Balkanica
New Series Vol. 28, 2014, Fasc. 1–2

Approximation Properties of Schurer-Stancu Type Polynomials

Sezgin Sucu [1], *Yeşim Döne* [2], *Ertan İbikli* [3]

Presented by Virginia Kiryakova

Considering four non-negative parameters α_1, α_2, β_1, β_2 which satisfy

$$0 \leq \alpha_2 \leq \alpha_1 \leq \beta_1 \leq \beta_2$$

and a given non-negative integer p, the generalized Schurer-Stancu type operators

$$S_{n,p,\alpha,\beta} : C\left[0, 1+p\right] \longrightarrow C\left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$$

$$S_{n,p,\alpha,\beta}\left(f;x\right) := \left(\frac{n+\beta_2}{n}\right)^{n+p} \sum_{k=0}^{n+p} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) C_{n+p}^{k} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^{k} \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n+p-k}$$

are constructed. Theorem on convergence and the degree of approximation are established by modulus of continuity.

MSC 2010: 41A25, 41A35

Key Words: Schurer-Stancu type operators, modulus of continuity, rate of convergence, Korovkin theorem

1. Introduction

Bernstein polynomials were introduced by Bernstein [3]. These polynomials have been studied intensively and defined variously generalized forms.

In 1962, F. Schurer [5] introduced and studied the linear positive operators called that Bernstein-Schurer polynomials,

$$B_{n,p} : C\left[0, 1+p\right] \longrightarrow C\left[0, 1\right]$$

defined by

$$B_{n,p}\left(f;x\right) := \sum_{k=0}^{n+p} f\left(\frac{k}{n}\right) C_{n+p}^{k} x^{k} \left(1-x\right)^{n+p-k} \quad . \tag{1}$$

D. D. Stancu [7] constructed the linear positive operators $B_n^{\alpha,\beta}(f;x)$ associated with a real function $f\in C[0,1]$, have the property that $\lim\limits_{n\to\infty} B_n^{\alpha,\beta}(f;x)=f(x)$ uniformly on $[0,1]$ in 1969. These operators are known Bernstein-Stancu polynomials.

Later, D. Bărbosu [2] constructed the linear positive operators as a generalization of Bernstein operators

$$P_{n,p,\alpha,\beta}:C\left[0,1+p\right]\longrightarrow C\left[0,1\right]$$

are defined by

$$P_{n,p,\alpha,\beta}\left(f;x\right):=\sum_{k=0}^{n+p} f\left(\frac{k+\alpha}{n+\beta}\right) C_{n+p}^{k} x^{k}\left(1-x\right)^{n+p-k} \tag{2}$$

and investigated approximation properties of these operators which are known Schurer-Stancu polynomials.

Then, A. D. Gadjiev and A. M. Ghorbanalizadeh [4] introduced a new construction of Bernstein-Stancu type polynomials with shifted knots. These linear positive operators

$$S_{n,\alpha,\beta}:C\left[0,1\right]\longrightarrow C\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$$

are defined by

$$\begin{aligned} S_{n,\alpha,\beta}\left(f;x\right) \ :&= \left(\frac{n+\beta_2}{n}\right)^{n} \\ &\times \sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) C_n^k \left(x-\frac{\alpha_2}{n+\beta_2}\right)^{k}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k} \end{aligned} \tag{3}$$

where $\frac{\alpha_2}{n+\beta_2}\le x\le \frac{n+\alpha_2}{n+\beta_2}$ and for $\alpha_k,\ \beta_k\,(k=1,2)$ are non-negative real numbers provided $0\le\alpha_2\le\alpha_1\le\beta_1\le\beta_2$. Moreover, A. D. Gadjiev and A. M. Ghorbanalizadeh investigated of convergence and the degree of approximation of polynomials $S_{n,\alpha,\beta}\left(f;x\right)$ in the space of continuous functions.

Let p be a non-negative integer and let

$$S_{n,p,\alpha,\beta}:C\left[0,1+p\right]\longrightarrow C\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$$

be defined for any $f\in C\left[0,1+p\right]$ and $n\in\mathbb{N}$, by

$$\begin{aligned} S_{n,p,\alpha,\beta}\left(f;x\right) \ &:= \left(\frac{n+\beta_2}{n}\right)^{n+p} \\ &\times \sum_{k=0}^{n+p} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) C_{n+p}^k \left(x-\frac{\alpha_2}{n+\beta_2}\right)^{k}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n+p-k} \end{aligned} \tag{4}$$

where $\frac{\alpha_2}{n+\beta_2} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}$ and for α_k, β_k $(k = 1, 2)$ are non-negative real numbers provided $0 \leq \alpha_2 \leq \alpha_1 \leq \beta_1 \leq \beta_2$. It is clear that for $p = 0$ we obtain Bernstein-Stancu type polynomials (3), for $\alpha_2 = \beta_2 = 0$ we get Schurer-Stancu polynomials (2) and if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ Bernstein-Schurer polynomials (1) are obtained.

This paper is about generalization of Schurer-Stancu type polynomials (4) with shifted knots symbolized by $S_{n,p,\alpha,\beta}(f; x)$. Convergence of the operators (4) is investigated with the help of the well known Bohman-Korovkin theorem. The degree of convergence of $S_{n,p,\alpha,\beta}(f; x)$ is established by using the modulus of continuity. Some numerical examples are given with the help of Maple13.

2. Approximation properties of $S_{n,p,\alpha,\beta}$ operators

Lemma 1. *Generalized Schurer-Stancu type operators, defined by* (4), *are linear and positive.*

The assertions follow from the definition of operators.

Lemma 2. *For* $\forall x \in [0, 1]$ *and* $\forall n \in \mathbb{N}$ *the generalized Schurer-Stancu type operators* (4) *have the following properties*

$$S_{n,p,\alpha,\beta}(1; x) = 1 \tag{5}$$

$$S_{n,p,\alpha,\beta}(t; x) = \left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{n+p}{n}\right)x + \frac{-p\alpha_2 + n(\alpha_1 - \alpha_2)}{n(n+\beta_1)} \tag{6}$$

$$\begin{aligned} S_{n,p,\alpha,\beta}(t^2; x) = {} & \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \frac{(n+p)(n+p-1)}{n^2}\left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 \\ & + \frac{n+\beta_2}{(n+\beta_1)^2}\left(\frac{n+p}{n}\right)\left(x - \frac{\alpha_2}{n+\beta_2}\right) \\ & + 2\alpha_1 \frac{n+\beta_2}{(n+\beta_1)^2}\left(\frac{n+p}{n}\right)\left(x - \frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2}. \end{aligned} \tag{7}$$

P r o o f. Let be

$$\widetilde{p}_{n,p,k,\alpha,\beta}(x) = \left(\frac{n+\beta_2}{n}\right)^{n+p} C_{n+p}^k \left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n+p-k}.$$

Using the definition (4), we easily get

$$S_{n,p,\alpha,\beta}(1; x) = \sum_{k=0}^{n+p} \widetilde{p}_{n,p,k,\alpha,\beta}(x) = 1,$$

where we use binomial expansion. Like this way, it is obvious that

$$\begin{aligned} S_{n,p,\alpha,\beta}(t;x) &= \sum_{k=0}^{n+p}\left(\frac{k+\alpha_1}{n+\beta_1}\right)\widetilde{p}_{n,p,k,\alpha,\beta}(x) \\ &= \frac{n+p}{n+\beta_1}\sum_{k=0}^{n+p}\frac{k}{n+p}\widetilde{p}_{n,p,k,\alpha,\beta}(x)+\frac{\alpha_1}{n+\beta_1}\sum_{k=0}^{n+p}\widetilde{p}_{n,p,k,\alpha,\beta}(x) \\ &= \left(\frac{n+p}{n+\beta_1}\right)\left(\frac{n+\beta_2}{n}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1}{n+\beta_1}. \end{aligned}$$

Then, we can conclude that

$$S_{n,p,\alpha,\beta}(t;x)=\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{n+p}{n}\right)x+\frac{-p\alpha_2+n(\alpha_1-\alpha_2)}{n(n+\beta_1)}\ ,$$

i.e. (6) holds. After simple calculation, we can write

$$\begin{aligned} S_{n,p,\alpha,\beta}(t^2;x) &= \sum_{k=0}^{n+p}\left(\frac{k+\alpha_1}{n+\beta_1}\right)^2\widetilde{p}_{n,p,k,\alpha,\beta}(x) \\ &= \frac{1}{(n+\beta_1)^2}\left\{\sum_{k=0}^{n+p}k^2\widetilde{p}_{n,p,k,\alpha,\beta}(x)\right. \\ &\quad \left.+2\alpha_1\sum_{k=0}^{n+p}k\widetilde{p}_{n,p,k,\alpha,\beta}(x)+\alpha_1^2\sum_{k=0}^{n+p}\widetilde{p}_{n,p,k,\alpha,\beta}(x)\right\} \\ &= \frac{1}{(n+\beta_1)^2}\left\{\left(\frac{n+\beta_2}{n}\right)^2(n+p)(n+p-1)\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2\right. \\ &\quad +\left(\frac{n+\beta_2}{n}\right)(n+p)\left(x-\frac{\alpha_2}{n+\beta_2}\right) \\ &\quad \left.+2\alpha_1\left(\frac{n+\beta_2}{n}\right)(n+p)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\alpha_1^2\right\}\ . \end{aligned}$$

Taking into account of the above equality, we get

$$\begin{aligned} S_{n,p,\alpha,\beta}(t^2;x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\frac{(n+p)(n+p-1)}{n^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 \\ &\quad +\frac{n+\beta_2}{(n+\beta_1)^2}\left(\frac{n+p}{n}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right) \\ &\quad +2\alpha_1\frac{n+\beta_2}{(n+\beta_1)^2}\left(\frac{n+p}{n}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1^2}{(n+\beta_1)^2}\ , \end{aligned}$$

i.e. (7) holds and the proof ends. ∎

Theorem 3. *Let f be a continuous function on $[0,1+p]$. Then*

$$\lim_{n\to\infty}\|S_{n,p,\alpha,\beta}(f;x)-f(x)\|_{C\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]}=0\ .$$

P r o o f. Taking into account of the equalities in Lemma 2, for $\nu=0,1,2$ we can write

$$\lim_{n\to\infty}\max_{x\in\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]}|S_{n,p,\alpha,\beta}(t^\nu;x)-x^\nu|=0. \tag{8}$$

Denoting

$$S_{n,p}(f;x)=\begin{cases} S_{n,p,\alpha,\beta}(f;x) & ;\quad x\in\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right] \\ f(x) & ;\quad x\in\left[0,\frac{\alpha_2}{n+\beta_2}\right)\cup\left(\frac{n+\alpha_2}{n+\beta_2},1\right] \end{cases},$$

we easily get

$$\|S_{n,p}f-f\|_{C[0,1]}=\max_{x\in\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]}|S_{n,p,\alpha,\beta}(f;x)-f(x)|\ . \tag{9}$$

On the other hand, using (8) and (9) we immediately get

$$\lim_{n\to\infty}\|S_{n,p}(t^\nu;x)-x^\nu\|_{C[0,1]}=0 \qquad \nu=0,1,2.$$

Applying the Korovkin's theorem to the sequence of linear positive operators $S_{n,p}$ we obtain

$$\lim_{n\to\infty}\|S_{n,p}(f;x)-f(x)\|_{C[0,1]}=0$$

for every continuous function f. Therefore (9) gives

$$\lim_{n\to\infty}\max_{x\in\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]}|S_{n,p,\alpha,\beta}(f;x)-f(x)|=0$$

and the proof is completed. ∎

3. The rate of convergence of $S_{n,p,\alpha,\beta}$ operators

Qualitative result known as a sequence of operators approximate the identity operator was obtained in Section 2 with the help of well-known Korovkin theorem. The other important problem in approximation theory is quantitative estimate which is about how quickly do the operators converge the identity operator. The aim of the present section is a study of quantitative result using modulus of continuity.

In order to obtain the degree of convergence, we are going to define the modulus of continuity of the function f as follows:

$$\omega(f;\delta) = \sup\left\{|f(x)-f(y)| : x,y \in [0,1], |x-y| \le \delta\right\} \tag{10}$$

for any positive number δ [1].

Before starting our main theorem, we give lemma and auxiliary result concerning the order of approximation.

Lemma 4. *The operators* (4) *verify*

$$\begin{aligned} S_{n,p,\alpha,\beta}((t-x)^2;x) = \frac{1}{(n+\beta_1)^2}\Bigg\{ & \frac{(n+\beta_2)^2(n+p)}{n^2} \\ & \times\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right) \\ & +\left(\left(\frac{n+p}{n}\beta_2-\beta_1\right)x+\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)\right)^2 \\ & +\left(p+2\beta_2\frac{n+p}{n}-2\beta_1\right)px^2 \\ & +\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)2px\Bigg\} . \end{aligned} \tag{11}$$

P r o o f. The linearity of $S_{n,p,\alpha,\beta}$ leads us to

$$S_{n,p,\alpha,\beta}((t-x)^2;x) = S_{n,p,\alpha,\beta}(t^2;x) - 2xS_{n,p,\alpha,\beta}(t;x) + x^2 S_{n,p,\alpha,\beta}(1;x) .$$

Taking into account of the equality in Lemma 2 , we obtain

$$
\begin{aligned}
S_{n,p,\alpha,\beta}((t-x)^2\,;x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2 \frac{(n+p)(n+p-1)}{n^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 \\
&+\frac{(n+\beta_2)}{(n+\beta_1)^2}(\frac{n+p}{n})\left(x-\frac{\alpha_2}{n+\beta_2}\right) \\
&+2\alpha_1\frac{(n+\beta_2)}{(n+\beta_1)^2}(\frac{n+p}{n})\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1^2}{(n+\beta_1)^2} \\
&-2x\left[\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{n+p}{n}\right)x+\frac{-p\alpha_2+n(\alpha_1-\alpha_2)}{n(n+\beta_1)}\right]+x^2 \\
&=\left(\frac{1}{n+\beta_1}\right)^2 \\
&\times\left\{(n+\beta_2)^2\,\frac{(n+p)(n+p-1)}{n^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2\right. \\
&+(n+\beta_2)\,(\frac{n+p}{n})\left(x-\frac{\alpha_2}{n+\beta_2}\right) \\
&+2\alpha_1\,(n+\beta_2)\,(\frac{n+p}{n})\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\alpha_1^2 \\
&-2x\left[(n+\beta_1)\,(n+\beta_2)\left(\frac{n+p}{n}\right)x\right. \\
&\left.\left.+(n+\beta_1)\,\frac{-p\alpha_2+n(\alpha_1-\alpha_2)}{n}\right]+x^2\,(n+\beta_1)^2\right\} \\
&=\frac{1}{(n+\beta_1)^2} \\
&\times\left\{\frac{(n+\beta_2)^2\,(n+p)}{n^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)\right. \\
&+\left(\left(\frac{n+p}{n}\beta_2-\beta_1\right)x+\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)\right)^2 \\
&\left.+\left(p+2\beta_2\frac{n+p}{n}-2\beta_1\right)px^2+\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)2px\right\}.
\end{aligned}
$$

Thus, we get the desired result. ■

It is well known the following result obtained by O. Shisha and B. Mond [6].

Theorem 5. *Let $(L_n)_{n\in\mathbb{N}}$, $L_n : C\,[a,b] \to B\,[a,b]$ be a sequence of linear positive operators, reproducing the constant functions. For any $f\in C\,[a,b]$,*

$x \in [a,b]$ and $\delta \in [0, b-a]$, the following

$$|L_n(f;x) - f(x)| \le \left\{1 + \delta^{-1}\sqrt{L_n\left((t-x)^2;x\right)}\right\}\omega(f;\delta)$$

holds.

Theorem 6. *For any $f \in C[0,1+p]$ and each $x \in \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$ generalized Schurer-Stancu type operators (4) verify*

$$|S_{n,p,\alpha,\beta}(f;x) - f(x)| \le 2\omega(f;\sqrt{\delta_{n,p,\alpha,\beta,x}}) \tag{12}$$

where

$$\begin{aligned}\delta_{n,p,\alpha,\beta,x} = \frac{1}{(n+\beta_1)^2}&\left\{\frac{(n+\beta_2)^2(n+p)}{n^2}\left(x - \frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)\right.\\ &+\left(\left(\frac{n+p}{n}\beta_2 - \beta_1\right)x + \left(\alpha_1 - \frac{n+p}{n}\alpha_2\right)\right)^2\\ &+\left(p + 2\beta_2\frac{n+p}{n} - 2\beta_1\right)px^2\\ &\left.+\left(\alpha_1 - \frac{n+p}{n}\alpha_2\right)2px\right\}\end{aligned} \tag{13}$$

and $\delta_{n,p,\alpha,\beta,x}$ tends to zero as $n \to \infty$.

Proof. The main step in the proof of this theorem consists in the following inequality given in Theorem 5

$$|S_{n,p,\alpha,\beta}(f;x) - f(x)| \le \left\{1 + \delta^{-1}\sqrt{S_{n,p,\alpha,\beta}\left((t-x)^2;x\right)}\right\}\omega(f;\delta)\ . \tag{14}$$

Applying Lemma 4 and (14) we get

$$\begin{aligned}|S_{n,p,\alpha,\beta}(f;x) - f(x)| \le &\left(1 + \frac{1}{\delta}\frac{1}{n+\beta_1}\right.\\ &\times\left\{\frac{(n+\beta_2)^2(n+p)}{n^2}\left(x - \frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2} - x\right)\right.\\ &+\left(\left(\frac{n+p}{n}\beta_2 - \beta_1\right)x + \left(\alpha_1 - \frac{n+p}{n}\alpha_2\right)\right)^2\\ &+\left(p + 2\beta_2\frac{n+p}{n} - 2\beta_1\right)px^2\\ &\left.\left.+\left(\alpha_1 - \frac{n+p}{n}\alpha_2\right)2px\right\}^{1/2}\right)\omega(f;\delta).\end{aligned}$$

In the above inequality by choosing

$$\delta=\sqrt{\delta_{n,p,\alpha,\beta,x}}=\frac{1}{n+\beta_1}\left\{\frac{(n+\beta_2)^2(n+p)}{n^2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)\right.$$
$$+\left(\left(\frac{n+p}{n}\beta_2-\beta_1\right)x+\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)\right)^2$$
$$\left.+\left(p+2\beta_2\frac{n+p}{n}-2\beta_1\right)px^2+\left(\alpha_1-\frac{n+p}{n}\alpha_2\right)2px\right\}^{1/2},$$

we arrive to (12) and the proof ends. ■

Remark 7. Theorem 6 expresses the order of local approximation of f by $S_{n,p,\alpha,\beta}f$. For getting the order of global approximation, we must take in (13) the maximum of $\delta_{n,p,\alpha,\beta,x}$ when $x\in\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$,

$$|S_{n,p,\alpha,\beta}(f;x)-f(x)|\leq 2\omega(f;\sqrt{\delta_{n,p,\alpha,\beta}})$$

where

$$\delta_{n,p,\alpha,\beta}=\frac{1}{(n+\beta_1)^2}\left\{\frac{n+p}{4}+\left((\beta_2-\beta_1)\frac{n+\alpha_2}{n+\beta_2}+(\alpha_1-\alpha_2)\right)^2\right.$$
$$+\left(p+2\frac{n+\beta_2}{n}(\beta_2-\beta_1)+2\frac{p}{n}\beta_2+\frac{p}{n^2}\beta_2^2\right)p\left(\frac{n+\alpha_2}{n+\beta_2}\right)^2$$
$$+\left(2\frac{n+\beta_2}{n}(\alpha_1-\alpha_2)-2\frac{1}{n}\alpha_2(\beta_2-\beta_1)-2\frac{p}{n}\alpha_2-2\frac{p}{n^2}\alpha_2\beta_2\right)$$
$$\left.\times p\frac{n+\alpha_2}{n+\beta_2}+\left(\frac{p}{n^2}\alpha_2^2-2\frac{1}{n}\alpha_2(\alpha_1-\alpha_2)\right)p\right\}.$$

Clearly, this maximum depends on the relations between α_k, β_k, p $(k=1,2)$.

Example 8. Let be $f(x)=\sin(2\pi(x+1)^{\frac{3}{2}})$, $p=1$, $\alpha_1=0.2$, $\alpha_2=0.1$, $\beta_1=0.3$ and $\beta_2=0.4$. If it is chosen $n=20$ and $n=100$, approximation to function f by $S_{n,p,\alpha,\beta}$ generalized Schurer-Stancu type operators has been shown in Figure 1. Furthermore, we give its algorithm after the figure.

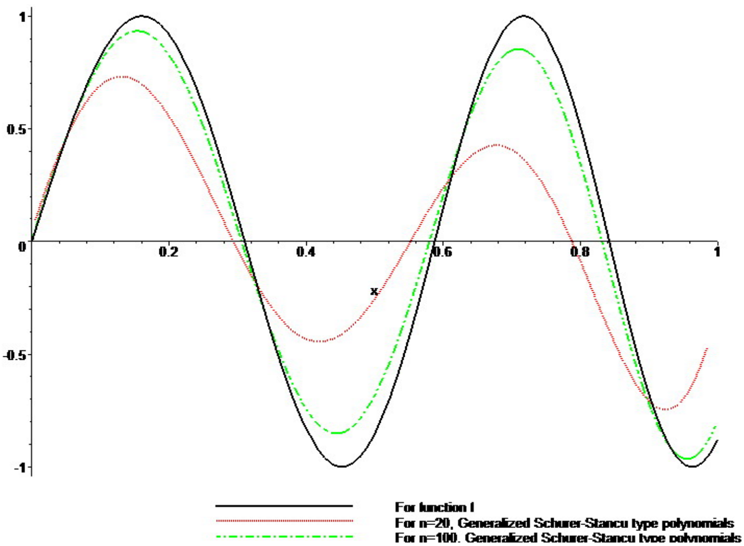

Figure 1: Approximation to $f(x) = \sin(2\pi(x+1)^{\frac{3}{2}})$ by generalized Schurer-Stancu type polynomials

```
with(plots):
> p:=1:
> alpha1:=0.2:
> alpha2:=0.1:
> beta1:=0.3:
> beta2:=0.4:
> f:=x-> sin(2*Pi*(x+1)^(3/2))
> S:=(n,p,alpha1,alpha2,beta1,beta2)-> ((n+beta2)/n)^(n+p)
*sum(f((k+alpha1)/(n+beta1))*binomial(n+p,k)
*(x-(alpha2/(n+beta2)))^k
*((n+alpha2)/(n+beta2)-x)^(n+p-k),k=0..n+p):
> p1:=plot(f(x),x=0..1,color=black):
> p2:=plot(evalf(S(20,p,alpha1,alpha2,beta1,beta2)),
x=(alpha2/(20+beta2))..(20+alpha2)/(20+beta2),
color=red, linestyle=2):
> p3:=plot(evalf(S(100,p,alpha1,alpha2,beta1,beta2)),
x=(alpha2/(100+beta2))..(100+alpha2)/(100+beta2),
color=green, linestyle=4):
> display([p1,p2,p3]);
```

Example 9. In the following Table 1 we calculate the error bound of function $f(x) = \sin(\pi x)$ with the help of modulus of continuity for $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 3$, $\beta_2 = 4$ and $p = 1$ by using Maple13.

n	Error estimate by $S_{n,p,\alpha,\beta}$ operators to the function $f(x) = \sin(\pi x)$
10	1.246331908
10^2	0.353109854
10^3	0.100809812
10^4	0.031463164
10^5	0.009936086
10^6	0.003141640
10^7	0.000993460
10^8	0.000314159
10^9	0.000099345
10^{10}	0.000031415

Table 1. The error bound of function $f(x) = \sin(\pi x)$.

References

[1] F. Altomare, M. Campiti. *Korovkin Type Approximation Theory and Its Application,* de Gruyter Studies in Mathematics, Vol. **17**, Walter de Gruyter and Co., Berlin, 1994.

[2] D. Bărbosu. Schurer-Stancu type operators. // *Studia Univ. Babes-Bolyai Math.*, **48**, No 3, (2003), 31–35.

[3] S. N. Bernstein. Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités. // *Commun. Soc. Math. Kharkow*, **13**, No 2, (1912), 1–2.

[4] A. D. Gadjiev, A. M. Ghorbanalizadeh, Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables. // *Appl. Math. Comput.* **216**, (2010), 890-901.

[5] F. Schurer. *Linear positive operators in approximation theory.* Math. Inst. Techn. Univ. Delft report, 1962.

[6] O. Shisha, B. Mond. The degree of convergence of linear positive operators. // *Proc. Nat. Acad. Sci. U.S.A.*, **60**, (1968), 1196–1200.

[7] D. D. Stancu. Asupraunei generalizări a polinoamelor lui Bernstein. // *Studia Univ. Babes-Bolyai Math. (Cluj)*, **14**, No 2 (1969), 31–45.

[1] *Department of Mathematics,*
Faculty of Science, Ankara University,
Tandoğan TR-06100,
Ankara, TURKEY
e-mail: ssucu@ankara.edu.tr

Received: May 7, 2012

[2] *Department of Statistic,*
Faculty of Science, Ankara University,
Tandoğan TR-06100,
Ankara, TURKEY
e-mail: yesim.done@ankara.edu.tr

[3] *Department of Mathematics,*
Faculty of Science, Ankara University,
Tandoğan TR-06100,
Ankara, TURKEY
e-mail: ibikli@ankara.edu.tr