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FINITE GROUPS AS THE UNION OF PROPER SUBGROUPS

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ABSTRACT. As is known, if a finite solvable group G is an n-sum group then $n - 1$ is a prime power. It is an interesting problem in group theory to study for which numbers n with $n - 1 > 1$ and not a prime power there exists a finite n-sum group. In this paper we mainly study finite nonsolvable n-sum groups and show that 15 is the first such number. More precisely, we prove that there exist no finite 11-sum or 13-sum groups and there is indeed a finite 15-sum group. Results in [7] and [15] are thus extended and further generalizations are possible.

It is a basic fact in group theory that a finite group is the set-theoretic union of proper subgroups if and only if the group is not cyclic. For a finite group G the least positive integer n such that G is the union of n proper subgroups is defined to be the covering number of G. Denote by $\alpha(G)$ the covering number for any finite group G and we call G an $\alpha(G)$-sum group. If G is cyclic we define

$\alpha(G) = \infty$. Covering numbers of finite groups have been well investigated in the past forty years, for references the reader is refered to [7], [10] and [15]. Haber and Rosenfield proved in 1959 that there are no finite 2-sum groups and a finite group G is a 3-sum group if and only if the Klein 4-group is a homomorphic image of G [10]. Along this line Cohn proves in [7] that a finite group G is a 4-sum group if and only if $Z_3 \times Z_3$ or S_3 is a homomorphic image of G and G is a 5-sum group if and only if A_4 is a homomorphic image of G. Tomkinson proves in [15] that there are no finite 7-sum groups and for finite solvable n-sum groups one has $n = 1 + p^m$ where p is a prime number, thus the two conjectures by Cohn [7] are confirmed. There are other ways to study how groups can be expressed as a union of proper subgroups, and similar arguments also apply to the study on unions of ideals in ring theory, see [3], [11] and [16] for references.

It is an important problem when a rational integer n is a covering number for some finite group G. For any prime power p^m, it is easy to verify that the group $Z_p^m : Z_{p^m-1}$ has $1 + p^m$ as its covering number where Z_p^m is the elementary abelian p-group of order p^m and Z_{p^m-1} is the cyclic group of order $p^m - 1$. Thus one needs only to determine whether or not there exists a finite n-sum group for any rational integer n with $n - 1$ not a prime power. For example, Tomkinson asked in [15] if there are 11, 13 or 15-sum groups. Of course it is very helpful and challenging to determine the covering numbers of finite simple groups. For convenience we call an integer n a solvable covering number if there exists a finite solvable n-sum group. Note that finite simple groups may have solvable covering numbers, for example, $\alpha(A_5) = 10 = 1 + 3^2$. However, for a non-solvable covering number n a finite n-sum group is not solvable. In this paper we continue the study on finite n-sum groups, in particular we prove that 15 is the least non-solvable covering number, thus we solve the problem posed by Tomkinson. We also make effort to determine the covering numbers of finite nonabelian simple groups. Our machinery can be possibly applied for further generalizations.

All groups in this paper are finite, notation and terminology are standard. Information on maximal subgroups of finite simple groups can be found in the Atlas of Finite Groups [6], sometimes we use the information implicitly.

For the convenience of the reader we quote without proof some basic results on n-sum groups.

Lemma 1 [7]. *Let G be a finite group and N a normal subgroup of G. Then $\alpha(G) \leq \alpha(G/N)$.*

Lemma 2 [7]. *If a finite group $G = \cup_{j=1}^n H_j$ such that $|G : H_j| \leq |G : H_{j+1}|$ for $j \leq n - 1$ then $|G| \leq \Sigma_{j \geq 2}|H_j|$ and $|G : H_2| \leq n - 1$.*

Lemma 3 [10]. *If G is an n-sum group and $G = \cup_{j=1}^{n} H_j$ then for any $1 \leq k \leq n$ we have $\cap_{j \neq k} H_j = \cap_{j=1}^{n} H_j$.*

Lemma 4 [15]. *Let N be a normal subgroup of a finite group G. Let $U_1, U_2, \ldots, U_h$ be proper subgroups of G containing N and $V_1, V_2, \ldots, V_k$ be proper subgroups such that $NV_j = G$ with $|G : V_j| = I_j$ and $I_1 \leq I_2 \leq \cdots \leq I_k$. If $G = U_1 \cup \cdots \cup U_h \cup V_1 \cup \cdots \cup V_k$ then $I_1 \leq k$. Furthermore, if $I_1 = k$ then $I_1 = I_2 = \cdots = I_k = k$ and $V_i \cap V_j \leq U_1 \cup U_2 \cup \cdots \cup U_h$.*

Lemma 5 [13]. *For the symmetric group S_n, $\alpha(S_n) = 2^{n-1}$ when n is odd unless $n = 9$; $\alpha(S_n) \leq 2^{n-2}$ when n is even; $\alpha(A_n) \geq 2^{n-2}$ for $n \neq 7, 9$, with equality if and only if $n \equiv 2$ modulo 4.*

Lemma 6 [5]. *$\mathrm{PSL}(2,7)$ is a 15-sum group.*

Lemma 7 [5]. *Let G be either $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$, or $\mathrm{PGL}_2(q)$. Then $\alpha(G) = q(q+1)/2$ for q even or $q(q+1)/2+1$ for q odd where $q \neq 5, 7, 9$.*

Lemma 8. *Let K be one of the following simple groups: $\mathrm{PSL}(2,11)$, M_{11}, M_{12}, A_j, $7 \leq j \leq 12$. If G is a subgroup of $\mathrm{Aut}(K)$ containing K then $\alpha(G) \geq 15$.*

P r o o f. Suppose that $G = \cup_{j=1}^{\alpha} H_j$ where $\alpha = \alpha(G)$ and H_j's are maximal subgroups of G. First we consider the case where $K \neq H_j$ for any j. For this case we have $\alpha(K) \leq \alpha(G)$ since $K = \cup(K \cap H_j)$, so we need only to prove that $\alpha(K) \geq 15$.

If $K = A_j$ with $7 \leq j \leq 12$ then by Lemma 5 we have $\alpha(K) \geq 2^6 > 15$

If $K = \mathrm{PSL}(2,11)$ then by Lemma 7 $\alpha(K) \geq 67 > 15$. If $K = M_{11}$ then by [13] $\alpha(M_{11}) = 23 > 15$.

If $K = M_{12}$ then there need at least 12 maximal subgroups of M_{12} to cover all cyclic Sylow 11-subgroups. Let E be a cyclic subgroup of M_{12} of order 10, then E is self-centralizing in M_{12} and has $|M_{12}|/10 = 2^5 \cdot 3^3 \cdot 11$ conjugates. Note that every maximal subgroup of M_{12} containing an element of order 11 does not contain elements of order 10. By checking the maximal subgroups M of M_{12} containing an element of order 10 we see that M is not isomorphic to M_{11} and contains at most $2^4 \cdot 3^2 \cdot 11$ conjugates of E. Hence we need at least 6 $(= (2^5 \cdot 3^3 \cdot 11)/(2^4 \cdot 3^2 \cdot 11))$ more maximal subgroups to cover elements of order 10 and thus $\alpha(M_{12}) \geq 12 + 6 = 18 > 15$.

Now we suppose that K is one of H_j's, say $j = 1$. Then K is a proper subgroup of G and $G = \mathrm{Aut}(K)$. Since $\mathrm{Out}(M_{11}) = 1$, K is not isomorphic to

M_{11}. If $K = \mathrm{PSL}(2, 11)$ we see the lemma is true by Lemma 7. If $K = M_{12}$ then all maximal subgroups of G other than K have index at least 24, thus by Lemma 2 we have that $|G| \le \Sigma_{j>1}|H_j| \le \alpha(G)|G|/24$, whence $\alpha(G) \ge 24 > 15$. If $K = A_j$ with $j = 7, 11$ then by Lemma 5 $\alpha(G) \ge 2^6 > 15$. If $K = A_9$, let $x = (1\,2\,3\,4\,5\,6\,7)(8\,9)$. Now x is not in K and the maximal subgroups of G containing a conjugate of x are isomorphic to $S_7 \times Z_2$. Thus $\alpha(G) \ge 1 + 9!/(2((9 - 2)!)) = 1 + 9(9 - 1)/2 \ge 37 > 15$. Finally we consider the case where $K = A_j$ with $j = 8, 10, 12$. Then $x := (1\,2\,\ldots\,j)$ is not in K and the maximal subgroups of G containing a conjugate of x have index in G at least $15(j = 8), 45(j = 10)$ or 66 ($j = 12$) respectively, since x does not fix any figures $i \le j$. Evidently $\langle x \rangle$ is self-centralizing, so any maximal subgroup M of G contains at most $|M|/|\langle x \rangle|$ conjugates of x. Since $|M| \le |G|/15$ and G has $|G|/|\langle x \rangle|$ conjugates of x, we need at least 15 maximal subgroups of G other than K to cover conjugates of x, so $\alpha(G) \ge 15$. $\square$

Lemma 9 . *Let H be a noncyclic solvable subgroup of* $\mathrm{GL}(3, 2)$ *or* $\mathrm{GL}(2, 3)$. *Then $\alpha(H) \le 8$.*

P r o o f. If $H \le \mathrm{GL}(3, 2)$ then H is isomorphic to a subgroup of S_4 or D_{21}. So every chief factor of H is of order at most 7. This is also true for $H \le \mathrm{GL}(2, 3)$. Thus $\alpha(H) \le 8$ by [15]. $\square$

Lemma 10. *Let G be a finite group and H a normal subgroup of G which is the direct product of nonsolvable minimal normal subgroups N_j's. Then any normal subgroup M of G contained in H is also a direct product of some of these N_j's.*

P r o o f. First note that each N_j is a direct product of isomorphic nonabelian simple groups. We need only to prove that for any $y \in M$ if $y = x_1 x_2 \ldots x_n$ with $x_i \in N_i (i \le n)$ then $N_j \le M$ whenever $x_j \ne 1$. Suppose that $x_j \ne 1$ then there is an element $x \in N_j$ such that $[x, x_j] \ne 1$. Now $[x, x_j] = [x, y] \in M, M \cap N_j \ne 1$, so $N_j \le M$. We are done. $\square$

Theorem 11. *There are no* 11- *or* 13-*sum groups.*

P r o o f. Suppose toward a contradiction that there are 11- or 13-groups and let $G = \cup_{j=1}^n H_j$ be an n-group of minimal possible order with H_j's maximal in G and $n = 11$ or 13. Then the core $\mathrm{Core}_G(\cap H_j) = 1$ by Lemma 1. It follows that the Frattini subgroup $\Phi(G) = 1$. Set $I_j = |G : H_j|, j = 1, 2, \ldots, n$. By Lemma 2 there are at least two of I_j's at most $n - 1$.

Step 1. If $I_j \le n - 1$ then either H_j is normal in G or $\overline{G} := G/\mathrm{Core}_G(H_j)$ is not solvable.

P r o o f. If this is not true let H_j be not normal in G with $I_j \leq n-1$ and $\overline{G}$ solvable. Since H_j is maximal in G, the Fitting subgroup $F(\overline{G})$ is a minimal normal subgroup of $\overline{G}$ and $\overline{G} = \overline{H_j}F(\overline{G})$ with $\overline{H_j} \cap F(\overline{G}) = 1$. Thus $F(\overline{G})$ is of order at most $I_j \leq 13-1 = 12$. Since H_j is not normal in G, $F(\overline{G})$ is not in the center of $\overline{G}$. Set $p^m = |F(\overline{G})|$ where p is a prime, then $p^m \leq n-2$ since $n = 11$ or 13. If $\overline{H_j}$ is cyclic then $\alpha(G) \leq \alpha(\overline{G}) \leq p^m + 1 \leq n-1$ by [15], a contradiction. Thus $\overline{H_j}$ is not cyclic. It follows that $p^m = 8$ or 9 and $\overline{H_j} \leq \mathrm{GL}(3,2)$ or $\mathrm{GL}(2,3)$. By Lemma 9 we have $\alpha(G) \leq \alpha(\overline{H_j}) = 9 < n$, again a contradiction. We are done.

Step 2. $F(G) = 1$.

P r o o f. If $F(G)$ is not trivial let N be a nontrivial solvable minimal normal subgroup of G. By Lemma 3 for any $1 \leq s \neq t \leq n$ we have $\cap_{j \neq s}H_j = \cap_{j \neq t}H_j = \cap H_j$. Since $\mathrm{Core}_G(\cap H_j) = 1$ there are at least two H_j's such that $NH_j = G$ and $N \cap H_j = 1$. Let I_t be the least of these I_j's with $NH_j = G$, then $|N| = I_t$ and by Lemma 4 we have $I_j \leq n-1$. Note that $|N|$ is a prime power and $n = 11$ or 13, so one has $|N| \leq n-2$. If H_t is normal in G then N is in the center of G, so $|N|$ is a prime and $\alpha(G) = |N|+1 \leq n-1$ by [7], which is a contradiction. Hence H_t is not normal in G. Since H_t is maximal in G, $F(G/\mathrm{Core}_G(H_t)) = N \mathrm{Core}_G(H_t)/\mathrm{Core}_G(H_t)$. By Step 1 $G/\mathrm{Core}_G(H_t)$ is not solvable, thus $|N| = 8$ and $G/\mathrm{Core}_G(H_t) \cong Z_2^3 : \mathrm{PSL}(3,2)$ because $|N| \leq n-2 \leq 11$ and all proper subgroups of $\mathrm{PSL}(3,2)$ are solvable. If for some $x \in N, H_t^x \neq H_j$ for any j then $H_t^x = \cup(H_j \cap H_t^x)$. Hence $n \geq \alpha(H_t^x) = \alpha(G/N)$, which is contradictory to the minimality of G. Thus $H_t^x = H_j$ for some j.

We claim that $\mathrm{Core}_G(H_t) \neq 1$. Suppose otherwise that $\mathrm{Core}_G(H_t) = 1$. Let y be an involution of H_t, then there is an involution u in N such that $o(yu) = 4$ and $(yu)^2 \in N$. If an H_s contains a conjugate of yu then $N \cap H_s \neq 1$, so $N \leq H_s$ (otherwise $NH_s = G$ and thus $N \cap H_s = 1$). It follows that H_s/N is a maximal subgroup of $\mathrm{PSL}(3,2)$ containing an involution, whence $H_s/N \cong S_4$ and $I_s = 7$ by [6]. So we need at least 7 H_j's containing N to cover all conjugates of yu, which implies that $13 \geq n \geq 8 + 7 = 15$, impossible. Thus we have $\mathrm{Core}_G(H_t) \neq 1$, as claimed. Let T be the generalized Fitting subgroup $F^*(\mathrm{Core}_G(H_t))$, then $T \neq 1$. Note that $T \leq H_t^x$ for any $x \in N$, so the number f of H_j's not containing T is at most $n - 8 \leq 5$. For convenience we may assume that $TH_1 = TH_2 = \cdots = TH_f = G$ and $I_1 \leq I_2 \leq \cdots \leq I_f$. By Lemma 4 we have $I_1 \leq f$. Since $f \leq 5$ and $|N| = 8$, $N \leq H_1$(otherwise $NH_j = G$ and $I_j = |N| = 8$, a contradiction). If $I_1 = f$ then by Lemma 4, $I_j = f (j = 1, 2, \ldots, f)$ and $H_1 \cap H_2 \leq H_t$. As above we have $N \leq H_j (1 \leq j \leq f)$. Thus $N \leq H_1 \cap H_2 \leq H_t$, which is contradictory to the choice of H_t. Therefore $I_1 < f \leq 5$. It follows that $G/\mathrm{Core}_G(H_1)$ is solvable.

By Step 1 H_1 is normal in G and thus $I_1 = 2$ or 3. Noticing that $\Phi(G) = 1$, we see that $F^*(G)$ is the direct product of minimal normal subgroups of G. Since $\text{Core}_G(H_t)$ is normal in G, $T \leq F^*(G)$. So there is a minimal normal subgroup S of G such that $SH_1 = TH_1 = G$. It follows immediately that S is solvable and $S \cap H_1 = 1$, so $G = S \times H_1$. Now S is in the center of G and by [7] we have $\alpha(G) \leq 3 + 1 = 4$, contradicting the assumption on G.

Step 3. $F^*(G) = N_1 \times N_2 \times \cdots \times N_m$ where N_j's are minimal normal subgroups of G and isomorphic to $\text{PSL}(2, 8)$, $\text{PSL}(3, 2)$, $\text{PSL}(2, 11)$, M_{11}, M_{12} or $A_i, 5 \leq i \leq 12$. And $G/F^*(G)$ is cyclic of order dividing 6.

P r o o f. Let N be an arbitrary minimal normal subgroup of G. By Step 2, N is a direct product of isomorphic nonabelian simple groups. By Lemma 3 there is an integer t such that $NH_t = G$ and $I_t \leq n - 1 \leq 12$. Since $N \cap \text{Core}_G(H_t) = 1$, N is isomorphic to a normal subgroup of $G/\text{Core}_G(H_t)$. Note that $G/\text{Core}_G(H_t)$ is a primitive permutation group of degree I_t. By [9] we know that N is simple and isomorphic to the simple groups listed above. Thus the outmorphism group $\text{Out}(N)$ is an elementary abelian 2- or 3-group [6]. Since $\Phi(G) = F(G) = 1$, $F^*(G)$ is the direct product of minimal normal subgroups N_j's of G. Now $G/F^*(G)$ acts by conjugation on N_j's and it follows that $G/F^*(G)$ is an abelian $\{2, 3\}$-group. If $G/F^*(G)$ is not cyclic then $\alpha(G/F^*(G)) \leq 4$ by [15], which is a contradiction.

Step 4. For each N_j there exists an H_i such that $N_j \leq H_i$.

P r o o f. If there is an N_j such that $N_jH_i = G$ for any i, then $m = 1, F^*(G) = N_1$ and $F^*(G) = \cup_i(F^*(G) \cap H_i)$. So $\alpha(F^*(G)) \leq 13$. By Lemmas 5, 6, 7 and 8, $F^*(G) \cong A_5$. Thus $G \cong S_5$, which is a contradiction since $\alpha(S_5) = 16$ by [7].

Step 5. $m \leq 2$.

P r o o f. Suppose $m \geq 3$. Set $J_r = \{i : N_rH_i = G\}$ for $1 \leq r \leq m$. For convenience and without loss of generality we may assume that $s := |J_1| = \max\{|J_r| : 1 \leq r \leq m\}$ and $N_1H_i = G$ for $1 \leq i \leq s$ with $I_1 \leq I_2 \leq \cdots \leq I_s$. Thus $N_1 \leq H_j$ for $j > s$. Set $M = N_2 \times N_3 \times \cdots \times N_m$, then $M \leq \cap_{j \leq s}H_j$.

We claim that $s < (n + 1)/2$. Suppose othertwise that $s \geq (n + 1)/2$. Re-label H_i's for $i > s$ such that $N_2H_{s+1} = N_2H_{s+2} = G$ since there are at least two H_i's not containing N_2. Thus $N_3 \leq H_{s+1} \cap H_{s+2}$. Now N_3 is contained in $s + 2$ of H_i's. Note that $s + 2 \geq 9$ for $n = 13$ and 8 for $n = 11$, so $n - s - 2 \leq 4$. By [15] for G and the normal subgroup N_3 we have $I_t \leq n - s - 2 \leq 4$ with $N_3H_t = G$

for some $t > s + 2$. Hence H_t is normal in G by Step 1, which is impossible since H_t is maximal in G and N_3 is nonabelian simple. Thus the claim holds true. By [15] for G and the normal subgroup N_1 we have $I_1 \leq s \leq (n-1)/2$. If $I_1 = s$ then $I_1 = I_2 = \cdots = I_s$ with $M \leq H_i$ for $i > s$. Thus M is contained in every H_i, which is contradictory to the assumption. So $I_1 < s$. Since $N_1 H_1 = G$, H_1 is not normal in G. By Step 1, $G \operatorname{Core}_G(H_1)$ is not solvable and $I_1 \geq 5$. From $5 \leq I_1 < s \leq (n-1)/2$ we see that $n = 13$, $s = 6$ and $I_1 = 5$. Re-labeling H_i's for $i > 6$ we may assume that $N_2 H_7 = N_2 H_8 = \cdots = N_2 H_e = G$ and $N_2 \leq H_i$ for $i > e$. Note that $e - 6 \geq 2$ and $N_3 \leq H_i$ for $i \leq e$. By the definition of s we have $e - s \leq s$, so $8 \leq e \leq 12$. By [15] for G and N_3, $5 \leq H_i \leq 13 - e$ for some $i > e$, thus $e = 8$ and $H_i = 5$ for all $i > 8$ and $H_9 \cap H_{10} \leq H_1 \cap H_2 \cap \cdots \cap H_8$, which is a contradiction since $N_1 \leq H_i$ for $i > 6$ and N_1 is not contained in H_1.

Step 6. Last contradiction.

P r o o f. We first prove that $m = 1$, so $F^*(G) = N_1$ is simple. Suppose otherwise that $m = 2$. Since each H_i contains either N_1 or N_2 we may assume that $N_1 \leq H_1 \cap H_2 \cap \cdots \cap H_s$ with $N_1 H_i = G$ for $i > s$ and $s \geq (n+1)/2 \geq 6$. Thus $N_2 \leq H_i$ and $|G : H_i| \geq 5$ for $i > s$ (see the proof in Step 5). By [15] for G and the normal subgroup N_1 and since N_2 is not contained in H_j for some $j \leq s$, $5 \leq |G : H_t| < n - s$ for some $t > s$. It follows that $n = 13, s = 7$ and $|G : H_t| = 5$, so $N_1 \cong A_5$. By the definition of s we know that there is at most one $H_i (i \leq 7)$ containing N_2. Now we consider cases.

Case I. $N_2 \leq H_i (i \leq 7)$, say $i = 1$. Again by [15] for G and N_2 we have $5 \leq |G : H_f| < 6$ for some $1 < f \leq 7$, thus $|G : H_f| = 5$ and $G / \operatorname{Core}_G(H_f) \leq S_5$ with $N_2 \cong A_5$. Now $F^*(G) \cong A_5 \times A_5$ and $|G/F^*(G)|$ is at most 2. Hence we have the following three possibilities: $G \cong A_5 \times A_5, A_5 \times S_5$ or $(N_1 \times N_2) : \langle x \rangle$ where x is of order 2 and $N_1 \langle x \rangle \cong N_2 \langle x \rangle \cong S_5$. Since $\alpha(A_5) = 10$ we see that $G = (N_1 \times N_2) : \langle x \rangle$. Let $x_i \in N_i$ such that $x x_i = x_i x$ with $o(x x_i) = 6$ for $i = 1, 2$. Evidently $x x_1 x_2$ is of order 6 and is not contained in $F^*(G)$ and $\langle x x_1 x_2 \rangle$ is self-centralizing in G. So $x x_1 x_2 \in H_j$ for some $j > 1$. Since H_j contains $N_i (i = 1$ or $2)$, H_j/N_i is isomorphic to a maximal subgroup of S_5 containing an element of order 6. Thus, by [6] $H_j/N_i \cong Z_2 \times S_3$ and H_j contains exactly $|H_j|/6 = (5!/2)(12/6) = 120$ conjugates of $x x_1 x_2$. Since $x x_1 x_2$ has $(5!/2)(5!/6) = 1200$ conjugates in G, we need at least 10 of these H_i's $(i > 1)$ to cover all conjugates of $x x_1 x_2$. Let $y_i \in N_i$ be an involution such that $x y_i$ is of order 4 $(i = 1, 2)$. Then $\langle x y_1 y_2 \rangle$ is self-centralizing of order 4 in G and is not contained in $F^*(G)$, so there are $(5!/2)(5!/4) = 1800$ conjugates of $x y_1 y_2$ in G. Let $x y_1 y_2$ be contained in H_k $(k > 1)$, then $H_k/N_j \cong S_4$ or $Z_5 : Z_4$ where $N_j \leq H_k$. It follows immediately

that H_k does not contain a conjugate of xx_1x_2. Since H_k contains at most $(5!/2)(4!/4) = 360$ conjugates of xy_1y_2 we need at least $5 = 1800/360$ of H_i's to cover all conjugates of xy_1y_2. Therefore $13 \geq \alpha(G) \geq 1 + 10 + 5 = 16$ which is absurd.

Case II. $N_2H_i = G$ for $i \leq 7$. By [15] for G and N_2 we have $5 \leq |G : H_f| < 7$ for some $1 \leq f \leq 7$, thus $|G : H_f| = 5$ or 6 and $G/\operatorname{Core}_G(H_f) \leq S_6$ with $N_2 \cong A_5$ or A_6. Now $F^*(G) \cong A_5 \times A_5$ or $A_5 \times A_6$ and $G/F^*(G)$ is of order at most 2. Hence we have the following possibilities: $G \cong A_5 \times A_5, A_5 \times S_5,$ $A_5 \times A_6, A_5 \times S_6$ or $(N_1 \times N_2) : \langle x \rangle$ where x is of order 2 and $N_1\langle x \rangle \cong S_5$ and $N_2\langle x \rangle \cong S_5, S_6$ or $A_6 \times Z_2$. Since $\alpha(A_5) = 10$, A_5 cannot be a direct factor of G. Thus $G = (N_1 \times N_2) : \langle x \rangle$. If $N_1 \cong N_2 \cong A_5$ then as proved in Case I we know that we need at least $10 + 5$ maximal subgroups to cover all elements of orders 4 and 6 outside $F^*(G)$, so $N_2 \cong A_6$. By [6] G contains exactly $(4!)(6!/5) = 24 \times 144 = 3456$ elements ab of order 5 with $a \in N_1 \setminus \{1\}$ and $b \in N_2 \setminus \{1\}$, and each maximal subgroup of G contains at most $144 \times 4 = 576$ such elements. Thus we need at least $3456/576 = 6$ maximal subgroups of G to cover all such elements of order 5. Since $N_2 \cong A_6$ there is an element $v \in N_2$ such that $\langle v \rangle$ is of order 3 corresponding to a product of two 3-cycles in A_6 and such that $xv = vx$. Let $u \in N_1$ be an element of order 3 such that $xu = ux$. Then $\langle xuv \rangle$ is of order 6 and self-centralizing in G. Now G contains $|G|/6$ conjugates of xuv and the maximal subgroups M of G containing a conjugate of xuv are isomorphic to $N_iL\langle x \rangle$ where $N_i \leq M$ with $i = 1$ or 2 and $L \cong Z_3^2 : Z_4$ or $S_3(L \leq N_j, j \leq 2, j \neq i)$. Hence M is of order at most $|G|/10$ and does not contain 5-elements of the type ab with $a \in N_1 \setminus \{1\}$ and $b \in N_2 \setminus \{1\}$. We now have $\alpha(G) \geq 10 + 6 > 13$, a contradiction. Therefore $m = 1$, as claimed.

Note that $F^*(G) \cong \mathrm{PSL}(3,2), \mathrm{PSL}(2,8), \mathrm{PSL}(2,11), M_{11}, M_{12}$ or $A_j, 5 \leq j \leq 12$, $G/F^*(G)$ is of order at most 3. By Lemma 8 and [7] we know that $F^*(G)$ is a proper subgroup of G and $F^*(G) \cong \mathrm{PSL}(2,7), \mathrm{PSL}(2,8)$ or A_6 (note that $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$). By Lemma 5 G is not isomorphic to S_6. Now by [6] G contains an element x of order m' outside $F^*(G)$ such that all maximal subgroups of G containing an element of order m' are conjugate to $N_G(\langle x \rangle)$. The information about $(G, m', N_G(\langle x \rangle), |G|/|N_G(\langle x \rangle)|)$ are as follows [6]:

$(\mathrm{PSL}(2,7) : Z_2,\ 8,\ D_{16},\ 21), (\mathrm{PSL}(2,8) : Z_3,\ 9,\ Z_9 : Z_6,\ 28),$

$(A_6.2_2,\ 10,\ D_{20},\ 36), (A_6.2_3,\ 9,\ D_{16},\ 45).$

Thus $\alpha(G) \geq |G|/|N_G(\langle x \rangle)| \geq 21 > 13$, which is contradictory to the assumption on G. We are done. $\square$

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