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L_p EXTREMAL POLYNOMIALS. RESULTS AND PERSPECTIVES

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Abstract. Let $\alpha = \beta + \gamma$ be a positive finite measure defined on the Borel sets of $\mathbb{C}$, with compact support, where β is a measure concentrated on a closed Jordan curve or on an arc (a circle or a segment) and γ is a discrete measure concentrated on an infinite number of points.

In this survey paper, we present a synthesis on the asymptotic behaviour of orthogonal polynomials or L_p extremal polynomials associated to the measure α. We analyze some open problems and discuss new ideas related to their solving.

1. Introduction.

Let α be a positive finite measure defined on the Borel set $B(\mathbb{C})$ of $\mathbb{C}$, with compact support F. The L_p extremal polynomials $T_{n,p,\alpha}$ associated to the measure α and the support F are defined as the solutions

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of extremal problems in the space $L^p(F,\alpha)$. Let $m_{n,p}(\alpha)$ $(n \in \mathbb{N},\ p > 0)$ denote the extremal constants associated with α and F:

$$
\begin{aligned}
m_{n,p}(\alpha) &= \|T_{n,p,\alpha}\|_{L_p(F,\alpha)} \\
&= \min\left\{ \begin{array}{c} \|Q_n\|_{L_p(F,\alpha)} : Q_n = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \\ a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}. \end{array} \right\}
\end{aligned} \tag{1}
$$

Let us remark that in the case $p = 2$, the polynomials $T_{n,2,\alpha}$ coincide exactly with the (monic) orthogonal polynomials, and satisfy the following relations

$$
T_{n,2,\alpha}(z) = z^n + \cdots; \qquad \int_F T_{n,2,\alpha}(z)\overline{T_{m,2,\alpha}(z)}d\alpha(z) = 0; \text{ if } n \neq m. \tag{2}
$$

Many schools have studied the properties of orthogonal polynomials or L_p extremal polynomials, particularly:

a) Schools which are interested with the formal or algebraic case of orthogonal polynomials or L_p extremal polynomials (recurrent properties, the zeros distribution, formal resolution of differential equations, ...).

b) Schools which are interested on the analytic or functional case of orthogonal polynomials or L_p extremal polynomials (representation of analytic functions by series of polynomials, interpolation, continued fractions, Sturm-Liouville operators, asymptotic behaviour, ...).

One of the fundamental problems of the orthogonal polynomials or L_p extremal polynomials theory which can be classified in **b)** is the asymptotic behaviour when $n \to \infty$. Among the present methods used to solve this problems, we can cite those which are based on the deep study of extremal problems in Hardy spaces of holomorphic functions. These methods have been initiated and developed by the Soviet school (Smirnov [45, 46], Gueronimus [10, 11], Korovkine [23], Souetine [47], ...), and the American one (Szegö [50, 51, 52, 53], Widom [58], Nevai [37, 38], ...).

The study of the asymptotic behaviour of orthogonal or extremal polynomials contribute to solve important problems in mathematics, especially:

(i) The convergence of Padé approximants or of continued fractions ($F = [-1,+1] \cup \{z_k\}$, $p = 2$, see Gonchar [12]).

(ii) The spectral theory ($F = [-1,+1] \cup \{z_k\}$, $p = 2$, see Gueronimus [9], Nikishin [39])

(iii) The distribution of zeros of orthogonal polynomials or L_p extremal polynomials ($F = \Gamma = \{z : |z| = 1\}$, $p \geq 1$; $F = \Gamma \cup \{z_k\}$, $p = 2$, see [28]).

(iv) The representation theory of analytic functions by series of polynomials ($F = \Gamma$ or $F = E$, E being a rectifiable Jordan curve, see Szegö [51], [53], Smirnov [45].)

In this survey paper, we consider a measure α of the following form: $\alpha = \beta + \gamma$, where β is concentrated on a rectifiable Jordan curve E (or on the unit circle Γ) and γ is a discrete measure concentrated on an infinite number of points which lay at the exterior of E. We establish a synthesis on the asymptotic behaviour of orthogonal polynomials or L_p extremal polynomials associated to the measure α. Some recent results will be exposed in this work. We analyze some open problems and we discuss new ideas related to their solving.

2. Synthesis of the studied cases.

Many mathematicians have studied the problem of the asymptotic behaviour of orthogonal polynomials or of L_p extremal polynomials, for instance: Stieltjes [48, 49], Darboux [6], Fejer [8], Szegö [50, 51, 52, 53], Smirnov [45, 46], Krein [25], Korovkine [23], Nevai [37, 38], Van Assche [55, 56], Marcellan [2, 3], Kaliaguine [14, 15], ...

In this section we present a synthesis of all cases of interest already studied depending on different measures α and their different supports F.

2.1. Orthogonal polynomials ($p = 2$).

(I) $F = \Gamma$, where Γ is the unit circle, α is a measure of the following form: α is concentrated on Γ, is absolutely continuous with respect to the measure $|d\xi|$ on the arc, and also satisfies the Szegö condition. This case was studied by Szegö [52, 53] in 1921 and the asymptotic formula he obtained is:

$$L_n(z) \approx \frac{z^n}{\overline{D_\rho}\left(\frac{1}{z}\right)}; \qquad |z| > 1,$$

where D_ρ is the Szegö function and will be defined later on. The case of α not absolutely continuous has been studied by Kolmogoroff [24], Krein [25] and by Gueronimus [11].

(II) $F = [-1, +1]$, α is absolutely continuous with respect to the Lebesgue measure on F (which is $d\alpha = \rho(x)dx$, $0 \le \rho \in L^1(F, dx)$), α satisfying the Szegö condition. This case was studied by Szegö [52, 53] in 1921.

Other mathematicians as Mehler [35], Heine [13], Darboux [6], Stieljes [48, 49], Fejer [8], Perron [41, 42], Adamov [1] and Mehler [35] studied the same problem before Szegö, but for the particular cases of classical orthogonal polynomials (Legendre, Chebychev, Jacobi, Laguerre, Hermite).

(III) $F = E$, where E is a rectifiable Jordan curve, α is absolutely continuous with respect to the measure $|d\xi|$ on the arc (i.e. $d\alpha = \rho(\xi)|d\xi|$, $0 \leq \rho \in L^1(E, |d\xi|)$). This case was studied by Szegö [52, 53] in 1921.

The same problem in the case where E is analytic and $\rho \equiv 1$ was studied by Smirnov [46] (1928), Korovkine [23] (1941), Gueronimus [10] (1952), Souetine [47] (1966). They considered other classes of measures α and curves E.

(IV) $F = \bigcup\limits_{i=1}^{n} E_i$, where E_i is a curve or an arc, α is absolutely continuous on E_i. This case was studied by H. Widom [58] (1968).

(V) $F = \Gamma \cup \{z_k\}_{k=1}^{l}$, where Γ is the unit circle, $z_k \in \mathrm{Ext}(\Gamma)$, α is a measure of the form $\alpha = \beta + \gamma$ where β is concentrated on Γ and γ is a discrete measure with the masses A_k at the points z_k, i.e. $\gamma = \sum\limits_{k=1}^{l} A_k \delta_{z_k}$ (δ_{z_k} is the Dirac measure at the point z_k). This case was studied by Li and Pan [28] (1994).

(VI) $F = E' \cup \{z_k\}_{k=1}^{l}$, where E' is a rectifiable arc, α is a measure of the form: $\alpha = \beta + \gamma = \beta + \sum\limits_{k=1}^{l} A_k \delta z_k$, β is concentrated on E' and is absolutely continuous with respect to the measure $|d\xi|$ on the arc and also satisfies the Szegö condition; γ is a discrete measure with the masses A_k at the points z_k, i.e. $\gamma = \sum\limits_{k=1}^{l} A_k \delta_{z_k}$. This case was studied by Gonchar [12] (1975) for the case $E' = [-1, +1]$, who applied its result to prove the convergence of Padé approximants for some classes of meromorphic functions. Kaliaguine [18] (1995) studied the case of arc.

(VII) $F = [-1, +1] \cup \{z_k\}_{k=1}^{\infty}$, $\alpha = \beta + \gamma$, β being concentrated on $[-1, +1]$, γ being a discrete measure with the masses A_k at the points z_k, i.e. $\gamma = \sum\limits_{k=1}^{\infty} A_k \delta_{z_k}$. This case was studied by Peherstorfer and Yuditskii [40] in 2001.

(VIII) $F = E \cup \{z_k\}_{k=1}^{l}$, where E is a rectifiable Jordan curve belonging to the Gueronimus class, $z_k \in \mathrm{Ext}(E)$, α is a measure of the form: $\alpha = \beta + \sum\limits_{k=1}^{l} A_k \delta_{z_k}$, β being concentrated on E and being of the Szegö type. This case was studied by Kaliaguine and Benzine [14] in 1989.

(IX) $F = E \cup \{z_k\}_{k=1}^{\infty}$, where E is a rectifiable Jordan curve belonging to the Gueronimus class, $z_k \in \mathrm{Ext}(E)$, α is a measure of the form: $\alpha = \beta + \sum_{k=1}^{\infty} A_k \delta_{z_k}$, $\sum_{k=1}^{\infty} A_k < +\infty$, β being concentrated on E and being of the Szegö type. The masses $\{A_k\}_{k=1}^{\infty}$ and the points $\{z_k\}_{k=1}^{\infty}$ satisfy some conditions. This case was studied by Benzine [4] in 1997.

(X) $F = E' \cup \{z_k\}_{k=1}^{\infty}$, where E' is a rectifiable arc, $z_k \in \mathrm{Ext}(E)$, α is a measure of the form: $\alpha = \beta + \sum_{k=1}^{\infty} A_k \delta_{z_k}$, $\sum_{k=1}^{\infty} A_k < +\infty$; β being concentrated on E' and being of the Szegö type. The masses $\{A_k\}_{k=1}^{\infty}$ and the points $\{z_k\}_{k=1}^{\infty}$ satisfy some conditions. This case was studied by Khaldi and Benzine [20] (2001).

(XI) $F = \Gamma \cup \{z_k\}_{k=1}^{\infty}$, where Γ is the unit circle, $z_k \in \mathrm{Ext}(\Gamma)$, α is a measure of the form: $\alpha = \beta + \sum_{k=1}^{\infty} A_k \delta_{z_k}$, $\sum_{k=1}^{\infty} A_k < +\infty$; β being concentrated on Γ and being of the Szegö type. The masses $\{A_k\}_{k=1}^{\infty}$ and the points $\{z_k\}_{k=1}^{\infty}$ verify only the natural condition: $\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$. This condition insures the convergence of the Blaschke product: $B(z) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \overline{z}_k z} \frac{|z_k|^2}{z_k}$. This case was studied by Khaldi and Benzine [19] in 2004.

2.2. L_p extremal polynomials ($p > 0$). In this section we present successive results concerning the asymptotic behaviour of L_p extremal polynomials depending on a considered measure and its support.

(I) $F = [-1, +1]$ and $d\alpha(x) = \rho(x)dx$, where ρ represents a non-negative, integrable on F weight function. The following cases of asymptotic behaviour of L_p extremal polynomials have been solved:

The case of $p = \infty$ and $\rho(x) \equiv 1$ which leads to the classical Chebyschev polynomials [54].

The case of $\rho(x) = t(x)/\sqrt{1 - x^2}$ where $\log t(x)$ represents a Riemann integrable function for which Bernstein [5] found the power asymptotics of the extremal constants $m_{n,p}(\alpha)$.

The case of $1/\rho(x) \in L_r([-1, +1])$, $\forall r > 1$. This important generalization was studied by Lubinsky and Saff [31].

(II) $0 < p < \infty$, F is a closed rectifiable Jordan curve satisfying some condition of smoothness. The absolutely continuous part of α satisfies the Szegö condition. This problem was studied by Gueronimus [10] in 1952.

(III) $0 < p < \infty$, $F = \cup_{k=1}^{\ell} E_k$, where E_k are smooth closed Jordan curves. This case has been studied by Widom [58].

(IV) $0 < p < \infty$, $F = E \cup \{z_k\}_{k=1}^{l}$, where E is a closed rectifiable Jordan curve with some smoothness condition, $z_k \in \mathrm{Ext}(E)$, $\alpha = \beta + \gamma$, where β is concentrated on E and of the Szegö type, and γ is a discrete measure with masses A_k at the points z_k. This problem generalizes the Gueronimus problem (II) where only the case of a curve was considered. It was solved in 1993 by Kaliaguine [15].

(V) $p \geq 1$, $F = E \cup \{z_k\}_{k=1}^{\infty}$, where E is a closed rectifiable Jordan curve with some smoothness condition, $z_k \in \mathrm{Ext}(E)$, $\alpha = \beta + \sum\limits_{k=1}^{\infty} A_k \delta_{z_k}$, where β is concentrated on E and β is of the Szegö type and γ is a discrete measure with masses A_k at the points z_k. This problem generalizes the Kaliaguine problem (IV) where only the case of a finite number of points was considered. It was solved by Laskri and Benzine [26] (see also [21]).

(VI) $0 < p < \infty$, $F = \Gamma \cup \{z_k\}_{k=1}^{\infty}$, where Γ is the unit circle, $z_k \in \mathrm{Ext}(\Gamma)$, the measure $\alpha = \beta + \sum\limits_{k=1}^{\infty} A_k \delta_{z_k}$, $\sum\limits_{k=1}^{\infty} A_k < +\infty$, where β is concentrated on Γ and is of the Szegö type. The masses $\{A_k\}_{k=1}^{\infty}$ and the points $\{z_k\}_{k=1}^{\infty}$ satisfy the conditions: $\sum\limits_{k=1}^{\infty} (1 - |z_k|) < \infty$ and $\dfrac{m_{n,p}(\alpha)}{m_{n,p}(\beta)} \leq \prod\limits_{k=1}^{\infty} |z_k|$, $n > N_0$. This problem generalizes the Khaldi and Benzine result [19], where the authors considered only the case of orthogonal polynomials ($p = 2$). It was solved by Laskri and Benzine [27].

3. Functional spaces used to solve the asymptotic behaviour problems. One can find in the literature several technics to solve the problem of the asymptotic behaviour of orthogonal or L_p extremal polynomials. The technic that we use consists to generate and to study some sequences of extremal problems in Hardy spaces. The limits of the optimal values associated to these extremal problems give us, in general, the asymptotic formula of the L_p extremal polynomials. These technics have been developed mainly by Gueronimus [10], Widom [58], Kaliaguine and Benzine [14], Kaliaguine [15], Benzine [4].

3.1. Hardy spaces inside or outside the unit disk. We denote by $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the interior of the unit disk and by $G = \{w \in \mathbb{C} : |w| > 1\}$ the exterior of the unit disk.

We start with the usual $H^p(\Delta)$ space, $1 \leq p \leq \infty$. A function $f(u) \in H(\Delta)$, analytic in Δ, belongs to the $H^p(\Delta)$ if

$$\|f\|^p_{H^p(\Delta)} = \sup \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \qquad (0 < r < 1). \tag{3}$$

In this case, f has limit values on the unit circle (almost everywhere) and the limit function is from the L_p class.

Changing the of variables by $w = \dfrac{1}{u}$, we can define the space $H^p(G)$. A function $f(w) \in H(G)$ (analytic in G) is from $H^p(G)$ space if $g(u) = f\left(\dfrac{1}{u}\right) \in H^p(\Delta)$.

$H^p(G)$ is a Banach space. Each function $f(w)$ from this space has limit values on the unit circle (almost everywhere) and

$$\|f\|^p_{H^p(G)} = \int_{|w|=1} |f(w)|^p |dw|. \tag{4}$$

For $0 < p < 1$, $H^p(\Delta)$ is not a normed space, but it is a metric space with the distance

$$d(f,g) = \|f-g\|^p_{H^p(\Delta)} = \sup \int_0^{2\pi} |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta < \infty \ (0 < r < 1), \tag{5}$$

and it is a complete space. Each function $f(w)$ of $H^p(G)$ has a decomposition

$$f = B(w)[h(w)]^{\frac{2}{p}},$$

where $B(w)$ is the Blaschke product associated with zeros of $f(w)$ and $h(w) \in H^2(G)$. The function $|f(w)|^p$ has limit values on the unit circle.

3.2. Conformal mapping. Let E be a Jordan closed curve, $\Omega = \operatorname{Ext}(E)$, $G = \{w \in \mathbb{C} : |w| > 1\}$. Let $w = \Phi(z)$ be a function which maps Ω conformally on G in such a manner that $\lim\limits_{z\to\infty} \dfrac{\Phi(z)}{z} > 0$ and $\Phi(\infty) = \infty$. In fact this limit is equal to $\dfrac{1}{C(E)}$, $C(E)$ is the logarithmic capacity of E. Let $\Psi : G \to \Omega$ be the inverse function of Φ. The two functions $\Phi(z)$ and $\Psi(w)$ have a continuous extension to

E and on the unit circle, respectively (Caratheodory Theorem). Their derivatives $\Phi'(z)$ and $\Psi'(w)$ have no zeros in Ω and G and have limit values on E and on the unit circle almost everywhere (with respect to the Lebesgue measure). The functions $\Phi'(z)$ and $\Psi'(w)$ are defined and integrable on E and on the unit circle. This gives the possibility to define analytic functions $(\Phi'(z))^{\frac{1}{p}}$ and $(\Psi'(w))^{\frac{1}{p}}$ for all $p : 0 < p < \infty$.

3.3. Szegö function. As previously we consider a measure of the following type: $\alpha = \beta + \gamma$, where β is concentrated on the curve E. Suppose that the absolutely continuous part $d\beta = \rho(\xi)|d\xi|$, $\xi \in E$, of α, satisfies the following Szegö condition:

$$\int_E (\log \rho(\xi)) Phi'(\xi)|\,|d\xi| > -\infty. \tag{6}$$

Then, one can construct the so-called Szegö function $D_{E,\rho}(z)$ associated with the curve E and the weight function $\rho(\xi)$ with the following properties:

(i) $D_{E,\rho}(z)$ is analytic in Ω, $D_{E,\rho}(z) \neq 0$ in Ω, $D_\rho(\infty) > 0$;

(ii) $D_{E,\rho}(z)$ has limit values (almost everywhere on E) and

$$|D_{E,\rho}(\xi)|^{-p}|\Phi'(\xi)| = \rho(\xi), \quad \xi \in E \text{ (almost everywhere on } E),$$

where $D_{E,\rho}(\xi) = \lim\limits_{z \to \xi} D_{E,\rho}(z)$ (almost everywhere on E), explicitly $D_{E,\rho}(z) = D_G(\Phi(z))$ and

$$D_G(w) = \exp\left\{-\frac{1}{2p\pi}\int_0^{2\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log \frac{\rho(\xi)}{|\Phi'(\xi)|}|\Phi'(\xi)|\,|d\xi|\right\} (\xi = \Psi(e^{i\theta})), \tag{7}$$

3.4. $H^p(\Omega, \rho)$ space. An analytic function in Ω belongs to the $H^p(\Omega, \rho)$ space if

$$\frac{f(\Psi(w))}{D_{E,\rho}(\Psi(w))} \in H^p(G). \tag{8}$$

The spase $H^p(\Omega, \rho)$ $(1 \leq p < \infty)$ is a Banach space. Each function $f(z)$ belonging to $H^p(\Omega, \rho)$ has limit values on E and

$$\|f\|^p_{H^p(\Omega,\rho)} = \int_E |f(\xi)|^p \rho(\xi)|d\xi| = \lim_{\substack{R \to 1 \\ R > 1}} \frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D_{E,\rho}(z)|^p}|\Phi'(z)dz|, \tag{9}$$

where $E_R = \{z \in \Omega : |\Phi(z)| = R\}$.

For $0 < p < 1$, $H^p(\Omega, \rho)$ is as above, a metric space with the quasi-norm

$$\|f\|^p_{H^p(\Omega,\rho)} = \sup \frac{1}{R} \int_{E_R} \frac{|f(z)|^p}{|D_{E,\rho}(z)|^p} \Phi'(z)dz| \tag{10}$$
$$= \lim_{\substack{R\to 1\\ R>1}} \frac{1}{R} \int_{E_R} \frac{|f(z)|^p}{|D_{E,\rho}(z)|^p} |\Phi'(z)dz|.$$

3.5. Extremal problems in $H^P(\Omega, \rho)$ spaces, $(0 < p < \infty)$. In this section we present three extremal problems in $H^P(\Omega, \rho)$ for $0 < p < \infty$ and their solutions.

(I) Let $F = E$, where E is a closed rectifiable Jordan curve and $\Omega = \mathrm{Ext}(E)$. The optimal solution φ^* of the following extremal problem

$$\inf\{\|\varphi\|^p_{H^p(\Omega,\rho)},\ \varphi \in H^p(\Omega, \rho),\ \varphi(\infty) = 1\}, \tag{11}$$

is given by

$$\varphi^*(z) = \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)}, \tag{12}$$

i.e., the infimum (11) denoted $\mu(\beta)$ is reached for $\varphi^* : \mu(\beta) = \|\varphi^*\|^p_{H^P(\Omega,\rho)}$.

(II) Let $F = E \cup \{z_k\}_{k=1}^l$, $z_k \in \Omega$. The optimal solution ψ_l^* of the following extremal problem

$$\inf \left\{ \begin{array}{c} \|\varphi\|^p_{H^p(\Omega,\rho)},\ \varphi \in H^p(\Omega, \rho),\ \varphi(\infty) = 1, \\ \varphi(z_k) = 0,\ k = 1, 2, \ldots, \ell \end{array} \right\}, \tag{13}$$

is given by

$$\psi_\ell^* = \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\ell} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)}, \tag{14}$$

i.e., the infimum (13) denoted $\mu(\alpha_l)$ is reached for $\psi_l^* : \mu(\alpha_l) = \|\psi_l^*\|^p_{H^P(\Omega,\rho)}$. The optimal values of the problems (11) and (13) are connected by

$$\mu(\alpha_\ell) = \left(\prod_{k=1}^{\ell} |\Phi(z_k)| \right)^p \cdot \mu(\beta). \tag{15}$$

(III) Let $F = E \cup \{z_k\}_{k=1}^\infty$, $z_k \in \Omega$. The optimal solution ψ^* of the following extremal problem

$$\inf \left\{ \begin{array}{c} \|\varphi\|^p_{H^p(\Omega,\rho)},\ \varphi \in H^p(\Omega, \rho),\ \varphi(\infty) = 1, \\ \varphi(z_k) = 0,\ k = 1, 2, \ldots \end{array} \right\}, \tag{16}$$

is given by

$$\psi^*(z) = \varphi^*(z) \cdot B_\infty(z) = \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)}, \tag{17}$$

i.e., the infi nimum (16) denoted $\mu(\alpha)$ is reached for $\psi^* : \mu(\alpha) = \|\psi^*\|^p_{H^P(\Omega,\rho)}$.

The optimal values of the problems (11) and (16) are connected by

$$\mu(\alpha) = \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right)^p \cdot \mu(\beta). \tag{18}$$

4. Asymptotic behaviour of L_p extremal polynomials.

Now, we are in position to give in details different results mentioned in Section 2.2, starting from the Gueronimus one [10] from 1952 until the latest one obtained in 2005. We start with some definitions needed.

Let us consider a rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$ [10] defined as follows:

Definition 1. *For a closed Jordan curve, the Faber polynomials $F_n(z)$ are defined by means of the following decomposition:*

$$\Phi^n(z) = F_n(z) + \lambda_n(z),$$

where

$$\lambda_n(z) = O(1/z), \ z \to \infty.$$

One says that a curve E belongs to the class $\mathcal{G}$, (notation $E \in \mathcal{G}$) if

$$\lambda_n \to 0 \textit{ uniformly on } E.$$

In [10] and [23], one can find examples of families of curves belonging to the class $\mathcal{G}$. For instance, such families are the analytic curves or the smooth curves and some others curves.

In what follows, we consider measures α (or α_ℓ) of the following type: α (α_ℓ) is concentrated on the set $E \cup \{z_k\}_{k=1}^{\infty}$ (or $E \cup \{z_k\}_{k=1}^{\ell}$), $z_k \in \Omega :$

$$\alpha = \beta + \gamma \text{ (or } \alpha_\ell = \beta + \gamma_\ell), \tag{19}$$

where β is concentrated on E and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$, i.e.,

$$(20)\qquad d\beta(\xi) = \rho(\xi)|d\xi|,\ \rho : E \to \mathbb{R}_+ \quad \text{and} \quad \int_E \rho(\xi) d\xi < +\infty,$$

γ (or γ_ℓ) is a discrete measure with the masses A_k at the points $z_k \in \mathrm{Ext}(E)$, i.e.,

$$(21)\qquad \gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k} \ (\text{ or } \ \gamma_\ell = \sum_{k=1}^{\ell} A_k \delta_{z_k}),\ A_k > 0, \quad \text{and} \quad \sum_{k=1}^{\infty} A_k < \infty,$$

where each δ_{z_k} is the Dirac measure at the point z_k.

As it has been mentioned in the introduction, we associate to the measures β, α_ℓ and α the extremal constants $m_{n,p}(\beta)$, $m_{n,p}(\alpha_\ell)$, $m_{n,p}(\alpha)$ and the L_p extremal polynomials $T_{n,p,\beta}(z)$, $T_{n,p,\alpha_\ell}(z)$, $T_{n,p,\alpha}(z)$ as follows:

$$(22)\quad m_{n,p}(\beta) = \|T_{n,p,\beta}\|_{L_p(\beta)} = \min\{\|Q_n(z)\|_{L_p(\beta)},\ Q_n(z) = z^n + \cdots\},$$

$$(23)\quad m_{n,p}(\alpha_\ell) = \|T_{n,p,\alpha_\ell}\|_{L_p(\alpha_\ell)} = \min\{\|Q_n(z)\|_{L_p(\alpha_\ell)},\ Q_n(z) = z^n + \cdots\},$$

$$(24)\quad m_{n,p}(\alpha) = \|T_{n,p,\alpha}\|_{L_p(\alpha)} = \min\{\|Q_n(z)\|_{L_p(\alpha)},\ Q_n(z) = z^n + \cdots\},$$

with

$$(25)\qquad \|g\|^p_{L_p(\alpha)} = \int_E |g(\xi)|^p \rho(\xi)|d\xi| + \sum_{k=1}^{\infty} A_k |g(z_k)|^p.$$

4.1. Case $0 < p < \infty$ and where the measure is supported by a curve.

Theorem 1 (1952, Gueronimus [10]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ ($d\beta(\xi) = \rho(\xi)|d\xi|$) satisfies the Szegö condition*

$$\int_E (\log \rho(\xi))|\Phi'(\xi)||d\xi| > -\infty,$$

then

(i) $\displaystyle \lim_{n\to\infty} \frac{m_{n,p}(\beta)}{C(E)^n} = \mu(\beta)^{\frac{1}{p}},$

(ii) $\displaystyle \lim_{n\to\infty} \left\| \frac{T_{n,p,\beta}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \right\|_{H^p(\Omega,\rho)} = 0,$

(iii) $$T_{n,p,\beta}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} [1 + \varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$ uniformly on the compact sets of Ω.

4.2. Case $p = 2$ and where the measure is supported by a curve plus a finite number of points.

Theorem 2 (1989, Kaliaguine and Benzine [14]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ ($d\beta(\xi) = \rho(\xi)|d\xi|$) satisfies the Szegö condition*

$$\int_E (\log \rho(\xi)) |\Phi'(\xi)| \; |d\xi| > -\infty,$$

then

(i) $$\lim_{n\to\infty} \frac{m_{n,2}(\alpha_\ell)}{C(E)^n} = \mu(\alpha_\ell)^{\frac{1}{2}},$$

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,2,\alpha_l}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\ell} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \right\|_{H^p(\Omega,\rho)} = 0,$$

(iii) $$T_{n,2,\alpha_l}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\ell} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} [1 + \varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$ uniformly on the compact sets of Ω.

4.3. Case $0 < p < \infty$ and where the measure is supported by a curve plus a finite number of points.

Theorem 3. (1993, Kaliaguine [15]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ ($d\beta(\xi) = \rho(\xi)|d\xi|$) satisfies the Szegö condition*

$$\int_E (\log \rho(\xi)) |\Phi'(\xi)| \; |d\xi| > -\infty,$$

then

(i) $$\lim_{n\to\infty} \frac{m_{n,p}(\alpha_\ell)}{C(E)^n} = \mu(\alpha_\ell)^{\frac{1}{p}},$$

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha_l}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\ell} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \right\|_{H^p(\Omega,\rho)} = 0,$$

(iii) $$T_{n,p,\alpha_l}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\ell} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} [1+\varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$ *uniformly on the compact sets of* Ω.

4.4. Case $p = 2$ and where the measure is supported by a curve plus an infinite number of points.

Theorem 4 (1997, Benzine [4]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ ($d\beta(\xi) = \rho(\xi)|d\xi|$) satisfies the Szegö condition*

$$\int_E (\log \rho(\xi))|\Phi'(\xi)| \; |d\xi| > -\infty,$$

$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$ *is such that*

(26) $$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1 \right) < \infty,$$

and

(27) $$\frac{m_{n,2}(\alpha)}{m_{n,2}(\beta)} \leq \left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right),$$

then

(i) $$\lim_{n\to\infty} \frac{m_{n,2}(\alpha)}{C(E)^n} = \mu(\alpha)^{\frac{1}{2}},$$

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,2,\alpha}(z)}{C(E)^n . \Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \right\|_{H^p(\Omega,\rho)} = 0,$$

(iii) $$T_{n,2,\alpha}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} [1+\varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$ *uniformly on the compact sets of* Ω.

Remark. In Benzine [4], we have found the condition

$$\frac{\int_E |T_{n,2}(\xi,\alpha)|^2 . \rho(\xi)|d\xi|}{\sum_{k=1}^{\infty} A_k |T_{n,2}(z_k,\alpha)|^2} \geq \frac{1}{\left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right)^2 - 1},$$

but we have not found the condition (27), due to a calculation mistake.

4.5. Case $p = 2$ and where the measure obeys weaker conditions and is supported by a curve plus an infinite number of points.

Theorem 5. (2000, Khaldi and Benzine [18]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ ($d\beta(\xi) = \rho(\xi)|d\xi|$) satisfies the Szegö condition*

$$\int_E (\log \rho(\xi))|\Phi'(\xi)|\ |d\xi| > -\infty,$$

$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$ *is such that*

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1\right) < \infty,$$

and

$$\frac{m_{n,2}(\alpha_\ell)}{m_{n,2}(\beta)} \leq \left(\prod_{k=1}^{\ell} |\Phi(z_k)|\right), \tag{28}$$

then

(i) $\displaystyle \lim_{n\to\infty} \frac{m_{n,2}(\alpha)}{C(E)^n} = \mu(\alpha)^{\frac{1}{2}},$

(ii) $\displaystyle \lim_{n\to\infty} \left\| \frac{T_{n,2,\alpha}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \right\|_{H^p(\Omega,\rho)} = 0,$

(iii) $\displaystyle T_{n,2,\alpha}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} [1 + \varepsilon_n(z)],$

$\varepsilon_n(z) \to 0$ *uniformly on the compact sets of* Ω.

Remark. The condition (28) is weaker than the condition (27) and allows to find the sets of the points $\{z_k\}$ and masses $\{A_k\}$ satisfying (28) and (27).

4.6. Case $p = 2$ and where the measure is supported by a circle plus an infnite number of points.

Theorem 6 (2004, Khaldi and Benzine [19]). *Let Γ denote the unit circle. Suppose that $\rho(\xi)$ $(d\beta(\xi) = \rho(\xi)|d\xi|)$ satisfies the Szegö condition*

$$\int_\Gamma (\log \rho(\xi))|d\xi| > -\infty,$$

and $\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$ is such that

$$\left(\sum_{k=1}^{\infty} |z_k| - 1 \right) < \infty,$$

then

(i) $\quad \lim_{n\to\infty} m_{n,2}(\alpha) = \mu(\alpha)^{\frac{1}{2}}$,

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,2,\alpha}(z)}{z^n} - \frac{D_{G,\rho}(z)}{D_{G,\rho}(\infty)} . \prod_{k=1}^{\infty} \frac{z - z_k}{z\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} \right\|_{H^p(G,\rho)} = 0,$$

(iii) $$T_{n,2,\alpha}(z) = z^n \cdot \frac{D_{G,\rho}(z)}{D_{G,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{z - z_k}{z\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} [1 + \varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$ uniformly on the compact sets of G.

4.7. Case $0 < p < \infty$ and where the measure is supported by a curve plus an infinite number of points.

Theorem 7 (2004, Laskri and Benzine [26]) and (2004, Khaldi [21]). *Let E be a closed rectifiable Jordan curve belonging to the Gueronimus class $\mathcal{G}$. Suppose that $\rho(\xi)$ $(d\beta(\xi) = \rho(\xi)|d\xi|)$ satisfies the Szegö condition*

$$\int_E (\log \rho(\xi))|\Phi'(\xi)|\ |d\xi| > -\infty,$$

$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$ is such that

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1 \right) < \infty,$$

and

$$n > N_0 : \frac{m_{n,p}(\alpha)}{m_{n,p}(\beta)} \leq \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right), \quad (\textit{Laskri and Benzine}) \tag{29}$$

or

$$\forall n,\ \forall \ell : \frac{m_{n,p}(\alpha_\ell)}{m_{n,p}(\beta)} \leq \left(\prod_{k=1}^{\ell} |\Phi(z_k)|\right), \quad (\textit{Khaldi}),$$

then

(i) $\displaystyle \lim_{n\to\infty} \frac{m_{n,p}(\alpha)}{C(E)^n} = \mu(\alpha)^{\frac{1}{p}},$

(ii) $\displaystyle \lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z)-\Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)}-1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \right\|_{H^p(\Omega,\rho)} = 0,$

(iii) $\displaystyle T_{n,p,\alpha}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{\Phi(z)-\Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)}-1} \cdot \frac{|\Phi(z_k)|^2}{\Phi(z_k)} [1+\varepsilon_n(z)],$

$\varepsilon_n(z) \to 0$ *uniformly on the compact sets of* Ω.

4.8. Case $0 < p < \infty$ and where the measure is supported by the circle plus an infinite number of points. Before giving Theorem 8, we define the class of measures $\mathcal{L}$.

Definition 2. *Let* $\alpha = \beta + \gamma$. *We say that the measure* α *belongs to the class* $\mathcal{L}$ *(notation* $\alpha \in \mathcal{L}$*) if the absolute part* β $(\beta(\xi) = \rho(\xi)|d\xi|)$ *and the discrete part* γ $\left(\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}\right)$ *of* α *satisfy*

$$\int_\Gamma (\log \rho(\xi))|d\xi| > -\infty,$$

$$\left(\sum_{k=1}^{\infty} |z_k| - 1\right) < \infty, \tag{30}$$

and

$$n > N_0 : \frac{m_{n,p}(\alpha)}{m_{n,p}(\beta)} \leq \left(\prod_{k=1}^{\infty} |z_k|\right), \tag{31}$$

The condition (30) guarantees the convergence of the Blaschke product $B_\infty(z)$ associated with the points $\{z_k\}_{k=1}^{\infty}$,

$$B_\infty(z) = \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k}.$$

If the condition (31) is satisfied, we have

$$\limsup_{n\to\infty} \; m_{n,p}(\alpha) \le (\mu(\alpha))^{1/p}. \tag{32}$$

More detailed discussion of the condition (32) is presented in the following section. Recently with this condition Laskri and Benzine have obtained the following result:

Theorem 8 (2004, Laskri and Benzine [7]). *Let Γ be the unit circle and $\alpha \in L$, then*

(i) $\lim\limits_{n\to\infty} m_{n,p}(\alpha) = (\mu(\alpha))^{1/p}$,

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha}(z)}{z^n} - \frac{D_{G,\rho}(z)}{D_{G,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} \right\|_{H^p(G,\rho)} = 0,$$

(iii) $$T_{n,p,\alpha}(z) = z^n \left[\frac{D_{G,\rho}(z)}{D_{G,\rho}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} + \varepsilon_n(z) \right],$$

$\varepsilon_n(z) \to 0$, *uniformly on the compact sets of G.*

5. Basic ideas used in the proofs and difficulties of the generalization of the finite set case to the infinite one and also $p = 2$ to $0 < p < \infty$.

5.1. Technics of proofs. The main step of the proofs of Theorems 1–8 consists in proving the asymptotic formulas of the extremal constants $m_{n,p}(\beta)$, $m_{n,2}(\alpha_\ell)$, $m_{n,p}(\alpha_\ell)$, $m_{n,2}(\alpha)$ and $m_{n,p}(\alpha)$.

We must carrefully follow the behaviour of these constants in the $H^p(\Omega,\rho)$ space to obtain finally their asymptotical form. For instance in the proof of the Theorem 1, since

$$\frac{m_{n,p}(\beta)}{C(E)^n} = \left\| \frac{T_{n,p,\beta}}{C(E)^n.\Phi^n(z)} \right\|_{H^p(\Omega,\rho)}, \tag{33}$$

and

$$\mu(\beta)^{\frac{1}{p}} = \left\| \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \right\|_{H^p(\Omega,\rho)},$$

then (i) implies

$$\left\| \frac{T_{n,p}(z,\beta)}{C(E)^n.\Phi^n(z)} \right\|_{H^p(\Omega,\rho)} \to \left\| \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \right\|_{H^p(\Omega,\rho)}. \tag{34}$$

It is relatively easy to establish the proofs of Theorems 1–8 from (i) to (ii). In the case $p = 2$ the proof is based on the applcation of the parallelogram rule. The case $p \neq 2$ is more difficult. For $1 \leq p < \infty$ it is based on the applcation of the Clarkson inequality. Different cases of $0 < p < 1$ where proved as follows: the case of a curve was solved by Gueronimus [10] using the Keldysh lemma [17], the case of a curve plus a finite set of points was solved by Kaliaguine [15, p. 235, Lemma 2.1] with help of the extension of the Keldysh lemma, the case of a circle plus an infinite set of points was solved recently by Bello Hernandez, Marcellan and Minguez [2, p. 430, Theorem 2] by using a new extension of the Keldysh lemma. A similar lemma can be also extented to solve the case of a curve (or a circle) with an infinite set of points.

It is easy to proove Theorems 1–8 from (ii) to (iii). The proofs are based on a direct application of the following

Lemma 1 [15]. *If $f(z) \in H^p(\Omega,\rho)$ (or $H^p(G,\rho)$) and $K \subset \Omega$ (or $K \subset G$), K is compact, then there exists a constant $C(K)$ ($C(K)$ depending only on K) such that:*

$$\sup_k |f(z| \leq C(K)\|f\|_{H^p(\Omega,\rho)} \quad (or \ \sup_k |f(z)| \leq C(K)\|f\|_{H^p(G,\rho)}).$$

Return to the asymptotic behaviour of extremal constants. The study of the asymptotic behaviour of extremal constants $m_{n,p}(\beta)$, $m_{n,2}(\alpha_\ell)$, $m_{n,p}(\alpha_\ell)$, $m_{n,2}(\alpha)$, and $m_{n,p}(\alpha)$ is fundamental for the research of the asymptotic formula of L_p extremal polynomials $T_{n,p,\beta}$, $T_{n,2,\alpha_\ell}$, T_{n,p,α_ℓ}, $T_{n,2,\alpha}$, $T_{n,p,\alpha}$. To establish asymptotic formulas of extremal constants, we always proceed by proving two inequalities. For example, in the case of $m_{n,p}(\alpha)$, we prove the following inequalities

$$\limsup_{n\to\infty} \frac{m_{n,p}(\alpha)}{(C(E))^n} \leq (\mu(\alpha))^{1/p}, \tag{35}$$

and

$$(\mu(\alpha))^{1/p} \leq \liminf_{n\to\infty} \frac{m_{n,p}(\alpha)}{(C(E))^n}. \tag{36}$$

The finite case

In Theorem 2 ($p = 2$, curve plus a finite number of points), it is difficult to obtain the formula (36).

In Theorem 3 ($0 < p < \infty$, curve plus a finite number of points), the proof is identical to the case $p = 2$.

The infinite case

In Theorems 4, 5, 6 ($p = 2$, curve or circle plus a finite number of points), it is difficult to obtain the formula (35).

In Theorem 7 ($0 < p < \infty$, curve plus an infinite number of points), it is also difficult to obtain the formula (35). We cannot proceed as in the case $p = 2$, because this technic ([19]) uses the orthogonality, which is not satisfied in the case $p \neq 2$.

However, we can use a new technic due to Peherstorfer and Yuditskii ([40]). To prove the asymptotic formula for the orthogonal polynomials, they use a measure concentrated on a segment plus an infinite number of points. These authors prove a similar formula to (35) by using the extremal properties of orthogonal polynomials.We think that with some modifications, this technic is available for establishing formula (35) in the case of L_p extremal polynomials. We will just give a sketch.

Peherstorfer and Yuditskii introduced the functions D_ε, $F_{\varepsilon,\eta}$ and the polynomials $P_{n,\varepsilon,\eta}$ as approximations, in a certain sense of D, $\dfrac{1}{D_\varepsilon}$ and $T_{n,2,\alpha}$ respectively. Without details, one can describe these functions as follows (see [40, p. 3217] for more precisions).

$|D_\varepsilon|$ is a smooth function such that $|D_\varepsilon| \geq 1$ and

$$\int_\Gamma \left||D|^2 - |D_\varepsilon|^2\right| |d\xi| < \varepsilon \qquad (\varepsilon > 0). \tag{37}$$

For $\eta > 0$, $|F_{\varepsilon,\eta}|$ is a smooth function which coincides with $\dfrac{1}{|D_\varepsilon|}$ on $\Gamma \setminus (\widetilde{E}_s \cup \widetilde{E}_+ \cup \widetilde{E}_-)$ and equals to η on $E_s \setminus (\widetilde{E}_+ \cup \widetilde{E}_-)$. Furthermore $|F_{\varepsilon,\eta}|$ coincides with $|\xi \pm 1|^2$, for $\xi \in E_\pm$. E_s, $\widetilde{E}_s$, $E_\pm$, $\widetilde{E}_\mp \subset \Gamma$ satisfy

$$E_\pm = \left\{\xi \in \Gamma : |\xi \pm 1|^2 \leq \frac{\eta}{2}\right\},$$

$$\begin{aligned}
\widetilde{E}_{\mp} &= \left\{\xi \in \Gamma : |\xi \pm 1|^2 \leq \eta\right\}, \\
|E_s| &= \int_{E_s} |d\xi| \leq \eta, \\
|\widetilde{E}_s| &= \int_{\widetilde{E}_s} |d\xi| \leq 2\eta.
\end{aligned}$$

$F_{\varepsilon,\eta}$ possess the fundamental properties (see [40], formula (2.13))

$$F_{\varepsilon,\eta} = \frac{1}{D(0)} + o(1) = \frac{1}{D_\rho(\infty)} + o(1), \qquad \eta \to 0, \qquad \varepsilon \to 0. \tag{38}$$

As in [40], we consider the polynomials $P_{n,\varepsilon,\eta}$, defined as follows:

$$P_{n,\varepsilon,\eta}(\xi) = \xi^{-n} Q_{n,\varepsilon,\eta}(\xi) + \xi^n Q_{n,\varepsilon,\eta}(1/\xi), \tag{39}$$

where

$$Q_{n,\varepsilon,\eta}(\xi) = q_{0,\varepsilon,\eta} + \cdots + q_{n,\varepsilon,\eta}\xi^n,$$

and

$$(BF_{\varepsilon,\eta})(\xi) = Q_{n,\varepsilon,\eta}(\xi) + \xi^{n+1} g_{n,\varepsilon,\eta}(\xi), \qquad g_{n,\varepsilon,\eta}(\xi) \in H^\infty, \tag{40}$$

$B(z)$ is the classical Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{z - \widetilde{z}_k}{z.\overline{\widetilde{z}_k} - 1} \cdot \frac{|\widetilde{z}_k|^2}{\widetilde{z}_k}, \qquad \widetilde{z}_k = \frac{1}{z_k}, \quad z \in U. \tag{41}$$

Conjecture 0. We will conjecture that it is possible to obtain (following a similar method as in [40, p. 3217])

$$\|P_{n,\varepsilon,\eta}\|_{L_p(\alpha,F)} \leq 1 + C\varepsilon + o(1), \quad n \to \infty, \quad C \text{ is a constant.} \tag{42}$$

If the formula (42) is true, then it implies the formula (32). Indeed, if we notice that

$$P_{n,\varepsilon,\eta}(z) = (B.F_{\varepsilon,\eta})(0).z^n + \cdots,$$

by using

$$m_{n,p}(\alpha) = \|T_{n,p,\alpha}\|_{L_p(\alpha,F)} = \inf_{Q_n \in \mathcal{P}_{n,1}} \|Q_n\|_{L_p(\alpha,F)},$$

we get, with the help of (42)

$$\begin{aligned}
m_{n,p}(\alpha) &\leq \left\| \frac{1}{(B.F_{\varepsilon,\eta})(0)} P_{n,\varepsilon,\eta} \right\|_{L_p(\alpha,F)} = \frac{1}{(B.F_{\varepsilon,\eta})(0)} \|P_{n,\varepsilon,\eta}\|_{L_p(\alpha,F)} \\
&\leq \frac{1 + C\varepsilon + o(1)}{(B.F_{\varepsilon,\eta})(0)}.
\end{aligned}$$

Finally, by using

$$\mu(\alpha)=\left(\prod_{k=1}^{\infty}|z_k|\right)^p\cdot\mu(\beta), \tag{43}$$

with

$$\mu(\beta)=\|\varphi^*\|^p_{H^p(G,\rho)}=\left\|\frac{D_\rho(\infty)}{D_\rho}\right\|^p_{H^p(G,\rho)}=[D_\rho(\infty)]^p=[D(0)]^p, \tag{44}$$

we obtain

$$\limsup_{n\to\infty} m_{n,p}(\alpha)\le\frac{D_\rho(\infty)}{B(0)}=D_\rho(\infty).\prod_{k=1}^{\infty}|z_k|=[\mu(\beta)]^{\frac{1}{p}}\left[\prod_{k=1}^{\infty}|z_k|\right]=(\mu(\alpha))^{1/p}.$$

To illustrate this section, we give the proof of Theorem 8 (see also [27] for more details). Before this, let us recall the lemma which we use many times in Hardy spaces $H^p(\Omega,\rho)$ (or $H^p(G,\rho)$).

Lemma 2 [15]**.** *Let $\{f_n\}$ be a sequence of functions in $H^p(G,\rho)$ and*

(i) $f_n\to f$ *uniformly on the compact sets of* G,

(ii) $\|f_n\|^p_{H^p(G,\rho)}\le M$ (const),

then

$$f\in H^p(G,\rho)\quad\textit{and}\quad\|f\|^p_{H^p(G,\rho)}\le\liminf_{n\to\infty}\|f_n\|^p_{H^p(G,\rho)}. \tag{45}$$

Proof of Theorem 8.
Proof of (i). Since $p>0$ and $\alpha\in\mathcal{L}$, then

$$\limsup_{n\to\infty} m_{n,p}(\alpha)\le(\mu(\alpha))^{1/p}. \tag{46}$$

It remains to prove that

$$(\mu(\alpha))^{1/p}\le\liminf_{n\to\infty} m_{n,p}(\alpha). \tag{47}$$

We will present two proofs of the above inequality.

First proof of (47). The extremal properties of $T_{n,p,\alpha}(z)$ and $T_{n,p,\alpha_\ell}(z)$ imply

$$m_{n,p}(\alpha)=\|T_{n,p,\alpha}\|_{L_p(\alpha,F)}\ge\|T_{n,p,\alpha}\|_{L_p(\alpha_\ell,F_\ell)} \tag{48}$$

$$\geq \|T_{n,p,\alpha_\ell}\|_{L_p(\alpha_\ell,F_\ell)} = m_{n,p}(\alpha_\ell).$$

By (48) we get

$$m_{n,p}(\alpha) \geq m_{n,p}(\alpha_\ell), \quad \forall p > 0, \quad \forall \ell. \tag{49}$$

Combining (49) with Theorem 2.2 of [15], we obtain

$$\liminf_{n\to\infty}(m_{n,p}(\alpha)) \geq (\mu(\alpha_\ell))^{1/p}, \quad \forall p > 0, \quad \forall \ell. \tag{50}$$

Now, by using the fact that

$$\mu(\alpha_\ell) = \mu(\beta) \cdot \left(\prod_{k=1}^{\ell} |z_k|\right)^p,$$

(see [15, formula (1.9)]), and letting $l \to \infty$, we have

$$\liminf_{n\to\infty}(m_{n,p}(\alpha)) \geq \mu(\beta)^{1/p} \cdot \left(\prod_{k=1}^{\infty} |z_k|\right) = (\mu(\alpha))^{1/p}. \tag{51}$$

S e c o n d p r o o f o f (47). By putting

$$\phi^*_{n,p} = T_{n,p,\alpha}(z)/z^n, \tag{52}$$

and by using (46), we get

$$\|\phi^*_{n,p}\|_{H^p(G,\rho)} \leq M = \text{const} . \tag{53}$$

Let $M^* = \liminf_{n\to\infty} \|\phi^*_{n,p}\|^p_{H^p(G,\rho)}$. We have

$$M^* = \lim_{n\to\infty, n\in N_1} \|\phi^*_{n,p}\|^p_{H^p(G,\rho)}. \tag{54}$$

This result and Lemma 1 imply that $\{\phi^*_{n,p},\ n \in N_1\}$ is a normal family in G. Then we can find a function $\psi(z)$ which is the uniform limit (on the compact subsets of G) of some subsequences $\{\phi^*_{n,p},\ n \in N_2\}$ of $\{\phi^*_{n,p},\ n \in N_1\}$.

From Lemma 2: $\psi \in H^p(G,\rho)$ and

$$\|\psi\|^p_{H^p(G,\rho)} \leq \liminf_{n\to\infty} \|\phi^*_{n,p}\|^p_{H^p(G,\rho)}. \tag{55}$$

On the other hand, $\psi(\infty) = 1$ and $\psi(z_k) = 0$, $k = 1, 2, \ldots$. Finally, with (55) we get

$$\mu(\alpha) \leq \|\psi\|^p_{H^p(G,\rho)} \leq \liminf_{n\to\infty} \|\phi^*_{n,p}\|^p_{H^p(G,\rho)} \leq \liminf_{n\to\infty}(m_{n,p}(\alpha))^p. \tag{56}$$

Consequently (46) and (56) imply

$$(\mu(\alpha))^{1/p} \leq \liminf_{n\to\infty} m_{n,p}(\alpha) \leq \liminf_{n\to\infty} m_{n,p}(\alpha) \leq (\mu(\alpha))^{1/p},$$

then (i) of Theorem 8 follows.

P r o o f o f (ii). The function

$$\Psi_n = \frac{1}{2}(\phi^*_{n,p} + \psi^*),$$

where

$$\|\psi^*\|^p_{H^p(G,\rho)} = \mu(\alpha),$$

tends to the following limits

$$\Psi_n(\infty) = 1 \text{ and } \lim_{n\to\infty} \Psi_n(z_k) = 0, \quad k = 1, 2, \ldots$$

As in (i), we have

$$\liminf_{n\to\infty} \|\Psi_n\|^p_{H^p(G,\rho)} \geq \mu(\alpha). \tag{57}$$

Finally, (ii) follows from the Clarkson inequality and an extension of the Keldysh lemma.

For $1 \leq p \leq 2$:

$$\begin{aligned}
&\left[\int_\Gamma \left|\frac{1}{2}(\phi^*_{n,p} + \psi^*)\right|^p \rho(\xi)|d\xi|\right]^{1/p-1} + \left[\int_\Gamma \left|\frac{1}{2}(\phi^*_{n,p} - \psi^*)\right|^p \rho(\xi)|d\xi|\right]^{1/p-1} \\
\leq\ & \left[\frac{1}{2}\int_\Gamma |\phi^*_{n,p}|^p \rho(\xi)|d\xi| + \frac{1}{2}\int_\Gamma |\psi^*|^p \rho(\xi)|d\xi|\right]^{1/p-1}.
\end{aligned}$$

For $2 \leq p < \infty$:

$$\begin{aligned}
&\int_\Gamma \left|\frac{1}{2}(\phi^*_{n,p} + \psi^*)\right|^p \rho(\xi)|d\xi| + \int_\Gamma \left|\frac{1}{2}(\phi^*_{n,p} - \psi^*)\right|^p \rho(\xi)|d\xi| \\
\leq\ & \frac{1}{2}\int_\Gamma |\phi^*_{n,p}|^p \rho(\xi)|d\xi| + \frac{1}{2}\int_\Gamma |\psi^*|^p \rho(\xi)|d\xi|.
\end{aligned}$$

For $0 < p < 1$: We use the extension of the Keldysh lemma due to Bello Hernandez, Marcellan and Minguez ([2], Theorem 2, p. 430) in the case of unit circle plus an infinite number of points. If we adapt this result to our case, we obtain the following version of the Keldysh lemma.

Lemma 3. *Let* $\{z_k\}_{k=1}^{\infty}$ *be a set of points in* G, $\alpha = \beta + \gamma$ *where* $\alpha \in \mathcal{L}$ *and* $\{f_n\} \subset H^p(G, \rho)$, $0 < p < \infty$. *Let*

$$\widetilde{f_n} = \frac{f_n}{\varphi^*}, \text{ where } \varphi^*(z) = \frac{D_\rho(\infty)}{D_\rho(z)}.$$

If

(a) $\lim\limits_{n\to\infty} \widetilde{f_n}(\infty) = 1,$

(b) $\lim\limits_{n\to\infty} \widetilde{f_n}(z_k) = 0,\ k = 1, 2, \ldots;$

(c) $\sum\limits_{k=1}^{\infty} (|z_k| - 1) < +\infty,$

(d) $\lim\limits_{n\to\infty} \|f_n\|_{H^p(G,\rho)} = D_\rho(\infty) \prod\limits_{k=1}^{\infty} |z_k|,$

then

$$\lim_{n\to\infty} \left\| f_n - \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} .\varphi^* \right\|_{H^p(G,\rho)} = \lim_{n\to\infty} \|f_n - (B_\infty.\varphi^*)\|_{H^p(G,\rho)} = 0.$$

We get (ii) of Theorem 8 in the case $0 < p < 1$, by applying Lemma 3 to the sequence $\{f_n = \phi^*_{n,p}\} \subset H^p(G, \rho)$.

We have

$$\phi^*_{n,p}(\infty) = 1 \text{ and } \varphi^*(\infty) = 1.$$

Hence (a) follows. On the other hand, (b) is a consequence of the fact that $\varphi^*(z_k) \neq 0$ and

$$\lim_{n\to\infty} \phi^*_{n,p}(z_k) = 0, \quad k = 1, 2, \ldots.$$

(c) is exactly the condition (30). We obtain (d) by considering (43), (44) and the fact that

$$\lim_{n\to\infty} \|\phi^*_{n,p}\|_{H^p(G,\rho)} = \lim_{n\to\infty} m_{n,p}(\alpha) = (\mu(\alpha))^{1/p}.$$

Then (ii) of Theorem 8 is proved.

P r o o f of (iii). It is clear that the formulas of type (ii) combined with Lemma 1 give the asymptotic formulas of orthogonal polynomials or L_p extremal polynomials. Indeed, applying Lemma 1 for the functions

$$\varepsilon_n(z) = \frac{T_{n,p,\alpha}(z)}{z^n} - \psi^*(z), \tag{58}$$

then for all compact $K \subset G$, we get

$$\sup_{z\in K} |\varepsilon_n(z)| \leq C(K)\|\varepsilon_n\|^p_{H^p|(G,\rho)} \to_{n\to\infty} 0.$$

This completes the proof of (iii).

5.3. Open problems and conjectures. In this section, we present some open problems, conjectures and ideas for their solving in question.

Open problem 1. *Consider the asymptotic behaviour of L_p extremal polynomials associated with a measure concentrated on a closed rectifiable Jordan curve E. Gueronimus* [10] *established the asymptotic formula of L_p extremal polynomials for a class $\mathcal{G}$ of curves* (*see Definition* 1 *and Theorem* 1). *The question is: Can we get the same result for a larger class than Gueronimus class $\mathcal{G}$. We denote by $\mathcal{BM}$ the class of curves for which we have*

$$\lim_{n\to\infty} \frac{m_{n,p}(\beta)}{C(E)^n} = \mu(\beta)^{\frac{1}{p}},$$

and by $\mathcal{BH}$ the set of all the rectifiable Jordan curve. We have

$$\mathcal{G} \subset \mathcal{BM} \subset \mathcal{BH}.$$

Can we obtain at the limit

$$\mathcal{BM} = \mathcal{BH}?$$

Conjecture 1. *$\mathcal{G} \subsetneq \mathcal{BM} \subsetneq \mathcal{BH}$ (i.e., $\mathcal{G} \subset \mathcal{BM}$ and $\mathcal{G} \neq \mathcal{BM}$; $\mathcal{BM} \subset \mathcal{BH}$ and $\mathcal{BM} \neq \mathcal{BH}$).*

One obtains the following result:

Theorem 1 (bis). *Let E be a closed curve belonging to the class $\mathcal{BM}$ and $0 < p < \infty$. Then*

1) $$\lim_{n\to\infty} \left\| \frac{T_{n,p,\beta}(z)}{C(E)^n.\Phi^n(z)} - \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} \right\|_{H^p(\Omega,\rho)} = 0,$$

2) $$T_{n,p,\beta}(z) = C(E)^n \cdot \Phi^n(z) \frac{D_{E,\rho}(z)}{D_{E,\rho}(\infty)} [1 + \varepsilon_n(z)],$$

$\varepsilon_n(z) \to 0$, *uniformly on the compact sets of Ω.*

Proof of Theorem 1 (bis). $E \in \mathcal{BM}$, then for $0 < p < \infty$ we have

$$\lim_{n\to\infty} \frac{m_{n,p}(\beta)}{C(E)^n} = \mu(\beta)^{\frac{1}{p}}.$$

The proofs of 2) and 3) are exactly identic to the proofs of (ii) and (iii) of Theorem 1 (see [10] for more details). Then the main problem of Conjecture 1 is to prove that $\mathcal{G} \neq \mathcal{BM}$, $\mathcal{BM} \neq \mathcal{BH}$ and to characterize $\mathcal{BM}$. □

Open problem 2. *Study of the asymptotic behaviour of L_p extremal polynomials associated with the measures α of the following form: $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$, β_a and β_s is the absolutely continuous and singular part respectively of β, β is concentrated on E or Γ, γ is a discrete measure concentrated on an infinite of points z_k belonging to the exterior of E or of Γ.*

Theorem 8 is a particular case of this problem. It corresponds to the case $\beta_s = 0$.

Before giving Conjecture 2, let us recall the definition of the Szegö function which corresponds to the measure $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$.

We suppose that the absolutely continuous part β_a of the measure β satisfies the following Szegö condition:

$$\int_E (\log \beta'_a(\xi))|\Phi'(\xi)||d\xi| > -\infty. \tag{59}$$

This allows us to construct, as in the absolutely continuous case, D_{E,β'_a} which is called the Szegö function, associated with the curve E and β and possesses the following properties:

a) D_{E,β'_a} is analytic in Ω, $D_{E,\beta'_a} \neq 0$ in Ω and $D_{E,\beta'_a} > 0$,

b) $D_{E,\rho}$ has a limit value almost everywhere on E and

$$\frac{|\Phi'(\xi)|}{|D_{E,\beta'_a}|^p} = |\beta'_a(\xi)|, \text{ (almost everywhere on } E). \tag{60}$$

Definition 3. *If the measure $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$ is such that β_a verifies the condition* (59) *and its discrete part verifies*

$$\left(\sum_{k=1}^{\infty} |z_k| - 1\right) < \infty,$$

then we say that $\alpha \in \mathcal{BA}$.

We are in position to present Conjecture 2.

Theorem 8 (bis) (Conjecture 2). *Let* Γ *be the unit circle and* $\alpha = \beta_a + \beta_s + \gamma$, *such that* $\alpha \in \mathcal{BA}$, *and* D_{Γ,β'_a} *the Szegö function associated with the circle* Γ *and the measure* β, *then*

(i) $\lim_{n\to\infty} m_{n,p}(\alpha) = (\mu(\alpha))^{1/p}$,

(ii) $$\lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha}(z)}{z^n} - \frac{D_{\Gamma,\beta'_a}(z)}{D_{\Gamma,\beta'_a}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} \right\|_{H^p(G,\rho)} = 0,$$

(iii) $$T_{n,p,\alpha}(z) = z^n \left[\frac{D_{\Gamma,\beta'_a}(z)}{D_{\Gamma,\beta'_a}(\infty)} \cdot \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} + \varepsilon_n(z) \right],$$

$\varepsilon_n(z) \to 0$, *uniformly on the compact sets of* G.

Idea on the proof of Theorem 8 (bis). (i) As in the absolutely continuous case (Theorem 8), for the proof of (i) we start establishing two inequalities:

$$\limsup_{n\to\infty} \frac{m_{n,p}(\alpha)}{(C(E))^n} \le (\mu(\alpha))^{1/p}, \tag{61}$$

and

$$(\mu(\alpha))^{1/p} \le \liminf_{n\to\infty} m_{n,p}(\alpha). \tag{62}$$

To prove (61) and as in the absolutely continuous case, we can follow closely the proof of Perherstofer and Yuditskii [40]. These authors use measures of the form $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$, β is concentrated on a segment and γ is a discrete measure concentrated on an infinite number of points outside the segment. It is more difficult to prove the formula (62).

(ii) If (i) can be established, it will be not difficult to prove (ii) because in this case we can use the extension of the Keldych Lemma (see [2]). This extension has been established in the case of the general measure α of the form: $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$, where β is concentrated on the unit circle and γ is a discrete measure concentrated on an infinite number of points outside the circle and $0 < p < \infty$.

Open problem 3. *Study of the asymptotic behaviour of* L_p *extremal polynomials* $(0 < p < \infty)$ *associated to the measures of the following form* $\alpha =$

$\beta+\gamma$, where β is concentrated on the arc (segment), and is absolutely continuous or not, and γ is a discrete measure concentrated on an infinite number of points $\{z_k\}_{k=1}^{\infty}$.

This is a difficult problem. It has been only solved for the particular case $p = 2$ for a finite number of points and an absolutely continuous measure by Kaliaguine [16].

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