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Serdica Math. J. **33** (2007), 47–58

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
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THE ASYMPTOTIC BEHAVIOUR OF THE FIRST EIGENVALUE OF LINEAR SECOND-ORDER ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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Communicated by V. Petkov

Abstract. The asymptotic of the first eigenvalue for linear second order elliptic equations in divergence form with large drift is studied. A necessary and a sufficient condition for the maximum possible rate of the first eigenvalue is proved.

1. Introduction. We investigate the asymptotic behavior of the first eigenvalue of linear second-order elliptic equations in divergence form with a large drift term, i.e.

$$Lu = -\left(a_j^m(x)u_{x_m} - Tc^j(x)u\right)_{x_j} + Tc^j(x)u_{x_j} + b(x)u \tag{1}$$

in a bounded smooth $\Omega \subset R^n$

$$\partial\Omega \in C^{1,1}, \quad a_j^m, c^j \in W^{1,\infty}(\Omega), \quad b \in L^\infty(\Omega), \quad a_j^m = a_m^j \tag{2}$$

Here T is a large positive parameter and the nonsymmetric operator L is a uniformly elliptic one, i.e.

2000 *Mathematics Subject Classification:* 35J70, 35P15.
Key words: linear elliptic equations, eigenvalue problem, asymptotic behavior, dynamical systems.

(3) $$a_j^m(x)\xi^j\xi^m \geq \mu\,|\xi|^2 \quad \text{for every} \quad x \in \bar\Omega, \quad \xi \in R^n, \quad \mu = \text{const} > 0.$$

Consider the eigenvalue problem for L with Dirichlet boundary conditions

(4) $$L\phi = \lambda\phi \;\text{ in }\; \Omega, \quad \phi = 0 \;\text{ on }\; \partial\Omega.$$

By the means of Krein-Ruttman Theorem, see [7] it is well known that there exists a simple real eigenvalue λ_1 and eigenfunction $\phi_1 \in W^{2,p}_{loc}(\Omega)$ for every $p > 1$, such that $\phi_1 > 0$ in Ω and it is unique up to multiplication with a constant. Moreover any eigenvalue λ of (4) satisfies $\operatorname{Re}\lambda \geq \lambda_1$ and λ_1 is called the first (or principle) eigenvalue.

Let us recall that the asymptotic of the first eigenvalue λ_1 of the operator L_ε

(5) $$L_\varepsilon u = -\varepsilon^2 a_j^k(x)u_{x_jx_k} + c^j(x)u_{x_j}$$

when $\varepsilon \to 0$ was investigated by Friedman [5]. Under the condition $c(x)\bullet\nu(x) < 0$ on $\partial\Omega$, where $c(x) = (c^1(x), \cdots, c^n(x))$, $\nu = (\nu^1(x), \cdots, \nu^n(x))$ is the unit outward normal to $\partial\Omega$, and "$\bullet$" denotes the scalar product in R^n, it is proved in [5] that $\lambda_1(\varepsilon) \to 0$ exponentially fast as $\varepsilon \to 0$. The same operator L_ε is studied by Devinatz, Ellis and Friedman in [2] where a sufficient condition for the estimate $m_1\varepsilon^{2(r-1)/(r+1)} \leq \lambda_1(\varepsilon) \leq m_2\varepsilon^{2(r-1)/(r+1)}$, m_1 , $m_2 = \text{const} > 0$ is given if the vector $c(x)$ vanishes at some point $x_0 \in \Omega$ to the order r, for some $r \geq 0$. Moreover, if there exists a function $w \in C^1(\bar\Omega)$ such that

(6) $$c\bullet\nabla w > 0 \;\text{ in }\; \bar\Omega$$

the authors prove in the same paper that $\lambda_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$ at a maximum possible rate ε^{-2}. Most of the results in [2] are proved by two different methods: i) L^2 a priori estimates and ii) maximum principle type arguments. A different probabilistic approach closed to the Markov's processes is used by Wentzel [8], who gives a formula for the first term in the asymptotic of λ_1 for operator (5). Let us also mention the paper of Berestycki, Hamel and Nadirashvili [1] where boundness of the first eigenvalue for the operator

(7) $$Lu = -\Delta u + Tc^j u_{x_j}, \quad \operatorname{div} c = 0$$

are investigated. They prove that $\lambda_1(T)$ are bounded as $T \to \infty$ if and only if there exists a function $w(x) \in H^1_0(\Omega)$, $w(x) \neq 0$, such that $c\bullet\nabla w = 0$ almost everywhere in Ω. Moreover, the first term κ in the asymptotic of $\lambda_1(T)$, $\lambda_1(T) = \kappa + o(1)$ as $T \to \infty$, is found in [1] Note that the choice $T = \varepsilon^{-2}$ makes the problem for the operators (5) and (7) equivalent with respect to the parameters ε and T.

2. Main result. In the present paper we give a formula for the principal term κ in the asymptotic of the first eigenvalue λ_1 of (1) at a maximum possible rate T^2, i.e.

$$\lambda_1 = \kappa T^2 + o(T^2) \ \text{ as } \ T \to \infty. \tag{8}$$

This formula is different from the formula of Wentzel in [8] and is obtained by means of pure partial differential equation's arguments. As a consequence $\kappa = \kappa(a, c, \Omega)$ is a monotone function on the matrix a, i.e. $\kappa(a, c, \Omega) \le \kappa(\bar{a}, c, \Omega)$ if $a \ge \bar{a}$, $a = \{a_j^m\}$, $\bar{a} = \{\bar{a}_j^m\}$, $a, \bar{a} \ge \mu I$. Moreover, for the whole class of uniformly elliptic operators L either $\kappa(a, c, \Omega) > 0$ or $\kappa(a, c, \Omega) = 0$ for all matrices $\{a_j^k\}$ satisfying (3). As for the dependence on Ω, κ is a monotone and bounded function, i.e. $\kappa(a, c, \Omega') \le \kappa(a, c, \Omega'') \le \sup_{\Omega} c \bullet Ac$, $A = a^{-1}$ when $\Omega \supset \Omega' \supset \Omega''$. Fimally, we give a necessary and a sufficient condition λ_1 to have a maximum possible rate T^2 as $T \to \infty$ which is exactly the sufficient condition (6) of Devinatz, Ellis and Friednam [2].

Theorem 1. *Suppose L satisfies (2) and (3). Then the first eigenvalue λ_1 of L has the asymptotic $\lambda_1 = \kappa T^2 + o(T^2)$ as $T \to \infty$ where*

$$\kappa = \inf_u \sup_z \int_\Omega (c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx, \tag{9}$$

$$\kappa = \inf_u \sup_z \left[\left(\int_\Omega c \bullet \nabla z \, u^2 dx \right)^2 \Big/ \int_\Omega \nabla z \bullet a \nabla z \, u^2 dx \right] \tag{10}$$

$$\kappa = \inf_{u,h} \int_\Omega (c - h) \bullet A(c - h) u^2 dx, \tag{11}$$

$$\kappa = \inf_{u,h} \left[\int_\Omega c \bullet Acu^2 dx - \left(\int_\Omega c \bullet Ahu^2 dx \right)^2 \Big/ \int_\Omega h \bullet Ahu^2 dx \right] \tag{12}$$

and the infinimum is taken over all functions $u \in H_0^1(\Omega)$, $\int_\Omega u^2 dx = 1$, $h \in L^\infty$, div $\left(h\, u^2\right) = 0$ in weak sense, while the supremum is over all Lipschitz functions $z \in C^{0,1}(\bar{\Omega})$.

P r o o f. From Corollary 4 in [4] we have the following formula for λ_1

$$\lambda_1 = \inf_u \left[\int_\Omega (\nabla u \bullet a \nabla u + b u^2) dx + \sup_z \int_\Omega (T c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx \right]$$

where $u \in H_0^1(\Omega)$, $\int_\Omega u^2 dx = 1$, $z \in C^{0,1}(\bar{\Omega})$. Hence if z is replaced with Tz in the above formula we get the inequality

$$\begin{aligned}\lambda_1 &= \inf_u \left[\int_\Omega (\nabla u \bullet a \nabla u + b u^2) dx + T^2 \sup_z \int_\Omega (c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx \right] \\ &\geq \inf_u \int_\Omega (\nabla u \bullet a \nabla u + b u^2) dx + T^2 \inf_u \sup_z \int_\Omega (c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx \\ &= \lambda_1^0 + T^2 \kappa\end{aligned}$$

where λ_1^0 is the first eigenvalue of the operator

$$L_0 u = -(a_j^m u_{x_m})_{x_j} + bu. \tag{13}$$

In order to prove the opposite inequality in (9), we use the estimate

$$\lambda_1 / T^2 \leq T^{-2} \int_\Omega (\nabla u \bullet a \nabla u + b u^2) dx + \sup_z \int_\Omega (c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx$$

for every $u \in H_0^1(\Omega)$, $\int_\Omega u^2 dx = 1$. After the limit $T \to \infty$ we obtain that

$$\lim_{T \to \infty} (\lambda_1 / T^2) \leq \sup_z \int_\Omega (c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z) u^2 dx$$

for every $u \in H_0^1(\Omega)$ and therefore the inequality $\lim_{T \to \infty} (\lambda_1 / T^2) \leq \kappa$ holds too.

As for the second formula (10) it is enough to maximize the expression under the $\sup_z$ in (9) replacing z with Nz, $N = \text{const}$. Simple computations show that this maximum with respect to N is attained for

$$N = \left(\int_\Omega c \bullet \nabla z \, u^2 dx \right)^2 \Big/ \int_\Omega \nabla z \bullet a \nabla z \, u^2 dx$$

which proves (10).

In order to prove (11) we will use the following formula for λ_1 given in Corollary 4 in [4]

$$\lambda_1 = \inf_{u,h} \left[\int_\Omega (\nabla u \bullet a \nabla u + bu^2)dx + \int_\Omega (Tc - h) \bullet A(Tc - h)u^2 dx \right]$$

where, $\operatorname{div}(hu^2) = 0$, $h \in L^\infty(\Omega)$, $u \in H_0^1(\Omega)$, $\int_\Omega u^2 dx = 1$.

Replacing h with Th we get the following chain of inequalities

$$\kappa = \lim_{T\to\infty} (\lambda_1/T^2) \leq \lim_{T\to\infty} T^{-2} \int_\Omega (\nabla u \bullet a \nabla u + bu^2)dx + \int_\Omega (c - h) \bullet A(c - h)u^2 dx$$

$$\leq s^{-2} \left[\int_\Omega (\nabla u \bullet a \nabla u + s^2 (c - h) \bullet A(c - h)u^2 + bu^2)dx - \int_\Omega (\nabla u \bullet a \nabla u + bu^2)dx \right]$$

$$\leq s^{-2} \left[\int_\Omega (\nabla u \bullet a \nabla u + s^2 (c - h) \bullet A(c - h)u^2 + bu^2)dx - \lambda_1^0 \right].$$

where as above λ_1^0 is the first eigenvalue of the operator L_0 in (12). The infinimum with respect to u and h in the above inequality and the limit $s \to \infty$ gives us

$$\kappa \leq \inf_{u,h} \int_\Omega (c - h) \bullet A(c - h)u^2 dx \leq \lim_{s\to\infty} (\lambda_1(s) - \lambda_1^0)/s^2 = \kappa$$

where $\lambda_1(s)$ is the first eigenvalue of the operator L in (1) for $T = s$.

Let us note that (10), (11) can be considered as infinimum on weight $u \in H_0^1(\Omega)$ of the norms of the linear functional over the spaces

$$L_u^2 = \left\{ z, \quad \int_\Omega \nabla z \bullet A \nabla z u^2 dx < \infty \right\}.$$

As for (12) the proof follows from (11) replacing h with Nh, where N is the optimal constant, $N = \int_\Omega c \bullet Ahu^2 dx \Big/ \int_\Omega h \bullet Ahu^2 dx$. $\square$

Corollary 1. *Under the assumptions of Theorem* 1, $\kappa(a, c, \Omega)$ *has the following properties* $\kappa(a, c, \Omega) \leq \kappa(\bar{a}, c, \Omega)$ *if* $a \geq \bar{a}$ *and* $\kappa(a, c, \Omega) < \kappa(\bar{a}, c, \Omega)$ *if* $a \geq \bar{a} + K.I$, $K = \text{const} > 0$. *Moreover,* $\kappa(a, c, \Omega) > 0$ *or* $\kappa(a, c, \Omega) = 0$ *for all matrices* a *satisfying* (3).

It is curious to mention that according to Proposition 2 in [3] and the above Corollary 1, $\lambda_1(T)$ is a monotone increasing function of a for a sufficiently small T and a monotone decreasing one with respect to a for a sufficiently large T. In general, for arbitrary finite $T > 0$ the monotonity of $\lambda_1(T)$ on a, is an open question (see for more details Proposition 2 in [3] and Proposition 5 in [4]).

Proposition 1. *Under the assumptions of Theorem 1 the following inequalities hold*

$$\sup_z \inf_x \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) \le \kappa \le \inf_x c \bullet Ac \tag{14}$$

Moreover, when $c = \dfrac{1}{2} a \nabla p$ *for some* $p \in W^{2,\infty}(\Omega)$, *then*

$$\kappa = \sup_z \inf_x \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) = \inf_x c \bullet Ac \tag{15}$$

P r o o f. The proof of (14) follows from the chain of inequalities

$$\begin{aligned}
\sup_z \inf_x \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) &\le \inf_u \sup_z \int_\Omega \inf_x \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) u^2 dx \\
&\le \inf_u \sup_z \int_\Omega \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) u^2 dx = \kappa \\
&= \inf_u \sup_z \int_\Omega \left(c \bullet Ac - (\frac{1}{2} a \nabla z - c) \bullet A (\frac{1}{2} a \nabla z - c) \right) u^2 dx \\
&\le \inf_u \int_\Omega c \bullet Acu^2 dx = \inf_x c \bullet Ac
\end{aligned}$$

To obtain the last inequality we use a sequence $u_n(x) \in H_0^1(\Omega)$, $\int_\Omega u_n^2 dx = 1$, where $B(u_n) = \operatorname{supp} u_n \to \{x_0\}$. Since

$$\begin{aligned}
\inf_x c \bullet Ac \le \inf_u \int_\Omega (\inf_x c \bullet Ac) u^2 dx &\le \inf_u \int_\Omega c \bullet Ac\, u^2 dx \\
&\le \lim_{n\to\infty} \int_\Omega c \bullet Acu_n^2 dx \le \lim_{n\to\infty} \sup_{x \in B(u_n)} c \bullet Ac.
\end{aligned}$$

So we get the inequality

$$\inf_x c \bullet Ac \le \inf_u \int_\Omega c \bullet Ac\, u^2 dx \le c(x_0) \bullet A(x_0) c(x_0)$$

for every $x_0 \in \Omega$. This proves (14).

If $c = \frac{1}{2} a\nabla p$ we get the estimates

$$\inf_x c \bullet Ac \le \frac{1}{4} \inf_x \nabla p \bullet a\nabla p \le \sup_z \inf_x \left(c \bullet \nabla z - \frac{1}{4} \nabla z \bullet a \nabla z \right) \le \kappa \le \inf_x c \bullet Ac$$

and (15) is proved. □

Remark 1. As a consequence of Proposition 1 we obtain that a necessary condition for the asymptotic (8) with the maximum possible rate T^2 as $T \to \infty$ is $c(x) \neq 0$ in $\bar{\Omega}$. Unfortunately, unless in the two dimensional case (see Proposition 3 below) this condition is not a sufficient one. The following theorem gives an answer of this question.

Theorem 2. *Suppose* (2), (3) *hold. Then* $\kappa > 0$ *if and only if* (6) *is satisfied.*

Proof. If (6) holds then the statement of Theorem 2 follows immediately from (10). As for the necessity of (6) we will apply Lemma 2.3 in [2] which we present here for completeness.

Lemma 1 (J. Frank and K. Robinson). *Suppose that* $c \in C^1$ *in some neighborhood* Ω_1 *of* $\bar{\Omega}$. *Every solution of the initial-value problem*

$$(16) \qquad \left| \begin{array}{l} \dot{\varphi}(t) = c(\varphi(t)) \\ \varphi(0) = x_0 \in \bar{\Omega} \end{array} \right.$$

remains in Ω_1 *for only a finite time in the time interval* $(-\infty, \infty)$ *if and only if there exist a function* $w \in C^1(\Omega)$ *such that* (6) *holds.*

Moreover we will use the following estimate from above for κ.

Lemma 2. *Suppose* (2) *and* (3) *are satisfied. Then for every* $C^{1,1}$ *regular closed curve* $\varphi(t) : [-N, N] \to \bar{\Omega}$ *the estimate*

$$(17) \qquad \kappa \le \frac{1}{M} \int_{-N}^{N} (c(\varphi) - \dot{\varphi}) \bullet A(\varphi)(c(\varphi) - \dot{\varphi}) \frac{dt}{|\dot{\varphi}|}$$

holds, where, $M = \int_{-N}^{N} \frac{dt}{|\dot{\varphi}(t)|}$.

Proof. Suppose that $\varphi(t)$ contains in Ω and $\rho(x)$ is the distance from x to the nearest point $t(x) \in \gamma = \{\varphi(t),\ -N \le t \le N\}$. Let us consider the domain

$$G(\varepsilon) = \{x \in \Omega,\ \rho(x) < \varepsilon\} = \{(t, y) \in [-N, N] \times D(\varepsilon, t)\},$$

and $D(\varepsilon,t) = \{y : r = |y-\varphi(t)| < \varepsilon, \ (y-\varphi)\bullet\dot{\varphi} = 0\}$ where $\varepsilon > 0$ is sufficiently small so that $\rho(x) \in C^{1,1}(\bar{G}(\varepsilon)\backslash\gamma)$and $G(\varepsilon) \subset \Omega$. For every $x \in G(\varepsilon)\backslash\gamma$ differentiating the equality $(x-\varphi(t(x)))\bullet\dot{\varphi}(t(x)) = 0$ with respect to $x's$ we get the identities

$$\dot{\varphi}(t) + [(x-\varphi(t))\bullet\ddot{\varphi}(t) - \dot{\varphi}^2(t)](\partial t/\partial x) = 0 \tag{18}$$

So $\dfrac{\partial t}{\partial x} = p(x)\dot{\varphi}$, where $p(x)$ is a scalar function. Then we have

$$\begin{aligned}\operatorname{div}\frac{\dot{\varphi}}{|\dot{\varphi}|} &= \frac{d}{dt}\left(\frac{\dot{\varphi}}{|\dot{\varphi}|}\right)\bullet\frac{\partial t}{\partial x} = \left(\frac{\ddot{\varphi}}{|\dot{\varphi}|} - \frac{\dot{\varphi}\bullet\ddot{\varphi}}{|\dot{\varphi}|^3}\dot{\varphi}\right)\bullet\frac{\partial t}{\partial x}\\ &= p(t(x))\left(\frac{\ddot{\varphi}}{|\dot{\varphi}|} - \frac{\dot{\varphi}\bullet\ddot{\varphi}}{|\dot{\varphi}|^3}\dot{\varphi}\right)\bullet\dot{\varphi} = 0\end{aligned} \tag{19}$$

Now we will apply (11) for the domain $G(\varepsilon)$ and for the functions $u = \dfrac{(\varepsilon-\rho)}{K_\varepsilon\,|\dot{\varphi}|^{1/2}}$ and $h(x) = \dot{\varphi}(t(x))$ where, $K_\varepsilon^2 = \int\limits_{G(\varepsilon)} (\varepsilon-\rho)^2\dfrac{dt}{|\dot{\varphi}|} = M\int\limits_{D(\varepsilon)} (\varepsilon-r)^2 dS$, $D(\varepsilon) = D(\varepsilon, 0)$.

Let us check that $\operatorname{div}(hu^2) = 0$ in $G(\varepsilon)$. Indeed, we have the equality $\operatorname{div}(hu^2) = \operatorname{div}\left(\dfrac{\dot{\varphi}}{|\dot{\varphi}|}\right)\dfrac{(\varepsilon-\rho)^2}{K_\varepsilon^2} - 2\dfrac{\dot{\varphi}}{|\dot{\varphi}|}\dfrac{\varepsilon-\rho}{K_\varepsilon^2}.\nabla\rho$. The first term is zero from (19) and the second term is zero from the geometry property $\dfrac{\dot{\varphi}}{|\dot{\varphi}|}\bullet\nabla\rho = 0$.

From (11) and the continuity properties of the coefficients in (2) we have the chain of inequalities

$$\begin{aligned}\kappa &\le K_\varepsilon^{-2}\int\limits_{G(\varepsilon)} (c(x)-\dot{\varphi}(t(x)))\bullet A(x)(c(x)-\dot{\varphi}(t(x)))(\varepsilon-\rho(x))^2\frac{dx}{|\dot{\varphi}(t(x))|}\\ &\le K_\varepsilon^{-2}\int\limits_{G(\varepsilon)} (c(\varphi(t(x)))-\dot{\varphi}(t(x)))\bullet A(\varphi(t(x)))(c(\varphi(t(x)))\\ &\quad -\dot{\varphi}(t(x)))(\varepsilon-\rho(x))^2\frac{dx}{|\dot{\varphi}(t(x))|} + K_\varepsilon^{-2}C\varepsilon\int\limits_{G(\varepsilon)} (\varepsilon-\rho(x))^2\frac{dx}{|\dot{\varphi}(t(x))|}\\ &\le K_\varepsilon^{-2}\int\limits_{\gamma} (c(\varphi(t))-\dot{\varphi}(t))\bullet A(\varphi(t))(c(\varphi(t))-\dot{\varphi}(t))\frac{dt}{|\dot{\varphi}(t)|}\int\limits_{D(\varepsilon)} (\varepsilon-r)^2 dS + \frac{C}{M}\varepsilon\end{aligned}$$

where the constant C is independent of ε. Since $K_\varepsilon^2 = M\int\limits_{D(\varepsilon)} (\varepsilon-r)^2 dS$ after the limit $\varepsilon \to 0$ we get (17).

In order to prove (17) for curves $\varphi(t) \in \bar{\Omega}$ it is enough to approximate φ in C^1 topology with curves $\varphi_n \in \Omega$. □

In the proof of Theorem 2 we will use, in the multidimensional case $n \geq 3$, the following simple corollary of Lemma 2.

Corollary 2. *Suppose* (2), (3) *are satisfied and* $n \geq 3$. *For every* $C^{1,1}$ *regular curve* $\varphi(t) : [-N, N] \to \bar{\Omega}$ *without self intersection points the estimate*

$$\kappa \leq \frac{K}{M} + \frac{1}{M} \int_{-N}^{N} (c(\varphi) - \dot{\varphi}) \bullet A(\varphi)(c(\varphi) - \dot{\varphi}) \frac{dt}{|\dot{\varphi}(t)|} \tag{20}$$

holds where the constant K *depends only on the maximum of* A, c *and the diameter* R *of* Ω.

P r o o f. Without loss of generality we assume that $\varphi(t) \in \Omega$. Now let us construct a closed curve with a length under control containing the curve $\varphi(t)$. Then applying Lemma 2 to the new closed curve we get the estimate (20). Suppose for this purpose that $S = \{|x - x_0| = \delta\} \subset \Omega$ is a sphere with sufficiently small radius δ such that $\gamma \cap S = \emptyset$, $\gamma = \{x \in \Omega;\, x = \varphi(t)\}$. If η_t are the lines through the points $\varphi(N)$ and $\varphi(t)$, $t \in [-N, N]$ we denote with γ_1 the set of all intersection points of η_t and S, i.e. $\gamma_1 = \{x \in \eta_t \cap S\}$. Correspondingly, γ_2 is the set $\gamma_2 = \{x \in S \cap q_t\}$ where q_t are the lines through the points $\varphi(-N)$ and $\varphi(t)$, $t \in [-N, N]$. Since $\text{mes}_S \gamma_1 = \text{mes}_S \gamma_2 = 0$, where mes_S is the $n-1$ dimensional Hausdorf measure on the sphere S, there exists point $y \in S$ such that the segments η and q connecting $\varphi(N)$ and y, $\varphi(-N)$ and y resp., do not intersect γ. In this way the curve $\tilde{\gamma}$ defined by γ, the segments η and q is a closed curve in $\bar{\Omega}$ without self intersecting points (see Figure 1). Moreover, we have the following estimate for the length $\tilde{M}$ of $\tilde{\gamma}$, $M \leq \tilde{M} \leq M + 2R$. By the means of approximation arguments, there is no problem to construct $\tilde{\gamma}$ as a $C^{1,1}$ smooth curve without self intersection points and length smaller than $M + 2R$ for instance.

Let us now finish the proof of the necessity in Theorem 2. Suppose $\kappa > 0$ but condition (6) fails. Note that according to Remark 1 $c(x) \neq 0$ in $\bar{\Omega}$. We extend the coefficients of L in a larger domain $\Omega_0 \supset \Omega$, $\Omega_0 = \bar{\Omega} \cup \{x \in R^n,\, \rho(x) < \rho_0\}$, where $\rho(x)$ is the distance from x to $\partial\Omega$ and ρ_0 is a sufficiently small positive constant. We shall use the notation $\tilde{c}$, $\tilde{a}$, $\tilde{b}$ for the even extensions of c, a, b in Ω_0 through the boundary $\partial\Omega$ in the normal direction ν of the coefficients of L. More precisely, if $x = y(x) + s.\nu(y(x))$, $0 < s \leq \rho_0$, ν is the unit outward normal to $\partial\Omega$ at the point $y(x) \in \partial\Omega$, nearest to x, then $\tilde{c}(x) = c(-s\nu(y(x))$, $\tilde{a}(x) = a(-s\nu(y(x))$, $\tilde{b}(x) = b(-s\nu(y(x))$ for $x \in \bar{\Omega}_0 \backslash \bar{\Omega}$.

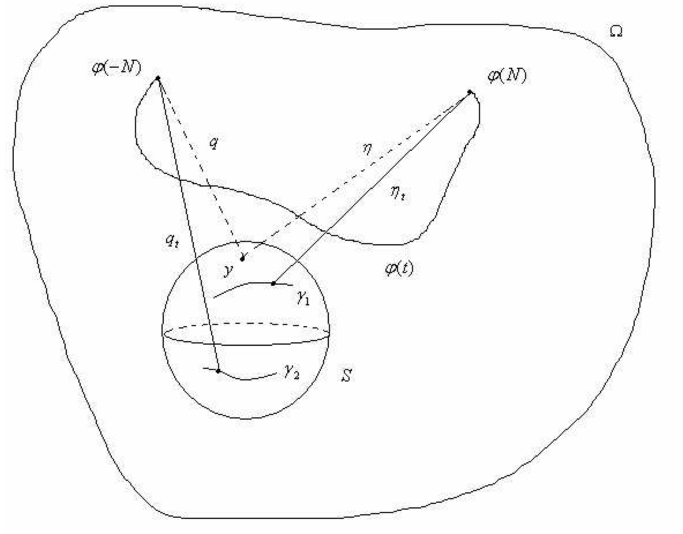

Fig. 1. Construction of the closed curve.

First, let us consider the multidimensional case $n \geq 3$. Since (6) fails, according to Lemma 1, there exists a solution $\varphi(t)$ of (16) such that the curve $\gamma = \{x = \varphi(t)\}$ remains in some fixed neighborhood Ω_1 of $\bar{\Omega}$, $\bar{\Omega} \subset \Omega_1 \subset \Omega_0$, for infinite time in the time interval $(-\infty, \infty)$. If γ or a part of γ with infinite length belongs to $\bar{\Omega}$ then by means of Corollary 2 it follows from (20) that

$$\kappa \leq \frac{K}{M} + \frac{1}{M} \int_{-N}^{N} (\tilde{c}(\varphi) - \dot{\varphi}) \bullet \tilde{A}(\varphi)(\tilde{c}(\varphi) - \dot{\varphi}) \frac{dt}{|\dot{\varphi}(t)|} = \frac{K}{M} \tag{21}$$

Here the curve γ is considered only on some finite time interval, $\varphi : [-N, N] \to \bar{\Omega}_1$. When N is chosen sufficiently large, i.e. the curve has sufficiently large length (for example $M > K/\kappa$) we get a contradiction, see (21).

If the part of γ with infinite length belongs to $\Omega_1 \backslash \bar{\Omega}$, then the even extension $\tilde{\varphi}$ of φ with respect to $\partial\Omega$ belongs to $\bar{\Omega}$ and is a solution of (16). Thus we can repeat the previous arguments and get a contradiction.

In the two dimensional case we will apply the theory of Poincare-Bendickson. More precisely, if there exists an infinite curve $\beta = \{x = \psi(t)\} \subset \bar{\Omega}_0$ which is a solution of (16), then from the even extension of $c(x)$ in Ω_0 it follows that there exists an infinite curve in $\bar{\Omega}$. Now according to theorem 4.3, chapter VII in [6], there exists a closed curve $\gamma = \{x = \varphi(t)\} \subset \bar{\Omega}$, where $\varphi(t)$ is a solution of (16). Using Lemma 2 instead of Corollary 2, the rest of the proof follows as in the multidimensional case.

We obtain also the following geometric necessary and sufficient condition for the maximum possible asymptotic (8) for λ_1 by the means of Theorem 2 and Lemma 1.

Proposition 2. *Suppose* (2), (3) *hold. Then* $\kappa > 0$ *if and only if every solution* $\varphi(t)$ *of* (16) *remains in some fixed neighborhood* $\Omega_1 \subset \Omega_0$ *of* $\bar{\Omega}$ *only for a finite time.*

3. Applications about the two-dimensional case. For $\Omega \in R^2$ the necessary and sufficient conditions in Theorem 2 and Proposition 2 can be replaced with an easy checkable and simple condition.

Proposition 3. *Suppose* (2), (3) *hold and* Ω *is a simply connected domain in* R^2. *Then* $\kappa > 0$ *if and only if* $c(x) \neq 0$ *in* $\bar{\Omega}$.

The proof of the Proposition 3 follows from Theorem 4.4, chapter VII in [6] and the Proposition 2.

The necessity of the condition that Ω is a simply connected domain in R^2 is shown by the following example.

Example 1. Consider in $G = \{1 < |x| < 2\} \subset R^2$ the operator

$$Lu = -(u_{x_1} + Tx_2 u)_{x_1} - (u_{x_2} - Tx_1 u)_{x_2} - Tx_2 u_{x_1} + Tx_1 u_{x_2}$$

In this case $c^1(x) = -x_2$, $c^2(x) = x_1$ and the system $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1$, $x_1(0) = y_1$, $x_2(0) = y_2$, $y = (y_1, y_2) \in \bar{G}$ has a periodic solution $\varphi(t) \in \bar{G}$ which is a circle with radius $|y|$. Note that $c(x) \neq 0$ in $\bar{G}$, but according to the Proposition 2 we get $\kappa = 0$. The reason is that G is not a simply connected domain in R^2.

A similar example in the multidimensional case illustrates that the necessary condition $c(x) \neq 0$ in $\bar{\Omega}$ for the positiveness of κ in Remark 1, is not sufficient one in the case $n \geq 3$ even in a simply connected domain.

Example 2. Let $G \subset R^3$ is a solid torus with radii 1 and 2, i.e. a solid of revolution, revolving a circle $\{x_1 = 0,\ (x_2 - 2)^2 + x_3^2 < 1\}$ about the axis Ox_3. Remain that the torus is a simply connected domain. Consider in G the operator

$$\begin{aligned} Lu = &-(u_{x_1} - T(-x_2 + x_3)u)_{x_1} - (u_{x_2} - T(-x_3 + x_1)u)_{x_2} \\ &-(u_{x_3} - T(-x_1 + x_2)u)_{x_3} + T(-x_2 + x_3)u_{x_1} \\ &+ T(-x_3 + x_1)u_{x_2} + T(-x_1 + x_2)u_{x_3} \end{aligned}$$

Since $\dot{x} = c(x)$, $x(0) = y \in \bar{G}$ has a periodic solution which remains in $\bar{G}$, then it follows that $\kappa = 0$ from the Proposition 2.

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Received September 25, 2006
Revised December 22, 2006