lambda_a > 1$, $\lambda_b < 1$, $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as one of them is in* $\Gamma_0(p)$*.*

Let the triplet $(L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$ *satisfy the same condition. Then there is a linear operator* $\mathcal{A}$*, constructed as a composition of the operators in Sections 3 and 4, such that* (1.2) *holds for every* $f \in L_p(w)(I)$ *and every* $t \in (0, 1]$.

Proof. If both triplets $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ and $(L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$ are in the class C_3 or in the class C_4, then the conclusion follows from Theorem 6.4 and if they are in the class C_5 or in the class C_6, then the conclusion follows from Theorem 6.1. When one of the triplets is in the class C_3 and the other – in C_4 we apply Theorem 6.6. When one of the triplets is in the class C_3 and the other – in C_5 we apply Theorem 6.8. When one of the triplets is in the class C_3 and the other – in C_6 we apply Theorem 6.11. When one of the triplets is in the class C_4 and the other – in C_5 we apply Theorem 6.11. When one of the triplets is in the class C_4 and the other – in C_6 we apply Theorem 6.8. When one of the triplets is in the class C_5 and the other – in C_6 we apply Theorem 6.3. □

Theorem 6.14. *Let* $r \in \mathbb{N}$ *and* $1 \le p \le \infty$*. Let the interval and the weights of the triplet* $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ *be in one of the following classes:*

C_7) (a, b), $\lambda_a > 1$, $\lambda_b > 1$, $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as one of them is in* $\Gamma_0(p)$;

C_8) (a, ∞), $\lambda_a > 1$, $\lambda_\infty < 1$, $\kappa_a \in \Gamma_i(p)$, $\kappa_\infty \in \Gamma_j(p)$ *as* $i \le j$.

Let the triplet $(L_p(\tilde{w})(\tilde{I}), AC_{loc}^{r-1}, \tilde{\varphi}^r D^r)$ *satisfy the same condition. Then there is a linear operator* $\mathcal{A}$*, constructed as a composition of the operators in Sections* 3 *and* 4*, such that* (1.2) *holds for every* $f \in L_p(w)(I)$ *and every* $t \in (0, 1]$.

Proof. The proof is the same as of Theorem 6.12. □

Let us consider the intervals and the exponents of the weight φ described in the classes C_1 – C_8. The operators from Section 3 do not change the classes. On the other hand the operators from Section 4 can vary the weight φ and the interval I, but always staying in the same set of classes (described in one of the three theorems above) as the original weight. Hence, the three sets of classes are mutually disjoint when treated by the operators studied in this article.

Let us turn our attention to the equivalence (1.3). For $1 \le p < \infty$ the weights $\varphi = 1$ and $w = 1$ are contained in the class C_1 for a finite interval and in the class C_3 for a semi-infinite interval. Hence, Theorem 6.12 gives for every triplet from C_1 or C_2 an operator $\mathcal{A}$ for which the equivalence (1.3) holds with a finite $\tilde{I}$. Similarly, Theorem 6.13 gives for every triplet from C_3, C_4, C_5 or C_6 an operator $\mathcal{A}$ for which the equivalence (1.3) holds with a semi-infinite $\tilde{I}$.

If $p = \infty$, then $0 \in \Gamma_{exc}(p)$. Now Theorem 6.2, items b) in Theorems 6.5, 6.7, 6.9 (together with Remark 6.8) and 6.10 imply that (1.3) holds when the corresponding κ's are 0.

Theorem 6.15. *Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. For $p < \infty$ let the interval and the weights of the triplet $(L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r)$ be in one of the classes $C_1 - C_6$. For $p = \infty$ let the interval and the λ's be in one of the classes $C_1 - C_6$ while the κ's be 0. Then there is a linear operator $\mathcal{A}$, constructed as a composition of the operators in Sections 3 and 4, such that (1.3) holds for every $f \in L_p(w)(I)$ and every $t \in (0, 1]$.*

For triplets from the classes C_7 and C_8 we cannot establish (1.3) using only the operators studied in this article. On the other hand, there are cases (not included in $C_1 - C_6$) when (1.3) is valid with such operators. Results in this respect will be given in [7].

7. Characterization of K-functionals by two moduli

7.1. Separating the singularities. The next lemma will be used for separating the singularities at the end-points of the interval. If ψ is a function defined on $I \subset \mathbb{R}$ and $J \subset I$, then we use the same notation ψ for the restriction $\psi|_J$ of ψ on J.

Lemma 7.1. *Let $I_1 = (\bar{a}, b_1)$ and $I_2 = (a_1, \bar{b})$ be two intervals on the real line such that $\bar{a} < a_1 < b_1 < \bar{b}$, where $\bar{a}$ is finite or $-\infty$ and $\bar{b}$ is finite or ∞. Let $I = (\bar{a}, \bar{b}) = I_1 \cup I_2$ and let w and φ be non-negative measurable on I weights such that $w \sim 1$ and $\varphi \sim 1$ on $[a_1, b_1]$. Then for $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le b_1 - a_1$ and $f \in L_p(w)(I)$ we have*

$$\begin{aligned} &K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \\ &\quad \sim K(f, t^r; L_p(w)(I_1), AC_{loc}^{r-1}, \varphi^r D^r) + K(f, t^r; L_p(w)(I_2), AC_{loc}^{r-1}, \varphi^r D^r). \end{aligned}$$

Assertions like this lemma are standard tools in K-functional theory. The proof follows the lines of the proof of [1, p.176, Lemma 2.3]. The restriction $t \le \bar{b} - \bar{a}$ (combined with the finite ratio $(\bar{b} - \bar{a})/(b_1 - a_1)$ requirement) in [1] is replaced by $t \le b_1 - a_1$ here.

7.2. Solutions of (1.4)**.** In order to characterize the K-functional (1.1) for a wider range of parameters we separate the singularities by means of Lemma 7.1. Then we can establish (1.4) with $1 \le p < \infty$ for all cases of intervals I, including $I = (-\infty, \infty)$, weights φ with exponents different than 1 and weights w with exponents not in $\Gamma_{exc}(p)$. This follows from the fact, that one of the ends of both intervals I_1 and I_2 obtained after the application of Lemma 7.1 is a finite point (b_1 and a_1 respectively) with exponents of φ and w equal to 0. So, we reduce the original problem to two problems, each of which falls into one of the classes C_1, C_2, C_3, C_5 or C_6.

Consider $I = (\bar{a}, \bar{b})$, where either $\bar{a} = a$ is finite or $\bar{a} = -\infty$ and either $\bar{b} = b$ is finite or $\bar{b} = \infty$. Depending on the values of λ, we define the intervals $\tilde{I}_1$, $\tilde{I}_2$ and the linear operators $\mathcal{A}_1$, $\mathcal{A}_2$ in Tables 4 and 5, where for $-\infty \le s \le \infty$ we have set

$$\sigma_s = \frac{1}{1-\lambda_s}, \quad \rho_s = \kappa_s + \frac{\lambda_s}{p}.$$

Table 4. Definition of $\mathcal{A}_1$

λ	$\mathcal{A}_1$	$\tilde{I}_1$
$\lambda_a < 1$	$B_1(\sigma_a; a, b_1; \xi_1) A_0(\rho_a; a, b_1; \xi_1)$	(a, b_1)
$\lambda_a > 1$	$B_r(\sigma_a; a, b_1; \infty, b_1; \xi_2) A_0(\rho_a; a, b_1; \xi_1)$	(b_1, ∞)
$\lambda_{-\infty} < 1$	$B_1(\sigma_{-\infty}; \infty, -b_1; *) A_0(\rho_{-\infty}; \infty, -b_1; *) \mathcal{S}(-1)$	$(-b_1, \infty)$
$\lambda_{-\infty} > 1$	$B_1(\sigma_{-\infty}; \infty, -b_1; -a_2, -b_1; \xi_4) A_r(\rho_{-\infty}; \infty, -b_1; \xi_3) \mathcal{S}(-1)$	$(-b_1, -a_2)$

Table 5. Definition of $\mathcal{A}_2$

λ	$\mathcal{A}_2$	$\tilde{I}_2$
$\lambda_b < 1$	$B_1(\sigma_b; b, a_1; \xi_5) A_0(\rho_b; b, a_1; \xi_5)$	(a_1, b)
$\lambda_b > 1$	$B_r(\sigma_b; b, a_1; \infty, a_1; \xi_6) A_0(\rho_b; b, a_1; \xi_5)$	(a_1, ∞)
$\lambda_\infty < 1$	$B_1(\sigma_\infty; \infty, a_1; *) A_0(\rho_\infty; \infty, a_1; *)$	(a_1, ∞)
$\lambda_\infty > 1$	$B_1(\sigma_\infty; \infty, a_1; b_2, a_1; \xi_5) A_r(\rho_\infty; \infty, a_1; \xi_6)$	(a_1, b_2)

Let us note that the domain of the functions on which $\mathcal{A}_1$ is defined is $I_1 = (\bar{a}, b_1)$, and the domain of the functions on which $\mathcal{A}_2$ is defined is $I_2 = (a_1, \bar{b})$. In the cases $\lambda_{-\infty} > 1$ and $\lambda_\infty > 1$, considered in the last rows of the two tables, one can choose arbitrary real points for the numbers a_2 and b_2 such that $a_2 < b_1$ and $b_2 > a_1$.

Using Lemma 7.1, Proposition 2.7, Theorems 6.2, 6.5, 6.9, 6.10 and Remark 6.8 we get

Theorem 7.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $-\infty \le \bar{a} < a_1 < b_1 < \bar{b} \le \infty$, $I = (\bar{a}, \bar{b})$ and $\lambda_{\bar{a}}, \lambda_{\bar{b}} \neq 1$. Let $\kappa_{\bar{a}}, \kappa_{\bar{b}} \notin \Gamma_{exc}(p)$ if $p < \infty$ or $\kappa_{\bar{a}}, \kappa_{\bar{b}} = 0$ if $p = \infty$. Let the operators $\mathcal{A}_1, \mathcal{A}_2$ and the intervals $\tilde{I}_1, \tilde{I}_2$ be defined in Tables 4 and 5. Then (1.4) holds for every $f \in L_p(w)(I)$ and $t \in (0, 1]$.*

P r o o f. First, we separate the singularities by means of Lemma 7.1. Further, we have:

1. If $\bar{b} = b \in \mathbb{R}$, $\lambda_b < 1$, we apply Theorem 6.2.
2. If $\bar{a} = a \in \mathbb{R}$, $\lambda_a < 1$, we apply Theorem 6.2.
3. If $\bar{b} = b \in \mathbb{R}$, $\lambda_b > 1$, we apply Theorem 6.9 with $p < \infty, j_1 = r$ and get that $\mathcal{A}_2^{\#} = A_0(\tilde{\rho}_\infty; \infty, a_1; *) B_r(\sigma_b; b, a_1; \infty, a_1; \xi_6) A_0(\rho_b^{\#}; b, a_1; \xi_5)$ with $\rho_b^{\#} = \kappa_b + \lambda_b/p - \tilde{\rho}_\infty(1 - \lambda_b)$ does the job for $\mathcal{A}_2$. We complete the proof by showing that $\mathcal{A}_2^{\#} f - \mathcal{A}_2 f \in \Pi_{r-1}$ for every $f \in L_p(\chi_b^{\kappa_b})(I_2)$: first we use that $A_0(\tilde{\rho}_\infty; \infty, a_1; *)F - A_r(\tilde{\rho}_\infty; \infty, a_1; \xi_6)F \in \Pi_{r-1}$ for every $F \in L_{1,loc}(a_1, \infty) \cap L_1(\chi_{a_1}^{\tilde{\rho}_\infty - 1})(a_1 + 1, \infty)$, next we interchange the order of A_r and B_r by property xvi) of Section 5 and, finally, we get a single operator A_0 by property iii) of the same section. For $p = \infty$ Theorem 6.9 gives $\mathcal{A}_2^{\#} = B_1(\sigma_b; b, a_1; \infty, a_1; *)$ and we have by definition $\mathcal{A}_2^{\#} f - \mathcal{A}_2 f \in \Pi_{r-1}$ for every $f \in L_\infty(I_2)$.
4. The case $\bar{a} = a \in \mathbb{R}$, $\lambda_a > 1$ is considered as Case 3, using also Remark 6.8.
5. If $\bar{b} = \infty$, $\lambda_\infty < 1$, we apply Theorem 6.5.
6. If $\bar{a} = -\infty$, $\lambda_{-\infty} < 1$, we apply operator $\mathcal{S}(-1)$ to reduce it to Case 5.
7. If $\bar{b} = \infty$, $\lambda_\infty > 1$, we apply Theorem 6.10 with $\tilde{\rho}_{a_1} = 0$ and $\tilde{\rho}_\infty$ such that $j_1 = r$ as well as properties xvii) and ii) of Section 5.
8. If $\bar{a} = -\infty$, $\lambda_{-\infty} > 1$, we apply operator $\mathcal{S}(-1)$ to reduce it to Case 7. □

Remark 7.1. Let us note that operator $\mathcal{A}_2$ used in the case $\lambda_b > 1$, is not bounded, but $\mathcal{A}_2 f \in L_p(a_1, \infty) + \Pi_{r-1}$ for every $f \in L_p(\chi_b^{\kappa_b})(I_2)$ and the r-th modulus of $\mathcal{A}_2 f$ is finite. The same is valid in the case $\lambda_a > 1$. On the other hand, the operators in the cases $\lambda_{\pm\infty} > 1$, obtained by similar procedures, are bounded.

In the cases of common hypotheses of Theorem 6.15 and Theorem 7.1 we note that the conclusion of Theorem 6.15 is stronger, while the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ in Theorem 7.1 are simpler than $\mathcal{A}$ in Theorem 6.15.

8. *K*-functionals for spaces of continuous functions. Let $-\infty \le \bar{a} < \bar{b} \le \infty$. So far in the case $p = \infty$ we have considered K-functionals (1.1) in which the infimum is taken over the functions $g \in AC_{loc}^{r-1}(\bar{a}, \bar{b})$ such that $wg, w\varphi^r g^{(r)} \in L_\infty(\bar{a}, \bar{b})$ – that is on the largest possible set. In this section we shall consider K-functionals of the type (1.1), in whose definition $L_\infty(w)(\bar{a}, \bar{b})$ is replaced by any of the spaces:

$$
\begin{aligned}
&C(w)(\bar{a}, \bar{b}) = \{f : wf \in C(\bar{a}, \bar{b})\},\\
&C(w)[\bar{a}, \bar{b}) = \{f \in C(w)(\bar{a}, \bar{b}) : \exists \lim_{x\to\bar{a}+0} (wf)(x)\},\\
&C(w)(\bar{a}, \bar{b}] = \{f \in C(w)(\bar{a}, \bar{b}) : \exists \lim_{x\to\bar{b}-0} (wf)(x)\},\\
&C(w)[\bar{a}, \bar{b}] = C(w)[\bar{a}, \bar{b}) \cap C(w)(\bar{a}, \bar{b}].
\end{aligned} \tag{8.1}
$$

The additional requirements at the end-points of the domain, e.g. the existence of $\lim_{x\to\bar{a}+0} w(x)f(x)$, are called “weighted limit conditions”.

Let us start with the space $C(w)(\bar{a}, \bar{b})$. For every $f \in L_\infty(w)(\bar{a}, \bar{b})$ all functions g in (1.1) obviously satisfy $g \in C(w)(\bar{a}, \bar{b})$. Hence the replacement of $L_\infty(w)$ by $C(w)$ in (1.1) means that we only require in addition $g^{(r)}$ to be locally continuous, i.e. $g^{(r)} \in L_\infty(w\varphi^r)(\bar{a}, \bar{b})$ is replaced by $g^{(r)} \in C(w\varphi^r)(\bar{a}, \bar{b})$.

From the fact that the operators $A(\rho; \xi)$ and $B(\sigma; \xi)$ map the space of locally continuous functions into itself and from properties (3.5) and (4.3) by straightforward arguments we have

Proposition 8.1. *All assertions in Sections* 3, 4, 6 *and* 7 *hold in the case $p = \infty$ with L_∞ replaced by C.*

In the proof of Proposition 8.1 for the validity of the assertions connected with the classical fixed-step moduli ω_r, including Theorems 6.15 and 7.1 with L_∞ replaced by C, one also needs the following auxiliary statement.

Proposition 8.2. *Let $-\infty \le \bar{a} < \bar{b} \le \infty$ and either $I = (\bar{a}, \bar{b})$, or $I = [\bar{a}, \bar{b})$, or $I = (\bar{a}, \bar{b}]$, or $I = [\bar{a}, \bar{b}]$. Then for $f \in C(I)$ and $0 < t < \bar{b} - \bar{a}$ we have*

$$K(f, t^r; C(I), AC_{loc}^{r-1}, D^r) \sim K(f, t^r; L_\infty(\bar{a}, \bar{b}), AC_{loc}^{r-1}, D^r) \sim \omega_r(f, t)_{\infty(\bar{a}, \bar{b})}.$$

Proof. The assertion follows from the inequalities

$$(8.2)\quad c\,\omega_r(f,t)_{\infty(\bar a,\bar b)} \le K(f,t^r;L_\infty(\bar a,\bar b),AC_{loc}^{r-1},D^r) \le K(f,t^r;C(I),AC_{loc}^{r-1},D^r) \le c\,\omega_r(f,t)_{\infty(\bar a,\bar b)}.$$

The first inequality in (8.2) is standard in the characterization of the unweighted K-functionals by moduli and follows from the properties of the moduli. The second inequality in (8.2) is trivial because the infimum in the second K-functional is taken over a narrower set than in the first K-functional. Finally, the combinations of r-iterated Steklov's means (see e.g. [1, p. 177]) and their r-th derivatives belong to $C(I)$ for $f \in C(I)$, which justifies the third inequality in (8.2). □

In particular, Proposition 8.2 says that for functions $f \in C(a,\infty]$ the space on which the infimum is taken in the definition of the unweighted K-functional $K(f,t^r;L_\infty(a,\infty),AC_{loc}^{r-1},D^r)$ can be restricted to $\{g \in AC_{loc}^{r-1}(a,\infty) : g, g^{(r)} \in C(a,\infty]\}$ and get an equivalent K-functional. Considering the *weighted* K-functional (1.1) in the framework of the operators studied in this article it is natural to assume that the infimum in $K(f,t^r;C(w)(a,\infty],AC_{loc}^{r-1},\varphi^r D^r)$ is taken on the set $\{g \in C(w)(a,\infty] \;:\; g^{(r)} \in C(w\varphi^r)(a,\infty]\}$ (and similarly for the other weighted limit conditions).

In order to treat the spaces other than $C(w)(\bar a,\bar b)$ in (8.1) we need additionally to trace out how the operators of type A and B preserve the existence of a weighted limit at the ends of the interval.

Lemma 8.1. *Let $\alpha,\sigma \ne 0$, $\beta \in \mathbb{R}$, $-\infty < a < \xi < \infty$ and $\gamma = (1-\alpha-\beta)/\sigma$.*

a) *Set $\zeta = a$ for $\sigma > 0, \alpha > 0$; $\zeta = \xi$ for $\sigma\alpha < 0$; $\zeta = \infty$ for $\sigma < 0, \alpha < 0$. Then for every $f \in C(\chi_a^\gamma)[a,\xi)$ we have*

$$\lim_{(x-a)^\sigma \to 0} \frac{1}{(x-a)^\alpha} \int_\zeta^x (y-a)^{-\beta} f(a+(y-a)^\sigma)\,dy = \frac{1}{\alpha} \lim_{u\to a} (u-a)^\gamma f(u).$$

b) *Set $\zeta = a$ for $\sigma < 0, \alpha > 0$; $\zeta = \xi$ for $\sigma\alpha > 0$; $\zeta = \infty$ for $\sigma > 0, \alpha < 0$. Then for every $f \in C(\chi_a^\gamma)(\xi,\infty]$ we have*

$$\lim_{(x-a)^\sigma \to \infty} \frac{1}{(x-a)^\alpha} \int_\zeta^x (y-a)^{-\beta} f(a+(y-a)^\sigma)\,dy = \frac{1}{\alpha} \lim_{u\to \infty} (u-a)^\gamma f(u).$$

Proof. In the case a) by standard analytic arguments for family of kernels with concentrated mass we have

$$\lim_{(x-a)^\sigma\to 0}\frac{1}{(x-a)^\alpha}\int_\zeta^x (y-a)^{-\beta} f(a+(y-a)^\sigma)\,dy$$
$$=\lim_{(x-a)^\sigma\to 0}\int_\zeta^x \frac{(y-a)^{\alpha-1}}{(x-a)^\alpha}\,dy\cdot\lim_{(y-a)^\sigma\to 0}(y-a)^{1-\alpha-\beta} f(a+(y-a)^\sigma).$$

Under the assumptions on ζ the first limit is α^{-1}. Then, in the second limit we set $u-a=(y-a)^\sigma$ and prove case a). The proof of case b) is similar. □

By means of this lemma we establish

Proposition 8.3. *In the case $p=\infty$ every of Propositions 3.7, 3.8, 3.9, 4.13, 4.14, 4.15, 4.16, 4.17, 4.18 holds as L_∞ is replaced by C with one or two weighted limit conditions at the ends of the domains of the initial and target triplets. Operators of type A and B with $\sigma>0$ preserve the end point $\bar a$ and/or $\bar b$ at which the weighted limit exists. Operators $B_{i,j}(\sigma;a,\infty;\xi)$ with $\sigma<0$ interchange the ends a and ∞ at which the weighted limit exists. Operators $B_i(\sigma;\infty,e;s,e;\xi)$ and $B_j(\sigma;s,e;\infty,e;\xi)$ with $\sigma<0$ interchange the ends s and ∞ and/or preserve the end e at which the weighted limit exists.*

Proof. In view of Proposition 8.1 it is enough to study the weighted limit behaviour of the operators of type A and B.

If an operator A or B treats a singularity at an end-point, at which the weighted limit exists (this is always the case for the operators with two indexes), then we apply Lemma 8.1. For the operators of type A we additionally set $\sigma=1$.

Otherwise, the point at which the weighted limit exists (in all such cases denoted by e) is finite and Af (or Bf) is continuous at this point by definition. Note that the exponent of the weight of the C-space to which Af (or Bf) belongs is in $\Gamma_+(\infty)=[0,\infty)$ according to the assumptions of the respective proposition.

Finally, the existence of a weighted limit of $(Af)^{(r)}$ (or $(Bf)^{(r)}$) is governed by the same rules in view of (3.5) and (4.3). This completes the proof. □

Now, Proposition 8.3 allows to enhance the results in Section 6.

Proposition 8.4. *All assertions in Section 6 hold in the case $p=\infty$ with initial space $L_\infty(w)(\bar a,\bar b)$ replaced by $C(w)[\bar a,\bar b]$ and target space $L_\infty(\tilde w)(\bar a',\bar b')$ replaced by $C(\tilde w)[\bar a',\bar b']$.*

Remark 8.1. Proposition 8.4 holds in the case of initial space $C(w)[\bar a,\bar b)$ when the target space of continuous functions satisfies the weighted limit condition at the end of $(\bar a',\bar b')$ which corresponds to $\bar a$ in the application of the operator $\mathcal{A}$. A similar assertion holds for $C(w)(\bar a,\bar b]$.

The next statement sums up the unweighted moduli characterization results for the spaces of continuous functions (8.1).

Theorem 8.1. *Theorems* 6.15 *and* 7.1 *hold in the case* $p = \infty$ *when* L_∞ *is replaced by* C *with any of the additional weighted limit conditions.*

REFERENCES

[1] R. A. DeVore, G. G. Lorentz. Constructive Approximation. Springer-Verlag, Berlin, 1993.

[2] Z. Ditzian, V. Totik. Moduli of Smoothness. Springer Verlag, New York, 1987.

[3] B. R. Draganov, K. G. Ivanov. A New Characterization of Weighted Peetre K-Functionals. *Constr. Approx.* **21** (2005), 113–148.

[4] B. R. Draganov, K. G. Ivanov. A Characterization of Weighted L_p Approximations by the Gamma and the Post-Widder Operators. In: Constructive Theory of Functions, Varna 2005 (Ed. B. Bojanov) Marin Drinov Academic Publishing House, Sofia, 2006, 80–87.

[5] B. R. Draganov, K. G. Ivanov. Characterizations of Weighted K-Functionals and Their Application. In: Constructive Theory of Functions, Varna 2005 (Ed. B. Bojanov) Marin Drinov Academic Publishing House, Sofia, 2006, 88–97.

[6] B. R. Draganov, K. G. Ivanov. A Characterization of Weighted Approximations by the Post-Widder and the Gamma Operators. *J. Approx. Theory*, to appear.

[7] B. R. Draganov, K. G. Ivanov. A New Characterization of Weighted Peetre K-Functionals (III), 2005, manuscript.

[8] G. H. Hardy, J. E. Littlewood, G. Polya. Inequalities, 2nd ed. Cambridge Univ. Press, Cambridge, 1951.

[9] K. G. Ivanov. Some characterizations of the best algebraic approximation in $L_p[-1,1]$ $(1 \le p \le \infty)$. *C. R. Acad. Bulgare Sci.* **34** (1981), 1229–1232.

[10] K. G. Ivanov. A characterization of weighted Peetre K-functionals. *J. Approx. Theory* **56** (1989), 185–211.

[11] N. X. Ky. A method for characterization of weighted K-functionals. *Ann. Univ. Sci. Budapest.* **38** (1995), 147–152.

[12] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.* **34** (1972), 31–38.

Borislav R. Draganov
Department of Mathematics and Informatics
The Sofia University
5 James Bourchier Blvd.
1164 Sofia, Bulgaria
e-mail: bdraganov@fmi.uni-sofia.bg
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria

Kamen G. Ivanov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: kamen@math.bas.bg

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