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HAUSDORFF MEASURES OF NONCOMPACTNESS AND INTERPOLATION SPACES

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Abstract. A new measure of noncompactness on Banach spaces is defined from the Hausdorff measure of noncompactness, giving a quantitative version of a classical result by R. S. Phillips. From the main result, classical results are obtained now as corollaries and we have an application to interpolation theory of Banach spaces.

Introduction. The notion of measure of noncompactness was introduced by K. Kuratowski and, with a convenient but equivalent modification, by F. Hausdorff. Subsequently it was used in numerous branches of functional analysis and theory of differential and integral equations. In this note we introduce a new measure of noncompactness to obtain a quantitative version of a classical result by R. S. Phillips [5, Thm. 3.7] (see also Dunford-Schwartz [4, Lemma IV.5.4, p. 259] and Brooks-Dinculeanu [3, Thm. 1]). We shall also give an application to interpolation spaces.

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1. Hausdorff measures of noncompactnes. Given a Banach space X, the closed unitary ball in X is denoted by U_X. The *Haudorff measure of noncompactness* of a bounded subset $B \subset X$ is defined by

$$\chi_X(B) = \inf\{\varepsilon > 0 : \text{there exists a finite set } F \text{ in } X \text{ such that } B \subset F + \varepsilon U_X\}.$$

For properties of χ see [1].

2. A Phillips-like estimate. We shall state a quantitative version, but sligthly more general, of Brooks-Dinculeanu's Theorem 1 [3].

If (X_n), $n \in \mathbb{N}$ is a sequence of Banach spaces, for $1 \leq p < \infty$, we denote by

$$X^p = {}^p\bigoplus_{n=1}^{\infty} X_n,$$

the Banach space of all sequences (x_n) in $\prod_{n=1}^{\infty} X_n$ such that

$$\|(x_n)\|_{X^p} = \left[\sum_{n=1}^{\infty} \|x_n\|_{X_n}^p\right]^{1/p} < \infty.$$

Given a sequence (x_n) in X^p, let us set $P_k(x_n) = (x_1, \ldots, x_k, 0, 0, \ldots)$ and $\pi_k((x_n)) = x_k$, the projection on the k^{th}-component.

Theorem 2.1. *For a bounded subset $B \subset {}^p\bigoplus_{n=1}^{\infty} X_n$ we set*

$$\nu(B) = \limsup_{k \to \infty} \left[\sup_{x \in B} \|P_k(x_n) - (x_n)\|_{X^p} + \chi(P_k(B))\right].$$

Then, if χ is the Hausdorff measure of noncompactness in X^p, we have

$$\chi(B) \leq \nu(B) \leq 2\,\chi(B),$$

for all bounded subset B in X^p.

P r o o f. For each bounded subset $B \subset X^p$ and $n \in \mathbb{N}$, we have

$$B \subset (Id - P_n)B + P_n B.$$

Since the Hausdorff measure of noncompactness is subadditive, taking in account the inequality

$$\chi((P_n - Id)B) \leq \sup_{x \in B} ||P_n x - x||,$$

we get

$$\begin{aligned} \chi(B) &\le \chi((P_n - Id)B) + \chi(P_n B) \\ &\le \sup_{x\in B} ||P_n x - x|| + \chi(P_n B), \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $\chi(B) \le \nu(B)$.

Conversely, since operators P_n are uniformly bounded, let us define $M := \limsup_{n\to\infty} \|P_n\|$. Then, since $\|P_n\| = 1$ ($\|P_n x\| \le \|x\|$ for all x and $\|P_n x\| = \|x\|$ for $x = (x_1, \ldots, x_n, 0, 0, \ldots)$), it follows $M = 1$.

Given a bounded fixed subset B in X, let $r = \chi(B)$ and, for $\varepsilon > 0$ arbitrary, let $r_\varepsilon := r + \varepsilon$. Thus, there is a finite set B_0 in X such that

$$B \subset B_0 + r_\varepsilon U_{X^p}.$$

And, since B_0 is finite, there exist $N \in \mathbb{N}$ such that

$$\|(P_n - Id)x_0\| < \varepsilon,$$

for all $n \ge N$ and $x_0 \in B_0$. Now, let x an arbitrary element in B and $x_0 \in B_0$ chosen such that $\|x - x_0\| < r_\varepsilon$. Since

$$\|(P_n - Id)x\| - \|(P_n - Id)x_0\| \le \|(P_n - Id)(x - x_0)\| \le 1 \,.\, r_\varepsilon,$$

it holds

$$\|(P_n - Id)x\| \le \|(P_n - Id)x_0\| + r_\varepsilon,$$

and, for all $x \in B$ and $n \ge N$, we have

$$\|P_n x - x\| \le \varepsilon + r_\varepsilon = r + 2\varepsilon.$$

Therefore, taking $\varepsilon \to 0$ one has

$$\limsup_{n\to\infty} \sup_{x\in B} ||P_n x - x|| \le \chi(B).$$

Finally, since $\chi(P_\lambda B) \le \|P_n\|\ \chi(B) \le M\ \chi(B) \le \chi(B)$ we get

$$\nu(B) \le \chi(B) + \chi(B) = 2\chi(B),$$

and the proof is complete. □

From the result of the Theorem 2.1 we can prove the measure ν has all the properties of χ, therefore ν is a measure of noncompactness too. And albeit ν is a measure equivalent to χ, from ν we get the new results which follows below.

The next result is necessary to get our main application.

Lemma 2.2. *For a bounded subset* $B \subset X^p = {}^p\bigoplus_{n=1}^{\infty} X_n$ *we have*

$$\chi_{X_n}(\pi_n(B)) \leq \chi(B).$$

P r o o f. We start verifying that $\pi_n(U_{X^p}) = U_{X_n}$. Let $x = (x_j)_{j=1}^{\infty} \in U_{X^p}$, then

$$\|x\|_{X^p} = \left(\sum_{j=1}^{\infty} \|x_j\|_{X_j}^p \right)^{1/p} \leq 1.$$

Thus, we have $\|x_j\|_{X_j} \leq \|x\|_{X^p} \leq 1$ for all j. Since $x_n = \pi_n(x)$ we obtain $\pi_n(x) \in U_{X_n}$ and finally $\pi_n(U_{X^p}) \subset U_{X_n}$. Now, given $z \in U_{X_n}$, we define a sequence $x = (x_j)_{j=1}^{\infty}$ by $x_j = 0$, if $j \neq n$, and $x_j = z$, if $j = n$. Then $x \in X^p$ and $\|x\|_{X^p} = \|z\|_{X_n} \leq 1$, which implies $x \in U_{X^p}$ and $\pi_n(x) = z$. Therefore, given $z \in X_n$, there exists $x \in X^p$ with $\pi_n(x) = z$, what means $U_{X_n} \subset \pi_n(U_{X^p})$ and the assertion follows.

Now, given $\varepsilon > \chi(B)$, there exist balls $B_1, \ldots, B_M \in X^p$ which $B_i = B(x_i, \varepsilon)$, such that

$$B \subset \bigcup_{i=1}^{M} B(x_i, \varepsilon).$$

Thus,

$$\pi_n(B) \subset \pi_n\left(\bigcup_{i=1}^{M} B(x_i, \varepsilon) \right) \subset \bigcup_{i=1}^{M} \pi_n(B(x_i, \varepsilon)).$$

Now, since

$$\pi_n(B(x_i, \varepsilon)) = \pi_n(x_i) + \varepsilon \pi_n(U_{X^p}) = \pi_n(x_i) + \varepsilon U_{X_n},$$

for each i, we see that there exist elements $y_1, \ldots, y_M$ such that

$$\pi_n(B) \subset \bigcup_{i=1}^{M} \{y_i + \varepsilon U_{X_n})\}.$$

Therefore, $\chi_{X_n}(\pi_n(B)) \leq \varepsilon$ and the result follows. $\square$

Corollary 2.3. *A set* $K \subset X^p = {}^p\bigoplus_{n=1}^{\infty} X_n$ *is relatively compact, if and only if:*

A) $\sum_{m \geq k} ||x_m||_{X_n}^p \longrightarrow 0, \quad k \to \infty,$ *uniformly for* $x \in K$.

B) *the set* $K(m) = \{x_m = \pi_m(x) \; ; \; x \in K\}$ *is relatively compact in the norm of* X_m*, for each* $m \in \mathbb{N}$*.*

P r o o f. If $K \subset X^p$ is relatively compact, we have $\chi(K) = 0$ and, by Theorem 2.1, we obtain

$$\nu(K) = \limsup_{k\to\infty} \left[\sup_{x\in B} \|P_k(x_n) - (x_n)\|_{X^p} + \chi(P_k(K)) \right] = 0.$$

From Lemma 2.2, we have for each n

$$\chi_{X_n}(\pi_n(K)) = \chi_{X_n}(\pi_n(P_n(K))) \leq \chi(P_n(K)) \leq \|P_n\|_{L(X^p,X^p)}\, \chi(K),$$

thus, **A)** and **B)** follow. □

In particular, if X is a fixed Banach space and $X_n = X$, for each $n \in \mathbb{N}$, we have

$$X^p = {}^p\bigoplus_{n=1}^{\infty} X_n = \ell^p_X.$$

Thus, we obtain from Corollary 2.3 a result stated by Brooks-Dinculeanu [1, Thm. 1].

Corollary 2.4. *A set* $K \subset \ell^p_X$, $1 \leq p < \infty$*, is relatively compact, if and only if:*

A) $\displaystyle\sum_{m\geq k} \|x_m\|^p \longrightarrow 0, \quad k \to \infty$*, uniformly for* $x \in K$*.*

B) *for each* $m \in \mathbb{N}$*, the set* $K(m) = \{x_m; x \in K\}$ *is relatively compact in the norm of* X*.*

3. An application to interpolation spaces. Given a Banach space E and a number $\alpha > 0$, we set $\alpha\, E$ for the space E equipped with the norm

$$\| \,.\, \|_{\alpha E} = \alpha \, \| \,.\, \|_E.$$

Let (E_0, E_1) be a Banach couple and $0 < \theta < 1$ (see [2] for the definitions on interpolation theory of Banach spaces). For each $n \in \mathbb{Z}$ we set

$$X_n^\theta := 2^{-\theta n} E_0 + 2^{-(\theta-1)n} E_1.$$

For $1 \leq p < \infty$, the K-interpolation space $(E_0, E_1)_{\theta,p,K}$ can be identified with the subspace of all constant sequences in ${}^p\widehat{\bigoplus}_{n\in\mathbb{Z}} X_n^\theta$. Then, for each $n \in \mathbb{N}$, setting I_n for the segment in $\mathbb{Z}$ from $-n$ to n and $I_n^{\mathbf{c}} = \mathbb{Z} \setminus I_n$, we see that the functional

$$\nu_\theta(B) := \limsup_{n\to\infty} [\sup_{x\in B} [\sum_{k\in I_n^{\mathbf{c}}} [2^{-k\theta} K(2^k, x)]^p]^{1/p} + \chi(P_{I_n}(B))]$$

can be estimate in $(E_0, E_1)_{\theta,p,K}$.

As a consequence of the main theorem, we have the following compactness criterion for bounded sets in interpolation spaces, which goes back to J. Peetre.

Theorem 3.2 (J.Peetre). *Let $(E_0, E_1)_{\theta,p,K}$ be an interpolation space with $0 < \theta < 1$ and $1 \leq p < \infty$. Then, a bounded subset B in $(E_0, E_1)_{\theta,p,K}$ is relatively compact if and only if*

A) $\limsup_{n\to\infty} \sum_{k\in I_N^{\mathbf{c}}} [2^{-k\theta} K(2^k, x)]^p = 0$, *uniformly in $x \in B$,*

and

B) *the subset B is relatively compact in $E_0 + E_1$.*

Indeed, $\nu_\theta(B)$ can be estimate by the Hausdorff measure of noncompactness $\chi(B)$. Further, if B is precompact in $E_0 + E_1$ is also precompact in $X_n^\theta = 2^{-\theta n} E_0 + 2^{-(\theta-1)n} E_1$.

REFERENCES

[1] J. Banaś, K. Goebel. Measures of Noncompactness in Banach Spaces. Marcel Dekker Inc. New York, 1980.

[2] J. Bergh, J. Löfströn. Interpolation Spaces: An Introduction. Springer-Verlag, 1976

[3] J. K. Brooks, N. Dinculeanu. Conditional expectation and weak and strong compactness in spaces of Bochner integrable functions. *J. Mult. Analysis* **9** (1979), 420–427.

[4] N. Dunford, J. Schwartz. Linear Operators. Part I. Interscience Pub. Inc. New York, 1967.

[5] R. S. Phillips. On linear transformations. *Trans. Amer. Math. Soc.* **48** (1940), 516-541.

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