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A CHARACTERIZATION THEOREM FOR THE *K*-FUNCTIONAL ASSOCIATED WITH THE ALGEBRAIC VERSION OF TRIGONOMETRIC JACKSON INTEGRALS

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Abstract. The purpose of this paper is to present a characterization of a certain Peetre K-functional in $L_p[-1,1]$ norm, for $1 \le p \le 2$ by means of a modulus of smoothness. This modulus is based on the classical one taken on a certain linear transform of the function.

1. Introduction.

1.1. Notations. Let X be a normed space. For a given "differential" operator D we set $X \cap D^{-1}(X) = \{g \in X : Dg \in X\}$. Let X be one of the spaces $L_p[-1,1], 1 \le p < \infty$ or $C[-1,1]$. In this case we denote the norm in X by $\|\cdot\|_p$, $1 \le p \le \infty$, where $\|\cdot\|_\infty$ means the uniform norm. Two examples of the operator D are

$$D_1 g := \varphi(\varphi g')', \qquad D_2 g := \varphi^2 g'', \qquad \text{where } \varphi(x) = \sqrt{1-x^2}.$$

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We define for every $f \in X$ and $t > 0$ the K-functionals

$$K(f,t;X,Y,D_1) := \inf\left\{\|f-g\|_p + t\left\|\varphi(\varphi g')'\right\|_p : g \in Y\right\}, \tag{1.1}$$

$$K(f,t;X,Y,D_2) := \inf\left\{\|f-g\|_p + t\left\|\varphi^2 g''\right\|_p : g \in Y\right\}, \tag{1.2}$$

where Y is a given subspace of $X \cap D_1^{-1}(X)$ or $X \cap D_2^{-1}(X)$, respectively.

The K-functional (1.1) with $X = C[-1,1]$, $Y = C^2[-1,1]$ is equivalent to the approximation error of Jackson type operator $G_{s,n}$ in uniform norm (such equivalence was established in [5]), while the K-functional (1.2) with $X = L_p[-1,1]$, $Y = C^2[-1,1]$ is well-known and is equivalent to the approximation error of Bernstain polynomials in the interval $[0,1]$ ($p = \infty$) and characterizes the best polynomial approximations ($1 \le p \le \infty$).

We recall that the operator $G_{s,n} : C[-1,1] \to \Pi_{sn-s}$ is defined by (see [4])

$$G_{s,n}(f,x) = \pi^{-1}\int_{-\pi}^{\pi} f(\cos(\arccos x + v))K_{s,n}(v)dv,$$

where

$$K_{s,n}(v) = c_{n,s}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s}, \qquad \pi^{-1}\int_{-\pi}^{\pi} K_{s,n}(v)dv = 1.$$

Π_r denotes the set of all algebraic polynomials of degree not exceeding r (r is natural number).

Notation $\Phi(f,t) \sim \Psi(f,t)$ means that there is a positive constant γ, independent of f and t, such that $\gamma^{-1}\Psi(f,t) \le \Phi(f,t) \le \gamma\Psi(f,t)$.

By c we denote positive constants, independent of f and t, that may differ at each occurrence.

For r – natural number we denote

$$C^r[a,b] = \left\{f : f, f', \dots, f^{(r)} \in C[a,b] \text{ (continuous function in } [a,b])\right\}$$

1.2. Known results. The idea for the equivalence of the approximation errors of a given sequence of operators and the values of proper K-functionals was studied systematically in [1]. Such equivalence was established for the algebraic version of trigonometric Jackson integrals $G_{s,n}$ and K-functionals (1.1) in uniform norm in [5] (see Theorem A).

Theorem A. *For $s \geqslant 2$ and every $f \in C[-1,1]$ we have*

$$\|f - G_{s,n}f\|_\infty \sim K\left(f, \frac{1}{n^2}; C[-1,1], C^2, D_1\right).$$

Using a linear transform of functions in [3] Ivanov compares the K-functional

$$K(f,t;X,Y,D_3) := \inf\left\{\|f-g\|_p + t\left\|(\psi g')'\right\|_p : g\in Y\right\}, \tag{1.3}$$

with the already characterized K-functional

$$K(f,t;X,Y,D_4) := \inf\left\{\|f-g\|_p + t\left\|\psi g''\right\|_p : g\in Y\right\}, \tag{1.4}$$

where $\psi(x)=x(1-x)$; X is one of the spaces $L_p[0,1]$, $1\le p<\infty$ or $C[0,1]$; Y is a given subspace of $X\cap D_3^{-1}(X)$ or $X\cap D_4^{-1}(X)$, respectively; $D_3g := (\psi g')', D_4g:=\psi g''$.

Ivanov proved the following

Theorem B. *For every $t\in(0,1]$ and $f\in L_1[0,1]$ we have*

$$K(f,t;L_1[0,1],C^2,D_3)\sim K(Bf,t;L_1[0,1],C^2,D_4)+tE_0(f)_1,$$

where
$$(Bf)(x)=f(x)+\int_{1/2}^{x}\left(\frac{x}{y^2}-\frac{1-x}{(1-y)^2}\right)f(y)dy$$

and $E_0(f)_1$ denotes the best approximation of f in $L_1[0,1]$ by constant.

1.3. New results. The aim of this paper is to define a modulus that is equivalent to the K-functional (1.1) for $1\le p\le 2$. We apply the method presented in [3].

First, let us note that the K-functionals $K(f,t;L_p[-1,1],C^2,D_1)$ and $K(f,t;L_p[-1,1],C^2,D_2)$ are not equivalent. The inequality $K(f,t;L_p[-1,1],C^2,D_2)\le cK(f,t;L_p[-1,1],C^2,D_1)$ is not true for a fixed c, every f, every $t\in(0,1]$ and $1\le p\le 2$ because of functions like (with small positive ε)

$$f_\varepsilon(x)=\begin{cases}\arcsin x, & x\in[-1+\varepsilon,1-\varepsilon];\\ ax^3+bx+d, & x\in[1-\varepsilon,1];\\ ax^3+bx-d, & x\in[-1,-1+\varepsilon];\end{cases}$$

where a, b, d are chosen such that $f\in C^2$.
But these K-functionals can become equivalent if in the one of them instead f stays Af for appropriate operator A.

Let $f\in L_1[-1,1]$. For every $-1<x<1$ we define the value of the operator A by

$$(Af)(x)=f(x)+\frac{1}{2}\int_{0}^{x}\left(\frac{1+x}{(1+y)^2}-\frac{1-x}{(1-y)^2}\right)f(y)dy. \tag{1.5}$$

Using operator (1.5) we prove

Theorem 1. *For every* $t \in (0,1]$ *and* $f \in L_p[-1,1]$, $1 \le p \le 2$, *we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],C^2,D_2).$$

We mention that in Theorem 1 there is no additional term $tE_0(f)_p$ in the equivalence relation, while in Theorem B there is. Moreover, the equivalence in Theorem B is valid only for $p=1$, while Theorem 1 holds for $1 \le p \le 2$. Although the operators D_1 and D_3 are similar we cannot reduce one to another. We can write the operator $D_1 g(x)$ of the form:

$$(D_1 g)(x) = (1-x^2)g''(x) - xg'(x).$$

On the other hand, the analogue of D_3 for the interval $[-1,1]$ is

$$\widetilde{D}_3 G(y) = (1-y^2)G''(y) - 2yG'(y),$$

i.e. $\widetilde{D}_3 G$ differs from $D_1 G$ by constant multiplier 2 in the term containing G'.

From Theorem 1 and characterizations of some weighted Peetre K-functionals in terms of weighted moduli established in [2, Ch. 2, Theorem 2.1.1] we get

Corollary 1. *For* $f \in L_p[-1,1]$, $t \in (0,1]$ *and* $1 \le p \le 2$ *with* $\phi = \sqrt{1-x^2}$ *we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim \omega_\phi^2(Af,\sqrt{t})_p,$$

where ω_ϕ^2 *is Ditzian-Totik modulus of smoothness, introduced in* [2].

The equivalence in Theorem 1 is no longer true for $2 < p < \infty$ as the following example shows. Let $F(x) = \arcsin x$. We have $E_0(F)_p \sim 1$ and thus $ct \le K(F,t;L_p[-1,1],C^2,D_1)$ for $2 < p < \infty$ (see Lemma 4). On the other hand $K(AF,t;L_p[-1,1],C^2,D_2) = 0$ for every p because $AF(x) = x$, i. e. $AF \in C^2[-1,1]$ and $D_2(AF) = \varphi^2(x)(AF)'' = 0$.

The connection between the K-functionals of f and Af with D_1 and D_2 as differential operators respectively, is not so satisfactory when $2 < p < \infty$. We have

Theorem 2. *For every* $t \in (0,1]$ *and* $f \in L_p[-1,1]$, $2 < p < \infty$, *we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \le c\left[K(Af,t^{\frac{1}{p}+\frac{1}{2}};L_p[-1,1],C^2,D_2) + t^{\frac{1}{p}+\frac{1}{2}}E_0(f)_p\right],$$

$$K(Af,t;L_p[-1,1],C^2,D_2) + tE_0(f)_p \le cK(f,t;L_p[-1,1],C^2,D_1).$$

The proof of Theorem 1 follows the scheme. First we establish in Lemma 2 the equivalence

$$K(f,t;L_p[-1,1],Z_1,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2) \text{ for } 1 \le p < \infty,$$

where Z_1 and Z_2 are suitable subspaces of C^2 (see Definition 2). On the other hand these variations of Y produce K-functionals equal to the K-functionals we compare in Theorem 1. In Lemma 1 and Lemma 3 b) respectively we prove that

$$K(f,t;L_p[-1,1],Z_1,D_1) = K(f,t;L_p[-1,1],C^2,D_1) \text{ for } 1 \le p < \infty \text{ and}$$

$$K(F,t;L_p[-1,1],Z_2,D_2) = K(F,t;L_p[-1,1],C^2,D_2) \text{ for } 1 \le p \le 2.$$

The last two relation we obtain using Lemma 2 from [3, p.116]. We state this lemma, as we use it several times.

Definition 1. *For given $Y \subset X \cap D^{-1}(X)$ and a positive number γ we define $S_\gamma(Y)$ as the set of all $g \in X \cap D^{-1}(X)$ such that for every $\varepsilon > 0$ there is $h \in Y$ such that $\|g-h\| < \varepsilon$ and $\|Dh\| < \gamma\|Dg\| + \varepsilon$.*

Lemma B. *Let $Y_1, Y_2 \subset X \cap D^{-1}(X)$ and $\rho > 0$. Then for a given positive γ the following statements are equivalent:*

i) $K(f,t;X,Y_1,D) \le K(f,\gamma t;X,Y_2,D)$ *for every* $f \in X$, $0 < t$.

ii) $K(f,t;X,Y_1,D) \le K(f,\gamma t;X,Y_2,D)$ *for every* $f \in X$, $0 < t \le \rho$.

iii) $Y_2 \subset S_\gamma(Y_1)$.

In particular, i) *with $\gamma = 1$ holds when $Y_2 \subset Y_1$.*

Theorems 1 and 2 are proved in Section 3.

2. Properties of the operators.

In the next statement we collect some properties of operator A.

Theorem 3. a) *A is a linear operator, satisfying $\|Af\|_p \le 2\|f\|_p$ for every $1 \le p \le \infty$.*

b) *$Af = f$ for every $f \in \Pi_0$.*

c) *If $f, f' \in AC_{loc}(-1,1)$, then $(Af)(0) = f(0)$, $(Af)'(0) = f'(0)$ and*

$$\varphi^2(x)(Af)''(x) = \varphi(x)(\varphi(x)f'(x))', \qquad -1 < x < 1.$$

Proof. We write (1.5) as

$$(Af)(x) = f(x) + \frac{1}{2}\int_{-1}^{1} R(x,y)f(y)dy,$$

where the kernel $R:(-1,1)\times(-1,1)\to\mathbb{R}$ is given as follows: $R(0,y)=0$; for $x\in(-1,0)$ we have $R(x,y)=(1-x)(1-y)^{-2}-(1+x)(1+y)^{-2}$ if $y\in(x,0)$ and $R(x,y)=0$ if $y\notin(x,0)$; for $x\in(0,1)$ we have $R(x,y)=(1+x)(1+y)^{-2}-(1-x)(1-y)^{-2}$ if $y\in(0,x)$ and $R(x,y)=0$ if $y\notin(0,x)$.

Set $x_0=\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$. Then for a fixed $x\in(-1,0)$ the kernel $R(x,y)$ is negative for $y\in(x,x_0)$ and positive for $y\in(x_0,0)$; for a fixed $x\in(0,1)$ the kernel $R(x,y)$ is positive for $y\in(0,x_0)$ and negative for $y\in(x_0,x)$. Thus

$$\int_{-1}^{1}|R(x,y)|\,dy=2-2\sqrt{1-x^2}\le 2.$$

Hence $\|Af\|_\infty\le 2\,\|f\|_\infty$.

Set $y_0=\dfrac{(1+y)^2-(1-y)^2}{(1+y)^2+(1-y)^2}$. Then for a fixed $y\in(-1,0)$ the kernel $R(x,y)$ is positive for $x\in(-1,y_0)$ and negative for $x\in(y_0,y)$; for a fixed $y\in(0,1)$ the kernel $R(x,y)$ is negative for $x\in(y,y_0)$ and positive for $x\in(y_0,1)$. Thus for $y\neq 0$ we have

$$\int_{-1}^{1}|R(x,y)|\,dx=1-\frac{4}{(1+y)^2+(1-y)^2}+\frac{2}{\max\{(1+y)^2,(1-y^2)\}}\le 1.$$

Hence $\|Af\|_1\le\dfrac{3}{2}\,\|f\|_1$. Now the Riesz-Thorin theorem proves a).

Part b) follows from $\displaystyle\int_{-1}^{1}R(x,y)dy=0$.

Part c) follows from (1.5) by direct computation. □

The operator A is invertible and we give an explicit formula for its inverse operator A^{-1} . Let every $f\in L_1[-1,1]$ and $-1<x<1$ we set

$$(A^{-1}f)(x)=f(x)+\int_{0}^{x}\left(\frac{y}{1-y^2}+\frac{\arcsin y-\arcsin x}{(1-y^2)^{\frac{3}{2}}}\right)f(y)dy.$$

In the next statement we collect some properties of A^{-1}.

Theorem 4. a) *A^{-1} is a linear operator, $\left\|A^{-1}f\right\|_p\le c\,\|f\|_p$ for every $1\le p<\infty$.*

b) *$A^{-1}f=f$ for every $f\in\Pi_0$*

c) *$A^{-1}Af = AA^{-1}f = f$ for every $f \in L_1[-1,1]$.*

d) *If $f,\ f' \in AC_{loc}(-1,1)$, then $(A^{-1}f)(0) = f(0)$, $(A^{-1}f)'(0) = f'(0)$ and*

$$\varphi(x)(\varphi(x)(A^{-1}f)'(x))' = \varphi^2(x)f''(x), \qquad -1 < x < 1.$$

P r o o f. a) We have

$$|\arcsin 1 - \arcsin z| \leq c\sqrt{1-z^2} \quad \text{for} \quad z \in [0,1]$$

and

$$|\arcsin(-1) - \arcsin z| \leq c\sqrt{1-z^2} \quad \text{for} \quad z \in [-1,0].$$

As $0 < y < x$ and $\arcsin z$ is increasing $|\arcsin y - \arcsin 1| > |\arcsin y - \arcsin x|$. Now we estimate

$$\int_0^1 \left| \int_0^x \frac{1}{(1-y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) f(y) dy \right|^p dx$$

$$\leq \int_0^1 \left(\int_0^x \frac{1}{(1-y^2)^{\frac{3}{2}}} \left|(\arcsin y - \arcsin 1) f(y)\right| dy \right)^p dx$$

$$\leq c \int_0^1 \left(\int_0^x \frac{\sqrt{1-y^2}}{(1-y^2)^{\frac{3}{2}}} |f(y)|\, dy \right)^p dx = c \int_0^1 \left(\int_0^x \frac{|f(y)|}{1-y^2} dy \right)^p dx$$

$$\leq c \int_0^1 \left(\int_0^1 \frac{|f(y)|}{1-y} dy \right)^p dx \quad (y \to 1-y)$$

$$= c \int_0^1 \left(\int_x^1 \frac{|f(1-y)|}{y} dy \right)^p dx \text{ (Hardy inequality)}$$

$$\leq c \int_0^1 \left(y \frac{|f(1-y)|}{y} \right)^p dy = c \int_0^1 |f(1-y)|^p\, dy = c\, \|f\|_{p[0,1]}^p .$$

Similarly, using Hardy inequality we get

$$\int_0^1 \left| \int_0^x \frac{y}{1-y^2} f(y)dy \right|^p dx \le c \int_0^1 \left(\int_0^x \frac{|f(y)|}{1-y} dy \right)^p dx \le c \, \|f\|^p_{p[0,1]} ,$$

$$\int_{-1}^0 \left| \int_0^x \frac{1}{(1-y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) f(y)dy \right|^p dx \le c \, \|f\|^p_{p[-1,0]} \quad \text{and}$$

$$\int_{-1}^0 \left| \int_0^x \frac{y}{1-y^2} f(y)dy \right|^p dx \le c \, \|f\|^p_{p[-1,0]} .$$

From these inequalities we get

$$\left\{ \int_{-1}^1 \left| \int_0^x \left(\frac{y}{1-y^2} + \frac{\arcsin y - \arcsin x}{(1-y^2)^{\frac{3}{2}}} \right) f(y)dy \right|^p dx \right\}^{\frac{1}{p}} \le c \, \|f\|_{p[-1,1]} .$$

This proves a).

Part b) follows from

$$\int_0^x \left(\frac{y}{1-y^2} + \frac{1}{(1-y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) \right) dy = 0.$$

Finally, c) and d) can be obtained by direct computation. □

The action of the operators A and A^{-1} on the function $f(x) = x$ is given bellow:

$$\begin{aligned} (A(.))(x) &= \frac{1}{2}(1+x)\ln(1+x) - \frac{1}{2}(1-x)\ln(1-x), \\ (A^{-1}(.))(x) &= \arcsin x. \end{aligned}$$

Definition 2. *Set* $Z_1 = \{ f \in C^2[-1,1] : f'(-1) = 0, f'(1) = 0 \}$,

$$Z_2 = \left\{ f \in C^2[-1,1] : \int_{-1}^0 \frac{x}{\sqrt{1-x^2}} f'(x)dx = 0, \; \int_0^1 \frac{x}{\sqrt{1-x^2}} f'(x)dx = 0 \right\}.$$

Theorem 5. a) $(Af)''(x)$ *is continuous at* $x = -1$ *and at* $x = 1$ *for every* $f \in Z_1$.

b) $\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}(Af)'(x)dx = \int_{0}^{1} \frac{x}{\sqrt{1-x^2}}(Af)'(x)dx = 0$ *for every* $f \in Z_1$.

c) $(A^{-1}f)''(x)$ *is continuous at* $x = -1$ *and at* $x = 1$ *for every* $f \in Z_2$.

d) $(A^{-1}f)'(-1) = (A^{-1}f)'(1) = 0$ *for every function* $f \in Z_2$.

e) $A(Z_1) = Z_2$ *and* $A^{-1}(Z_2) = Z_1$.

Proof. For every function $f \in Z_1$ we have that $f'(x) = (x-1)f''(1) + o(1-x)$ and $f'(x) = (x+1)f''(-1) + o(1+x)$. From Theorem 3 c) we have

$$(Af)''(x) = f''(x) - \frac{x}{1-x^2}f'(x),$$

which together with the above representations gives $(Af)''(x) = \frac{3}{2}f''(1) + o(1)$ for x close to 1 and $(Af)''(x) = \frac{3}{2}f''(-1) + o(1)$ for x close to -1. This proves a).

We write the derivative of Af as

$$\begin{aligned}(Af)'(x) &= f'(x) - \frac{x}{1-x^2}f(x) + \int_0^x \frac{1+y^2}{(1-y^2)^2}f(y)dy \\ &= f'(x) - \frac{x}{1-x^2}f(x) + \int_0^x f(y)d\frac{y}{1-y^2} = f'(x) - \int_0^x \frac{y}{1-y^2}f'(y)dy.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_0^1 \frac{x}{\sqrt{1-x^2}}(Af)'(x)dx \\ &= \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}}(Af)'(x)dx \\ &= \lim_{\varepsilon\to 0+} \left(\int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}}f'(x)dx - \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} \int_0^x \frac{y}{1-y^2}f'(y)dydx \right).\end{aligned}$$

We consider the last term.

$$\begin{aligned}\int_0^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}\int_0^x\frac{y}{1-y^2}f'(y)dydx &= \int_0^{1-\varepsilon}\frac{y}{1-y^2}f'(y)\left(\int_y^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}dx\right)dy\\ &= \int_0^{1-\varepsilon}\frac{y}{1-y^2}(\sqrt{1-y^2}-\sqrt{2\varepsilon-\varepsilon^2})f'(y)dy.\end{aligned}$$

Then

$$\int_0^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}(Af)'(x)dx = \sqrt{2\varepsilon-\varepsilon^2}\int_0^{1-\varepsilon}\frac{y}{1-y^2}f'(y)dy.$$

For every $f\in Z_1$, $\left|\int_0^1\frac{y}{1-y^2}f'(y)dy\right|\le c$ and hence eventually

$$\lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}(Af)'(x)dx = \int_0^1\frac{x}{\sqrt{1-x^2}}(Af)'(x)dx = 0.$$

Similar arguments prove the other claim of b).

We have

$$\begin{aligned}(A^{-1}f)'(x) &= f'(x)-\tfrac{1}{\sqrt{1-x^2}}\int_0^x\tfrac{f(y)}{(1-y^2)^{\frac{3}{2}}}dy+\tfrac{x}{1-x^2}f(x)\\ &= f'(x)+\tfrac{1}{\sqrt{1-x^2}}\int_0^x\tfrac{y}{\sqrt{1-y^2}}f'(y)dy.\end{aligned}\tag{2.1}$$

$$\begin{aligned}(A^{-1}f)''(x) &= f''(x)+\tfrac{x}{1-x^2}f'(x)+\tfrac{x}{(1-x^2)^{\frac{3}{2}}}\int_0^x\tfrac{y}{\sqrt{1-y^2}}f'(y)dy\\ &= f''(x)+\tfrac{x}{1-x^2}(A^{-1}f)'(x).\end{aligned}\tag{2.2}$$

For every $f\in Z_2$ and $x\in[0,1]$ we rewrite (2.1) as

$$(A^{-1}f)'(x) = f'(x)-\frac{1}{\sqrt{1-x^2}}\int_x^1\frac{y}{\sqrt{1-y^2}}f'(y)dy.$$

Using Taylor's expansion of f' around 1 we get from the above

$$(A^{-1}f)'(x) = f'(x)-\frac{1}{\sqrt{1-x^2}}\int_x^1\frac{y}{\sqrt{1-y^2}}(f'(1)+(y-1)f''(1)+o(1-y))dy.$$

Now we compute the last integral.

$$\int_x^1 \frac{y(1-y)}{\sqrt{1-y^2}}dy = \frac{1}{2}(-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2}).$$

As

$$\lim_{x\to 1-0} \frac{-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{3}$$

we can write

$$-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2} = \frac{1}{3}(1-x^2)^{\frac{3}{2}} + o((1-x^2)^{\frac{3}{2}}).$$

Above computations and Taylor's expantion of f' around 1 imply

$$(A^{-1}f)'(x) = -\frac{2}{3}(1-x)f''(1) + o(1-x). \tag{2.3}$$

Equations (2.2) and (2.3) give $(A^{-1}f)''(x) = \frac{2}{3}f''(1) + o(1)$ for x close to 1. In a similar way we get

$$(A^{-1}f)'(x) = \frac{2}{3}(1+x)f''(-1) + o(1+x). \tag{2.4}$$

Hence $(A^{-1}f)''(x) = \frac{2}{3}f''(-1) + o(1)$ for x close to -1, which proves c).

Part d) follows from (2.3) and (2.4).

For every $f \in Z_1$ from a) we get $(Af)'$, $Af \in AC[-1,1]$ and hence $Af \in C^2$. Now using b) we get $Af \in Z_2$, i.e., $A(Z_1) \subset Z_2$. Similarly, from c) and d) we get $A^{-1}(Z_2) \subset Z_1$. Using Theorem 4 c) we get $Z_1 = A^{-1}(A(Z_1)) \subset A^{-1}(Z_2)$ and $Z_2 = A(A^{-1}(Z_2)) \subset A(Z_1)$. Hence $A^{-1}(Z_2) = Z_1$ and $A(Z_1) = Z_2$. $\square$

3. Proofs of the Theorems.

Lemma 1. a) *For every $t > 0$ and $f \in L_p[-1,1]$, $1 \le p < \infty$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) = K(f,t;L_p[-1,1],Z_1,D_1).$$

b) *For every $t > 0$ and $f \in C[-1,1]$ we have*

$$K(f,t;C[-1,1],C^2,D_1) \sim K(f,t;C[-1,1],Z_1,D_1)$$

Proof. Let $\mu \in C^\infty(\mathbb{R})$ be such that $\mu(x) = 1$ for $x \le 0$, $\mu(x) = 0$ for $x \ge 1$ and $0 < \mu(x) < 1$ for $0 < x < 1$. For given $\delta \in \left(0, \frac{1}{2}\right)$ we set $\mu_{-1}(x) = \mu\left(\frac{1+x}{\delta}\right)$ and $\mu_1(x) = \mu\left(\frac{1-x}{\delta}\right)$ for every $x \in [-1,1]$. Thus,

$\operatorname{supp} \mu_{-1}(x) = [-1, -1+\delta]$, $\operatorname{supp} \mu_1(x) = [1-\delta, 1]$ and $\left\| \mu_j^{(k)} \right\|_\infty = O(\delta^{-k})$ for $j = -1, 1$ and $k = 1, 2$.

Let $g \in C^2[-1,1]$. For $x \in [-1,1]$ set

$$G(x) = [1 - \mu_{-1}(x) - \mu_1(x)]g(x) + \mu_{-1}(x)g(-1) + \mu_1(x)g(1) \tag{3.1}$$

Then $G \in Z_1$. From $G(x) - g(x) = \mu_{-1}(x)[g(-1) - g(x)] + \mu_1(x)[g(1) - g(x)]$ we get $\|G - g\|_p \le 2^{1/p} \|G - g\|_\infty \le 2^{1/p} \omega_1(g, \delta)_\infty = O(\delta)$.

From (3.1) we obtain

$$\begin{aligned} (D_1 G)(x) = {} & [1 - \mu_{-1}(x) - \mu_1(x)](D_1 g)(x) \\ & - 2\varphi^2(x)[\mu'_{-1}(x) - \mu'_1(x)]g'(x) \\ & + (D_1\mu_1)(x)[g(1) - g(x)] + (D_1\mu_{-1})(x)[g(-1) - g(x)]. \end{aligned} \tag{3.2}$$

From (3.2) for $1 \le p < \infty$ we get $\|D_1 G\|_p \le \|D_1 g\|_p + O(\delta^{1/p})$, which proves part a) in view of Lemma 2 in [3, p. 116].

For $p = \infty$ (3.2) implies

$$\|D_1 G\|_\infty \le \|D_1 g\|_\infty + c\big[|g'(-1)| + |g'(1)|\big] + O(\delta) \le c\,\|D_1 g\|_\infty + O(\delta),$$

because of $|g'(-1)| = |(D_1 g)(-1)|$ and $|g'(1)| = |(D_1 g)(1)|$. Applying Lemma 2 again in [3, p. 116] we prove part b). □

Lemma 2. *For every $t > 0$ and $f \in L_p[-1,1]$, $1 \le p < \infty$, we have*

$$K(f, t; L_p[-1,1], Z_1, D_1) \sim K(Af, t; L_p[-1,1], Z_2, D_2).$$

Proof. For a given $g \in Z_1$ we set $G = Ag \in Z_2$ (see Theorem 5 e)). Then Theorem 4 c), a) and d) implies $\|f - g\|_p = \left\| A^{-1}(Af - Ag) \right\|_p \le c\,\|Af - G\|_p$ and $\|D_1 g\|_p = \left\| D_1 A^{-1} G \right\|_p = \|D_2 G\|_p$. Hence,

$$\|f - g\|_p + t\,\|D_1 g\|_p \le c(\|Af - G\|_p + t\,\|D_2 G\|_p),$$

which gives $K(f, t; L_p, Z_1, D_1) \le cK(Af, t; L_p, Z_2, D_2)$.

For a given $G \in Z_2$ we set $g = A^{-1}G \in Z_1$ (see Theorem 5 e)). Using Theorem 3 a), c) and Theorem 4 c), we get

$$\|Af - G\|_p = \left\| A(f - A^{-1}G) \right\|_p \le 2\,\|f - g\|_p\,, \quad \|D_2 G\|_p = \|D_2 Ag\|_p = \|D_1 g\|_p\,.$$

Hence,

$$\|Af - G\|_p + t\,\|D_2 G\|_p \le 2(\|f - g\|_p + t\,\|D_1 g\|_p),$$

which gives $K(Af, t; L_p[-1,1], Z_2, D_2) \le 2K(f, t; L_p[-1,1], Z_1, D_1)$. □

From Lemmas 1, 2 we obtain

Corollary 2. *For every $t > 0$ and $f \in L_p[-1,1]$, $1 \le p < \infty$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2).$$

Lemma 3. a) *For every $t \in (0,1]$ and $F \in L_p[-1,1]$, $2 < p < \infty$, we have*

$$K(F,t;L_p[-1,1],Z_2,D_2) \le c\left[K(F,t^{\frac{1}{p}+\frac{1}{2}};L_p[-1,1],C^2,D_2) + t^{\frac{1}{p}+\frac{1}{2}}E_0(F)_p\right].$$

b) *For every $t \in (0,1]$ and $F \in L_p[-1,1]$, $1 \le p \le 2$, we have*

$$K(F,t;L_p[-1,1],Z_2,D_2) = K(F,t;L_p[-1,1],C^2,D_2).$$

Proof. For $\delta \in (0,1/2)$ we set $\mu(x) = (1-x\delta^{-1})^3_+$, where $(y)_+ = y$ if $y \ge 0$ and $(y)_+ = 0$ if $y \le 0$. For $g \in C^2[-1,1]$ we set

$$G(x) = g(x) + \alpha\mu(x+1) + \beta\mu(1-x) \tag{3.3}$$

where

$$\alpha = \frac{\delta}{3}\frac{\int\limits_{-1}^{0}\frac{y}{\sqrt{1-y^2}}g'(y)dy}{\int\limits_{-1}^{0}\frac{y}{\sqrt{1-y^2}}\left(1-\frac{y+1}{\delta}\right)^2_+dy}, \qquad \beta = -\frac{\delta}{3}\frac{\int\limits_{0}^{1}\frac{y}{\sqrt{1-y^2}}g'(y)dy}{\int\limits_{0}^{1}\frac{y}{\sqrt{1-y^2}}\left(1-\frac{1-y}{\delta}\right)^2_+dy}.$$

Parameters α, β and δ are chosen in such way that $G \in Z_2$. From (3.3) we get $\|G-g\|_p \le c\delta^{1/p}(|\alpha|+|\beta|)$ and

$$G''(x) = g''(x) + 6\delta^{-2}\left[\alpha\left(1-\frac{x+1}{\delta}\right)_+ + \beta\left(1-\frac{1-x}{\delta}\right)_+\right].$$

Hence $\left\|\varphi^2 G''\right\|_p \le \left\|\varphi^2 g''\right\|_p + c\delta^{-1+1/p}(|\alpha|+|\beta|)$, and

$$\begin{aligned} K(F,t;L_p,Z_2,D_2) &\le \|F-G\|_p + t\|D_2G\|_p \\ &\le \|F-g\|_p + t\|D_2 g\|_p + c\delta^{1/p}(1+t\delta^{-1})(|\alpha|+|\beta|). \end{aligned}$$

In order to estimate $|\alpha|+|\beta|$ we calculate

$$\begin{aligned}
\int_0^1 \frac{y}{\sqrt{1-y^2}}\left(1-\frac{1-y}{\delta}\right)_+^2 dy &= \frac{1}{\delta^2}\int_{1-\delta}^1 \frac{y}{\sqrt{1-y^2}}(\delta-1+y)^2 dy \\
&\geq \frac{c}{\delta^2}\int_{1-\delta}^1 \frac{1}{\sqrt{1-y}}(\delta-(1-y))^2 dy \\
&= \frac{c}{\delta^2}\int_0^\delta \frac{(\delta-t)^2}{\sqrt{t}}dt = c\delta^{1/2}.
\end{aligned}$$

In a similar way we get

$$\left|\int_{-1}^0 \frac{y}{\sqrt{1-y^2}}(1-\frac{y+1}{\delta})_+^2 dy\right| \geq c\delta^{1/2}.$$

Then

$$|\alpha| \leq c\sqrt{\delta}\left|\int_{-1}^0 \frac{y}{\sqrt{1-y^2}}g'(y)dy\right|, \qquad |\beta| \leq c\sqrt{\delta}\left|\int_0^1 \frac{y}{\sqrt{1-y^2}}g'(y)dy\right|.$$

Using Hölder inequality we estimate

$$\begin{aligned}
|\alpha| &\leq c\sqrt{\delta}\left|\int_{-1}^0 \frac{y}{\sqrt{1-y^2}}g'(y)dy\right| \\
&\leq c\sqrt{\delta}\left\{\int_{-1}^0 \left|\frac{y}{\sqrt{1-y^2}}\right|^q dy\right\}^{1/q}\left\{\int_{-1}^0 |g'(y)|^p dy\right\}^{1/p} \\
&\leq c\sqrt{\delta}\left\{\int_{-1}^0 |g'(y)|^p dy\right\}^{1/p} \\
&= c\sqrt{\delta}\,\|g'\|_{p[-1,0]} \text{ for } p>2, \frac{1}{p}+\frac{1}{q}=1. \text{ Similarly} \\
|\beta| &\leq c\sqrt{\delta}\,\|g'\|_{p[0,1]} \text{ for } p>2. \text{ Hence } |\alpha|+|\beta| \leq c\sqrt{\delta}\,\|g'\|_p \text{ for } p>2.
\end{aligned}$$

In order to estimate the norm of g' we apply the inequality

$$\|g'\|_p \leq c\left(\|D_2 g\|_p + E_0(g)_p\right),$$

which, for instance, follows from [2, p. 135, assertion (a)]. Then we get

$$|\alpha| + |\beta| \le c\sqrt{\delta}\left(\|D_2 g\|_p + E_0(g)_p\right) \le c\sqrt{\delta}\left(\|D_2 g\|_p + \|F-g\|_p + E_0(F)_p\right).$$

Now we take $\delta = t/2$. Thus

$$K(F,t;L_p,Z_2,D_2) \le c\left(\|F-g\|_p + t^{1/p+1/2}\|D_2 g\|_p + t^{1/p+1/2}E_0(F)_p\right)$$

for every $g \in C^2[-1,1]$, which proves part a).

In order to prove part b) it is sufficient to show (see Lemma 2 in [3, p. 116]) that for every $g \in C^2[-1,1]$ and every $\varepsilon > 0$ there exists $G \in Z_2$ such that $\|G-g\|_p < \varepsilon$ and $\left\|\varphi^2 G''\right\|_p < \left\|\varphi^2 g''\right\|_p + \varepsilon$. For $1 \le p < 2$ we can define G by (3.3). We have

$$\begin{aligned}\|G-g\|_p &\le c\delta^{1/p}(|\alpha|+|\beta|)\\ &\le c\delta^{1/p+1/2}\left(\left|\int_{-1}^{0}\frac{y}{\sqrt{1-y^2}}g'(y)dy\right| + \left|\int_{0}^{1}\frac{y}{\sqrt{1-y^2}}g'(y)dy\right|\right) \overset{\delta\to 0}{\to} 0.\end{aligned}$$

$$\begin{aligned}\left\|\varphi^2 G''\right\|_p &\le \left\|\varphi^2 g''\right\|_p + c\delta^{-1+1/p}(|\alpha|+|\beta|)\\ &\le \left\|\varphi^2 g''\right\|_p + c\delta^{1/p-1/2}\left(\left|\int_{-1}^{0}\frac{y}{\sqrt{1-y^2}}g'(y)dy\right| + \left|\int_{0}^{1}\frac{y}{\sqrt{1-y^2}}g'(y)dy\right|\right).\end{aligned}$$

When $\frac{1}{p} - \frac{1}{2} > 0$ the last term tends to zero as $\delta \to 0$, which proves part b) in case $1 \le p < 2$.

The case $p = 2$ needs special consideration and different definition of G.

Let $\delta \in \left(0, \frac{1}{2}\right)$. We set

$$\psi_\delta''(x) = \begin{cases} 0 & \text{for } x \in [-1,0];\\ \dfrac{x}{(1-x^2)^{3/2}} & \text{for } x \in (0,1-\delta];\\ \dfrac{1-\delta}{(2\delta-\delta^2)^{3/2}} & \text{for } x \in (1-\delta,1].\end{cases}$$

By integration we have

$$\psi_\delta'(x) = \begin{cases} 0 & \text{for } x \in [-1,0];\\ \dfrac{1}{\sqrt{1-x^2}} - 1 & \text{for } x \in (0,1-\delta];\\ \dfrac{1}{\sqrt{2\delta-\delta^2}} - 1 + \dfrac{1-\delta}{(2\delta-\delta^2)^{3/2}}[x-(1-\delta)] & \text{for } x \in (1-\delta,1].\end{cases}$$

$$\psi_\delta(x) = \begin{cases} 0 & \text{for } x \in [-1,0]; \\ \arcsin x - x & \text{for } x \in (0, 1-\delta]; \\ \arcsin(1-\delta) - (1-\delta) & \\ + \left(\frac{1}{\sqrt{2\delta-\delta^2}} - 1\right)[x-(1-\delta)] & \text{for } x \in (1-\delta, 1]; \\ + \frac{1-\delta}{2(2\delta-\delta^2)^{3/2}}[x-(1-\delta)]^2 & \end{cases}$$

$\psi''_\delta(x)$, $\psi'_\delta(x)$ and $\psi_\delta(x)$ are continuous and increasing functions.

We set now $\mu(x) = \psi_\delta(x)$. For $g \in C^2[-1,1]$ we set

$$G(x) = g(x) + \alpha\mu(x) + \beta\mu(-x). \tag{3.4}$$

Parameters α and β are chosen in such way that $G \in Z_2$:

$$0 = \int_0^1 \frac{xG'(x)}{\sqrt{1-x^2}}dx = \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}}dx + \alpha \int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}}dx. \quad \text{Hence}$$

$$\alpha = -\frac{\int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}}\,dx}{\int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}}\,dx}. \text{ Similarly } \beta = -\frac{\int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}}\,dx}{-\int_{-1}^0 \frac{x\psi'_\delta(-x)}{\sqrt{1-x^2}}\,dx} = -\frac{\int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}}\,dx}{\int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}}\,dx}.$$

From (3.4) we get

$$\|G-g\|_2 \le (|\alpha|+|\beta|)\|\psi_\delta\|_2 \text{ and}$$
$$\left\|\varphi^2 G''\right\|_2 \le \left\|\varphi^2 g''\right\|_2 + (|\alpha|+|\beta|)\left\|\varphi^2\psi''_\delta\right\|_2.$$

In order to estimate the last expressions we use some properties of ψ_δ given in the following

Assertion 1. *Let* $\delta \in \left(0, \frac{1}{2}\right)$. *Then we have*

a) $\displaystyle\int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}}dx \sim \ln\frac{1}{\delta}$.

b) $\left\|\varphi^2\psi''_\delta\right\|_2 \sim \sqrt{\ln\frac{1}{\delta}}$.

c) $\|\psi_\delta\|_2 \sim 1$.

Using Assertion 1 we obtain

$$\begin{aligned}
\|G-g\|_2 &\le (|\alpha|+|\beta|)\,\|\psi_\delta\|_2 = \frac{\left|\int\limits_{-1}^{0}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right| + \left|\int\limits_{0}^{1}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right|}{\int\limits_{0}^{1}\frac{x\psi_\delta'(x)}{\sqrt{1-x^2}}dx}\|\psi_\delta\|_2 \\
&\le \frac{c}{\ln\frac{1}{\delta}}\left(\left|\int\limits_{-1}^{0}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right| + \left|\int\limits_{0}^{1}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right|\right).
\end{aligned}$$

$$\begin{aligned}
\left\|\varphi^2G''\right\|_2 &\le \left\|\varphi^2g''\right\|_2 + (|\alpha|+|\beta|)\left\|\varphi^2\psi_\delta''\right\|_2 \\
&= \left\|\varphi^2g''\right\|_2 + \left(\left|\int\limits_{-1}^{0}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right| + \left|\int\limits_{0}^{1}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right|\right)\frac{\left\|\varphi^2\psi_\delta''\right\|_2}{\int\limits_{0}^{1}\frac{x\psi_\delta'(x)}{\sqrt{1-x^2}}dx} \\
&\le \left\|\varphi^2g''\right\|_2 + \frac{c}{\sqrt{\ln\frac{1}{\delta}}}\left(\left|\int\limits_{-1}^{0}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right| + \left|\int\limits_{0}^{1}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right|\right).
\end{aligned}$$

Let $g \in C^2[-1,1]$, $\varepsilon > 0$ is a small number. For a given function g and $\varepsilon > 0$ we may choose $\delta > 0$ such that

$$\frac{c}{\sqrt{\ln\frac{1}{\delta}}}\left(\left|\int\limits_{-1}^{0}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right| + \left|\int\limits_{0}^{1}\frac{xg'(x)}{\sqrt{1-x^2}}dx\right|\right) \le \varepsilon,$$

which proves b) in case $p = 2$ in view of [3, Lemma 2, p.116]). □

Lemma 4. *For every $t \in (0,1]$ and $f \in L_p[-1,1]$, $2 < p < \infty$, we have*

$$tE_0(f)_p \le cK(f,t;L_p[-1,1],C^2,D_1).$$

P r o o f. For every $g \in C^2[-1,1]$ we have

$$|g(x)-g(0)| \le |\arcsin x|\left\|\varphi g'\right\|_\infty.$$

Hence $\|g-g(0)\|_p \le c\,\|\varphi g'\|_\infty$. Using that $\varphi(1)g'(1) = \varphi(-1)g'(-1) = 0$ and

Hölder inequality we get for every $x \in [-1,1]$

$$\begin{aligned}
\left|\sqrt{1-x^2}g'(x)\right| &= \left|\int_{-1}^{x}\left(\sqrt{1-t^2}g'(t)\right)'dt\right| \\
&= \left|\int_{-1}^{x}\frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}\left(\sqrt{1-t^2}g'(t)\right)'dt\right| \\
&\le \left\{\int_{-1}^{x}\left(\frac{1}{\sqrt{1-t^2}}\right)^q dt\right\}^{1/q}\left\{\int_{-1}^{x}\left|\sqrt{1-t^2}\left(\sqrt{1-t^2}g'(t)\right)'\right|^p dt\right\}^{1/p} \\
&\le c\left\|D_1g\right\|_p \text{ for } p>2 \text{ and } \frac{1}{p}+\frac{1}{q}=1. \text{ Thus,}
\end{aligned}$$

$$tE_0(f)_p \le t\left\|f-g(0)\right\|_p \le t\left\|f-g\right\|_p + t\left\|g-g(0)\right\|_p \le c\left[\left\|f-g\right\|_p + t\left\|D_1g\right\|_p\right],$$

which proves the lemma. □

P r o o f o f T h e o r e m s 1 a n d 2. From parts a) and b) of Theorems 3 and 4 we get $E_0(f)_p \sim E_0(Af)_p$. Using Corollary 2 and Lemma 3 part a) with $F = Af$ we get

$$\begin{aligned}
&K(f,t;L_p[-1,1],C^2,D_1) \\
\le\ & c\left[K(Af,t^{\frac{1}{p}+\frac{1}{2}};L_p[-1,1],C^2,D_2) + t^{\frac{1}{p}+\frac{1}{2}}E_0(f)_p\right], \quad \text{for } 2<p<\infty.
\end{aligned}$$

From Corollary 2 and Lemma 3 part b) we obtain

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],C^2,D_2), \qquad \text{for } \ 1\le p\le 2.$$

From Corollary 2 and Lemma 4 we obtain for $2<p<\infty$

$$\begin{aligned}
K(Af,t;L_p[-1,1],C^2,D_2) + tE_0(f)_p &\le K(Af,t;L_p[-1,1],Z_2,D_2) + tE_0(f)_p \\
&\le cK(f,t;L_p[-1,1],C^2,D_1),
\end{aligned}$$

which proves the theorems. □

4. Generalization. The results can be dealt with in a generalized case as in the K-functional (1.1) $D_1g := \varphi^{2-2\lambda}(\varphi^{2\lambda}g')'$ for $\lambda \in (0,1)$, while in the K-functional (1.2) D_2 remains the same. The corresponding linear operators A and A^{-1} for the general case are:

$$(Af)(x) := f(x) + \int_0^x f(y)\left[(y-x)\frac{\theta'(y)}{\theta(y)}\right]' dy$$

$$(A^{-1}f) := f(x) + \int_0^x f(y)\left[\theta''(y)\int_y^x \frac{dt}{\theta(t)} - \frac{\theta'(y)}{\theta(y)}\right] dy,$$

where $\theta(y) = \varphi^{2\lambda}(y) = (1-y^2)^\lambda$.

Then the analogue of Theorem 1 is

Theorem 1′. *Let* $\lambda \in (0,1)$. *Then for every* $t \in (0,1]$ *and* $f(x) \in L_p[-1,1]$, $1 \le p \le \dfrac{1}{\lambda}$ *we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],C^2,D_2).$$

The proof of Theorem 1′ follows the same pattern. The analogues of Therems 3 and 4 are the same to the value of absolute constant in the inequality of the norm. In the analogue of Theorem 5 the space Z_1 is the same, while the space

$$Z_2 = \left\{ f \in C^2[-1,1] : \int_{-1}^0 (\varphi^{2\lambda}(x))' f'(x)dx = 0, \int_0^1 (\varphi^{2\lambda}(x))' f'(x)dx = 0 \right\}.$$

Lemmas 1 and 2 and Corollary 2 remain the same. The conclusion of Lemma 3 b) is the same under assumption $1 \le p \le \dfrac{1}{\lambda}$.

Theorem 1′ is not true for $\lambda = 1$ – see Theorem B. To make it true on the right hand side of the relation we have to add the term $tE_0(f)_1$, what is exactly the result in [3] for $p = 1$. That is not strange, because for $\lambda = 1$ after we integrate by parts the integral conditions (describing the space Z_2 in the general case) we obtain the conditions considered by Ivanov in [3, p. 120] and for $\lambda = 1$ the differential operator $(D_1 g)(x) = (1-x^2)g'' - 2xg'(x)$ what is exactly the analogue of D_3 in the inteval $[-1,1]$.

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