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AN EXTENSION OF LORENTZ'S ALMOST CONVERGENCE AND APPLICATIONS IN BANACH SPACES

S. Mercourakis and G. Vassiliadis

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Abstract. We investigate an extension of the almost convergence of G. G. Lorentz requiring that the means of a bounded sequence converge uniformly on a subset M of $\mathbb{N}$. We also present examples of sequences $\alpha \in \ell^\infty(\mathbb{N})$ whose sequences of translates $(T^n\alpha)_{n\geq 0}$ (where T is the left-shift operator on $\ell^\infty(\mathbb{N})$) satisfy:
(a) $T^n\alpha, n \geq 0$ generates a subspace $E(\alpha)$ of $\ell^\infty(\mathbb{N})$ that is isomorphically embedded into c_0 while α is not almost convergent.
(b) $T^n\alpha, n \geq 0$ admits an ℓ^1-subsequence and a nontrivial weakly Cauchy subsequence while α is almost convergent.
Finally we show that, in the sense of measure, for almost all real sequences taking values in a compact set $K \subseteq \mathbb{R}$ (with at least two points), the sequence $(T^n\alpha)_{n\geq 0}$ is equivalent in the supremum norm to the usual ℓ^1-basis and (hence) not almost convergent.

0. Introduction. A sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of real numbers is said to be <u>Cesaro</u> summable if the sequence of its arithmetic means $\dfrac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}$,

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$n \geq 1$ is convergent in $\mathbb{R}$. If the sequence $\dfrac{\alpha_j + \cdots + \alpha_{j+n-1}}{n}$, $n \geq 1$ converges uniformly in $j \in \mathbb{N}$ to some $x \in \mathbb{R}$, then we say that α is an almost convergent sequence. This notion was introduced by G. G. Lorentz in [10]. Let us denote by $\ell^\infty(\mathbb{N})$ the Banach space of bounded real sequences with supremum norm and let $T : \ell^\infty(\mathbb{N}) \longrightarrow \ell^\infty(\mathbb{N})$ be the shift operator, defined by $T(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = (\alpha_2, \alpha_3, \ldots, \alpha_{n+1}, \ldots)$.

Our aim in this paper is the investigation of some (weaker) analogues of almost convergence for sequences $\alpha \in \ell^\infty(\mathbb{N})$, in connection with isomorphic invariant properties of the Banach space $E(\alpha)$, the closed linear span of the sequence $T^n\alpha$, $n \geq 0$ in $\ell^\infty(\mathbb{N})$. To give a taste of what we mean, let us note that if the set $\{T^n\alpha : n \geq 0\}$ is a weakly relatively compact subset of $\ell^\infty(\mathbb{N})$, then an easy application of the Mean Ergodic Theorem implies that the sequence α is almost convergent. To the opposite direction, if the sequence $T^n\alpha$, $n \geq 0$ is equivalent (in the supremum norm) to the usual basis of ℓ^1, then α is not almost convergent (see the beginning of section 3). A natural question is what happens if we assume that every subsequence of $T^n\alpha$, $n \geq 0$ admits a weakly Cauchy subsequence. This happens for instance, by the ℓ^1-Theorem of Rosenthal [4, p. 201], if the Banach space $E(\alpha)$ does not contain ℓ^1. Questions similar to the previous one were the motivation for the present paper.

We now briefly describe our main results. In the preliminary Section 1, we collect some standard definitions and results and fix the notation. In section 2 we introduce the notion of M-almost convergence (Definition 1) depending on a given subset $M \subseteq \mathbb{N}$, which generalizes the almost convergence of Lorentz (corresponding to the case $M = \mathbb{N}$). This notion was implicitly defined in the article [1] of P. C. Baayen and G. Helmberg. We extend Banach density and obtain the functional d_M^+ which characterizes M-almost convergence (see Theorem 1). We also provide examples which clarify this notion, in particular ones which distinguish M-almost convergence from the almost convergence of Lorentz and from the Cesaro summability.

In the third section we investigate what happens with respect to the almost convergence when:

(a) the set $T^n\alpha$, $n \geq 0$ is conditionally weakly compact, i.e. every subsequence of $T^n\alpha$, $n \geq 0$ has a further weakly Cauchy subsequence and (what happens when)

(b) the sequence $T^n\alpha$, $n \geq 0$ admits an ℓ^1-subsequence.

We also investigate the (measure of the set of) sequences $\alpha \in K^{\mathbb{N}}$ (where K is a compact subset of $\mathbb{R}$ and the cardinality of K, denoted by $|K|$ is at least 2) with the property that the sequence $T^n\alpha$, $n \geq 0$ is equivalent to the usual

ℓ^1-basis.

So the main results of this section are the following:

Theorem 2. *Let $(\Delta_n)_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into intervals of the positive integers such that:*

1. $\max \Delta_n + 1 = \min \Delta_{n+1}$
2. $\mid \Delta_n \mid \to_n \infty$.

Set $A = \bigcup_{j=0}^{\infty} \Delta_{2j+1}$ and let $\alpha = \chi_A \in \ell^\infty(\mathbb{N})$ and $E(\alpha)$ be the closed linear span of $\{T^n\alpha : n \geq 0\}$ in $\ell^\infty(\mathbb{N})$. Then $E(\alpha)$ is isomorphic to a subspace of c_0.

The characteristic function of A may not even be Cesaro summable, it can be M-almost convergent (to 0 or 1) but it is never almost convergent (see Remarks 4.(1), (2) and (3) and Examples (1) and (2)).

Theorem 3. *There is a set $A \subseteq \mathbb{N}$ with the following properties:*

1. *A is an infinite disjoint union of arithmetic progressions of $\mathbb{N}$.*
2. *For every $\varepsilon > 0$ there are D, $E \subseteq \mathbb{N}$ with D, E finite unions of arithmetic progressions, $D \subseteq A \subseteq E$ and $d(E \setminus D) < \varepsilon$.*
3. *$T^n\chi_A$, $n = 0, 1, \ldots$ has a subsequence equivalent (in the supremum norm) to the usual ℓ^1 basis.*
4. *$T^n\chi_A$, $n = 0, 1, \ldots$ has a non trivial weakly Cauchy subsequence.*

The set A is "regular" in a strong way because of condition (2). This condition also imposes that χ_A is almost convergent and yet the shift-sequence contains a subsequence equivalent to the usual ℓ^1-basis. Note that $E(\alpha)$ is not isomorphic to a subspace of ℓ^1 in this case. This is due to the Schur property of ℓ^1 (i.e. that every weakly convergent sequence in ℓ^1 is norm convergent) which would mean that the subsequence satisfying property (4) would be norm convergent (see [6, Th. 99, p. 74]), leading to a contradiction.

Theorem 2 concerns the case when the shift-sequence admits no ℓ^1-subsequence and Theorem 3 the case when there is an ℓ^1-subsequence of $T^n\alpha$, $n \geq 0$. These two results show that how "regular" the sequence α is, is not always related to how "regular" the space $E(\alpha)$ generated by $(T^n\alpha)$ is.

We close this section by showing that, given a compact subset K of $\mathbb{R}$ with at least two points and a strictly positive and regular Borel probability measure μ on K, the following holds:

Theorem 5. *μ_∞-almost every sequence $\alpha = (\alpha_n) \in K^{\mathbb{N}}$ satisfies:*

1. *The sequence $T^n\alpha, n \geq 0$ is equivalent in the supremum norm to the usual ℓ^1-basis and (hence) α is not almost convergent.*

2. *For any $\delta \in (0,1)$ there exists $M \subseteq \mathbb{N}$ with density $d(M) \geq 1-\delta$ such that the sequence α is M-almost convergent.*

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1. Preliminaries. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, then we set

$$d^+(\alpha) = \limsup_n \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \text{ and } d^-(\alpha) = \liminf_n \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}.$$

When α is the characteristic function of $A \subseteq \mathbb{N}$, we also write $d^+(A)$ and $d^-(A)$ and these values are called upper and lower density of A respectively. We denote

$$\mathcal{D} = \{A \subseteq \mathbb{N} : d^+(A) = d^-(A)\}.$$

When $A \in \mathcal{D}$, the common value of the upper and lower density is denoted by $d(A)$ and is called the density of A. Clearly a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is Cesaro summable in $\mathbb{R}$ if and only if the limit, $\lim_n \dfrac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = d^+(\alpha) = d^-(\alpha)(= d(\alpha))$ exists.

When $\alpha \in \ell^\infty(\mathbb{N})$ we set $f_n^\alpha = \dfrac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}$, $n \geq 1$; then it is easy to see that the sequence α is almost convergent (to the value $x \in \mathbb{R}$) if and only if the sequence of functions (f_n^α) converges uniformly on $\mathbb{N}$ (to the constant function x).

Throughout this paper, when κ, $\lambda \in \mathbb{N}$ we use the interval notation

$$[\kappa, \lambda] = \{n \in \mathbb{N} : \kappa \leq n \leq \lambda\} \text{ and } [\kappa, \lambda) = \{n \in \mathbb{N} : \kappa \leq n < \lambda\}.$$

We also denote $[\kappa, \infty) = \{n \in \mathbb{N} : n \geq k\}$.

Let $\beta\mathbb{N}$ denote the set of ultrafilters on $\mathbb{N}$. When $A \subseteq \mathbb{N}$ let $\overline{A} = \{u \in \beta\mathbb{N} : A \in u\}$. Then $\beta\mathbb{N}$, considered as a topological space with basis $\{\overline{A} : A \subseteq \mathbb{N}\}$ coincides with the Stone-Cech compactification of the discrete space $\mathbb{N}$ (see [14, p. 63]). It is well known that $\ell^\infty(\mathbb{N})$ is isometric to the space $C(\beta\mathbb{N})$ (the space of continuous real functions on $\beta\mathbb{N}$). The dual space $\ell^\infty(\mathbb{N})^*$ can be identified with the Banach space $\mathcal{M}(\mathbb{N})$ of all bounded finitely additive measures defined on all subsets $A \subseteq \mathbb{N}$. Thus when considering a $\mu \in \ell^\infty(\mathbb{N})^*$ as a finitely additive measure we may write $\mu(A) = \mu(\chi_A)$ (see also [4, p. 77]).

A positive normed linear functional L on $\ell^\infty(\mathbb{N})$ is called Banach limit if $L(\alpha) = L(T\alpha)$, $\forall \alpha \in \ell^\infty(\mathbb{N})$ (i.e. if L is shift-invariant). It is easy to check that the set of Banach limits $\mathcal{BL}$ is a convex and weak-$*$ compact subset of the unit ball of $\mathcal{M}(\mathbb{N})$. G. G. Lorentz has proved that a sequence $\alpha \in \ell^\infty(\mathbb{N})$ is almost convergent to $x \in \mathbb{R}$ if and only if $L(\alpha) = x$ for every $L \in \mathcal{BL}$ (see [10, Th. 1, p. 170]).

2. M-almost convergent sequences.

Definition 1. *Let M be a nonempty subset of $\mathbb{N}$. We say that the sequence $\alpha \in \ell^\infty(\mathbb{N})$ is M-almost convergent to $x \in \mathbb{R}$ if the sequence of functions $f_n^\alpha : \mathbb{N} \to \mathbb{R}$ converges uniformly on M to the constant function x, i.e.* $\lim_n \sup_{j \in M} |f_n^a(j) - x| = 0$.

Equivalently we request that $\forall \varepsilon > 0$ there is an $n(\varepsilon) \in \mathbb{N}$ so that

$$\left| \frac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n} - x \right| < \varepsilon \ \forall j \in M$$

when $n \geq n(\varepsilon)$.

We note that after defining the M-almost convergence, we found out that the notion of almost μ-well distribution (see Definition 11) introduced by P. C. Baayen and G. Helmberg in their article [1] concerning uniform distribution of sequences, was defined using the notion of M-almost convergence (although they did not refer to it explicitly).

Remarks 1.

(1) It is obvious that an M-almost convergent sequence is Cesaro summable. On the other hand if $(\alpha_n)_{n\in\mathbb{N}}$ is Cesaro summable, then it is obviously M-almost convergent for any M finite subset of $\mathbb{N}$. In the sequel we will assume that M is an infinite subset of $\mathbb{N}$, unless stated otherwise.

(2) A sequence $(\alpha_n)_{n\in\mathbb{N}}$ is $\mathbb{N}$-almost convergent if and only if it is almost convergent in the sense of G. G. Lorentz.

(3) So we have the following chain of implications for any $\alpha \in \ell^\infty(\mathbb{N})$: almost convergence $\Rightarrow$ M-almost convergence for (any) $M \subseteq \mathbb{N}$, $M \neq \emptyset \Rightarrow$ Cesaro summability.

We shall see that the converse of the above implications fail.

Example 1. An M-almost convergent sequence is not always almost convergent.

Let $M = \{n^3 : n = 1, 2, \dots\}$ and α be the characteristic function of the set $A = \bigcup_{n=1}^{\infty}[n^3, n^3 + 3n^2]$. Then it is easy to check that the means $f_n^{\alpha}, n \in \mathbb{N}$ converge uniformly on M to 1 (see also Example 2), hence α is M-almost convergent to the value 1. Considering the points $n^3 + 3n^2$, the corresponding n-means have the value 0 (and therefore α is not almost convergent).

We shall give later on an example of a bounded sequence which is Cesaro summable and not M-almost convergent for any infinite $M \subseteq \mathbb{N}$ (see Proposition 5, Example 3). The following Proposition and its Corollary essentially were proved in [1, p. 264]. We omit the simple proof of the following:

Proposition 1. *Let $\alpha \in \ell^{\infty}(\mathbb{N})$ be an M-almost convergent sequence, where M is any subset of $\mathbb{N}$. Then α is also $(M - h)$-almost convergent $\forall h \in \mathbb{N}$ (where $M - h = \{m - h : m \in M, m > h\}$).*

Corollary 1. *Let α be an M-almost convergent sequence, where $M = \{m_1 < m_2 < \cdots < m_n < \cdots\}$ is a syndetic subset of $\mathbb{N}$ (i.e. the set $\{m_{n+1} - m_n : n \in \mathbb{N}\}$ is finite). Then α is almost convergent.*

Proof. It is easy to check that if α is M_i-almost convergent, $i = 1, 2, \dots, n$ ($M_i \subseteq \mathbb{N}$) then α is $(\bigcup_{i=1}^{n} M_i)$-almost convergent. Let $h = \max\{m_{n+1} - m_n : n \in \mathbb{N}\}$. Since $\mathbb{N} = \bigcup_{i=0}^{h}(M - i)$, α is almost convergent. $\square$

Definition 2. *Given $\alpha = (\alpha_n) \in \ell^{\infty}(\mathbb{N})$, a sequence (t_n) in M and a strictly increasing sequence (k_n) in $\mathbb{N}$, we set*

$$J(\alpha, (t_n), (k_n)) = \inf_n \frac{\alpha_{t_n} + \alpha_{t_n+1} + \cdots + \alpha_{t_n+k_n-1}}{k_n}.$$

Then we define

$$d_M^+(\alpha) = \sup_{(t_n),(k_n)} J(\alpha, (t_n), (k_n)) \text{ and } d_M^-(\alpha) = -d_M^+(-\alpha).$$

Remarks 2.

(1) It is an easy exercise to see that if we considered only strictly increasing sequences (t_n) in the definition, the value of the supremum would still be the same. It is also easy to check that the function $d_M^+ : \ell^{\infty}(\mathbb{N}) \to \mathbb{R}$ is a sublinear functional, i.e. properties $d_M^+(\alpha + \beta) \leq d_M^+(\alpha) + d_M^+(\beta)$ $\forall \alpha, \beta \in \ell^{\infty}(\mathbb{N})$ and $d_M^+(\lambda\alpha) = \lambda \cdot d_M^+(\alpha)$, $\forall \alpha \in \ell^{\infty}(\mathbb{N})$ $\forall \lambda \geq 0$ are satisfied.

(2) When α is the characteristic function of a set $A \subseteq \mathbb{N}$ and $M = \mathbb{N}$ then $d_{\mathbb{N}}^+(A)$ and $d_{\mathbb{N}}^-(A)$ are (well known and) called upper and lower Banach density

of A respectively (see [5, p. 72]). If $d^+_{\mathbb{N}}(A) = d^-_{\mathbb{N}}(A)$ then the set A is said to have Banach density. If M is a subset of $\mathbb{N}$, we speak of the upper and lower M-Banach density of A when we refer to the values $d^+_M(A)$ and $d^-_M(A)$ respectively. Furthermore we introduce the following sets:

$$\mathcal{D}_M = \{A \subset \mathbb{N} : d^+_M(A) = d^-_M(A)\}.$$

When $A \in \mathcal{D}_M$ we denote the common value of the upper and lower M-Banach density by $d_M(A)$ (the M-Banach density of A). In particular $\mathcal{D}_{\mathbb{N}}$ is the family of all subsets of $\mathbb{N}$ which have Banach density, i.e. $d_{\mathbb{N}}(A) = d^+_{\mathbb{N}}(A) = d^-_{\mathbb{N}}(A)$.

(3) Clearly $\mathcal{D} = \mathcal{D}_M$, for any $M \subseteq \mathbb{N}$ finite and nonempty, where $\mathcal{D}$ is the set of $A \subseteq \mathbb{N}$ having density (see the preliminaries). It is obvious that $\mathcal{D}_{\mathbb{N}} \subseteq \mathcal{D}_M \subseteq \mathcal{D}$ for any $M \subseteq \mathbb{N}$, $M \neq \emptyset$ (see Remarks 1).

We can now prove that d^+_M has another expression related to the expression L. Sucheston provided in [13, p. 23] for the maximal value of Banach limits ($\tau(\alpha) = \sup\{L(\alpha) : L$ is a Banach limit$\}$, $\alpha \in \ell^\infty(\mathbb{N})$), which was the following:

$$\tau(\alpha) = \lim_n \sup_j \frac{1}{n} \sum_{i=j}^{j+n-1} \alpha_i.$$

Proposition 2. *Let $M \subseteq \mathbb{N}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. Then*

$$d^+_M(\alpha) = \limsup_n \sup_{j\in M} \frac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n}.$$

P r o o f. Let s be the value of the right hand side of the equation. We first prove the inequality $\underline{s \leq d^+_M(\alpha)}$.

Let $\varepsilon > 0$. Consider a subsequence (k_n) of $\mathbb{N}$ such that

$$\sup_{j\in M} \frac{\alpha_j + \cdots + \alpha_{j+k_n-1}}{k_n} > s - \varepsilon \ \forall n \in \mathbb{N}.$$

This implies that for $n \in \mathbb{N}$ there is a $j_n \in M$ satisfying

$$\frac{\alpha_{j_n} + \alpha_{j_{n+1}} + \cdots + \alpha_{j_n+k_n-1}}{k_n} > s - \varepsilon \ \forall n \in \mathbb{N}.$$

Hence $d^+_M(\alpha) \geq s - \varepsilon$ and the inequality holds.

To prove the reverse inequality, let $\varepsilon > 0$ and consider a sequence (j_n) in M and a subsequence (k_n) of $\mathbb{N}$ such that:

$$\frac{\alpha_{j_n} + \alpha_{j_{n+1}} + \cdots + \alpha_{j_n+k_n-1}}{k_n} \geq d^+_M(\alpha) - \varepsilon \ \forall n \in \mathbb{N}.$$

Then

$$\sup_{j\in M} \frac{\alpha_j + \cdots + \alpha_{j+k_n-1}}{k_n} \geq d^+_M(\alpha) - \varepsilon \ \forall n \in \mathbb{N}$$

and therefore

$$\limsup_n \sup_{j\in M} \frac{\alpha_j + \cdots + \alpha_{j+n-1}}{n} \geq d^+_M(\alpha) - \varepsilon$$

which implies the result. □

Remarks 3.

(1) The limit $\lim_n \sup_{j\in M} \dfrac{\alpha_j + \cdots + \alpha_{j+n-1}}{n}$ does not always exist.

Let for instance $M = \bigcup_{n\in\mathbb{N}}[n^4, n^4+4n^3+6n^2+3n+1]$ and A be a subset of $\mathbb{N}$ with $d^-(A) = 0, d^+(A) = 1$. We consider the sequence $\alpha \in \ell^\infty(\mathbb{N})$ which is constructed as follows:
$\alpha = \chi_B$, with $\chi_B(i) = \chi_A(j)$ if $i = n^4 + 4n^3 + 6n^2 + 3n + j$, $j = 1, 2, \ldots, n$ for some $n \in \mathbb{N}$ and $\chi_B(i) = 0$ otherwise (i.e. χ_B restricted on the last n elements of each interval $[n^4, (n+1)^4]$ is χ_A restricted on $[1, n]$).
Consider now subsequences $(k_n), (\ell_n)$ of $\mathbb{N}$ such that $d^+(A) = \lim_n \dfrac{|\, A\cap[1,k_n]\,|}{k_n} = 1$ and $d^-(A) = \lim_n \dfrac{|\, A\cap[1,\ell_n]\,|}{\ell_n} = 0$. Let $m_n = n^4 + 4n^3 + 6n^2 + 3n + 1$ and observe that the supremum of the means of α starting from points $j \in M$ is the supremum of the means of α starting from m_n.
It is now easy to check that $\liminf_n \sup_{j\in M} \dfrac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n} = 0 < 1 = \limsup_n \sup_{j\in M} \dfrac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n}$.

(2) One can easily check that $d^+_{\mathbb{N}}(\alpha) = \tau(\alpha)\ \forall \alpha \in \ell^\infty(\mathbb{N})$.

We now obtain a characterization of M-almost convergence:

Theorem 1. *Let α be a bounded sequence and $M \subseteq \mathbb{N}$. Then $d^-_M(\alpha) = d^+_M(\alpha)$ if and only if α is M-almost convergent.*

Proof. It is a direct consequence of Proposition 2, since

$$d^-_M(\alpha) = d^+_M(\alpha) = x \Leftrightarrow \liminf_n \inf_{j\in M} (f^\alpha_n(j) - x) =$$
$$\limsup_n \sup_{j\in M} (f^\alpha_n(j) - x) = 0 \Leftrightarrow \lim_n \sup_{j\in M} |f^\alpha_n(j) - x| = 0. \qquad \square$$

Proposition 3. *Let α be a bounded sequence satisfying $C_1 \le \alpha_n \le C_2$ $\forall n \in \mathbb{N}$ and M be any nonempty subset of $\mathbb{N}$. Then we have*

$$C_1 \le d^-_{\mathbb{N}}(\alpha) \le d^-_M(\alpha) \le d^-(\alpha) \le d^+(\alpha) \le d^+_M(\alpha) \le d^+_{\mathbb{N}}(\alpha) \le C_2.$$

P r o o f. It is an easy consequence of Proposition 2, since when $M_0 \subset M \subset \mathbb{N}$ we have

$$\limsup_n \sup_{j\in M_0} f^\alpha_n(j) \le \limsup_n \sup_{j\in M} f^\alpha_n(j) \le \limsup_n \sup_{j\in \mathbb{N}} f^\alpha_n(j).$$

Take $M_0 = \{m_0\} = \{\min M\}$. One can verify that $\limsup_n f^\alpha_n(m_0) = d^+(\alpha)$ (see Remarks 2(3)). $\square$

Proposition 4. *Let M be an infinite subset of $\mathbb{N}$. If $A \in \mathcal{D}_M$ and $d_M(A) = x > 0$, then there is an $n_0 \in \mathbb{N}$ such that any interval $[j, j+n_0-1]$ of length n_0 with $\underline{j \in M}$ intersects A. We say then that A is "concentrated" around M.*

P r o o f. Let $\varepsilon = \dfrac{x}{2} > 0$ and $\alpha = \chi_A$. Then there is an $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n} - x\right| < \varepsilon \;\forall j \in M$$

when $n \ge n_0$. Hence

$$\frac{\alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+n-1}}{n} > \frac{x}{2} > 0 \;\forall j \in M$$

and for $n \ge n_0$. So $A \cap [j, j+n-1] \ne \emptyset$ $\forall j \in M$ and $n \ge n_0$ and the result follows. $\square$

The following are immediate Corollaries of the previous Proposition:

Corollary 2. *Let $A \subseteq \mathbb{N}$ be infinite with arbitrarily large gaps $[m_n, M_n]$. If $M \subseteq \{m_n : n \in \mathbb{N}\}$ is infinite and $A \in \mathcal{D}_M$ then $d_M(A) = 0$.*

Corollary 3. *If $A \in \mathcal{D}$ with $d(A) > 0$ and A has arbitrarily large gaps $[m_n, M_n]$, then $A \notin \mathcal{D}_M$, for any infinite $M \subseteq \{m_n : n \in \mathbb{N}\}$.*

Example 2. We already know by Example 1 that $\mathcal{D}_{\mathbb{N}} \subsetneq \mathcal{D}_M$ for an infinite subset M of $\mathbb{N}$. We now show the stronger result that $\mathcal{D}_{\mathbb{N}} \subsetneq \mathcal{D}_M$ even for a set M of density 1.

Let $M = \bigcup_{n=1}^{\infty}[n^4, n^4+4n^3)$ and $A = \bigcup_{n=1}^{\infty}[n^4, n^4+4n^3+6n^2)$. Consider $\alpha = \chi_A$. One can easily verify that $d(A) = d(M) = 1$, calculating the limits $\lim_k \frac{1}{N_k}\sum_{n=1}^{N_k}\chi_A(n)$ and $\lim_k \frac{1}{N_k}\sum_{n=1}^{N_k}\chi_M(n)$, where $N_k = k^4, k \in \mathbb{N}$ (because $\lim_k \frac{N_{k+1}}{N_k} = 1$).

Let now (t_n) be a sequence in M and (k_n) be a subsequence of the positive integers. We claim that

$$\lim_n \frac{\alpha_{t_n} + \cdots + \alpha_{t_n+k_n-1}}{k_n} = 1 \tag{1}$$

If (t_n) has a finite set of values, then the means in (1) converge to $d(A) = 1$. Otherwise we can assume that (t_n) is strictly increasing. If $t_n \in [N^4, N^4+4N^3]$, the k_n-mean may include at most $4N+1$ points of the interval $[N^4, (N+1)^4]$ not belonging to A and in this case it necessarily also includes $6N^2$ points of this interval, elements of A. So the means in (1) exceed the value $\frac{6N^2}{6N^2+N+1} \to_N 1$. Consequently $A \in \mathcal{D}_M$ and $d_M(A) = 1$.

Finally it is easy to see that $A \notin \mathcal{D}_{\mathbb{N}}$, since $d_{\mathbb{N}}^{-}(A) = 0 < d_{\mathbb{N}}^{+}(A) = 1$ (consider the mean of length $4n+1$ starting from the point $n^4+4n^3+6n^2, n \in \mathbb{N}$).

Proposition 5. *Consider a partition $(\Delta_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ into intervals of the positive integers with the properties:*

1. $\max \Delta_n + 1 = \min \Delta_{n+1}$
2. $\mid \Delta_n \mid \to_n \infty$.

Let $A = \bigcup_{n=0}^{\infty}\Delta_{2n+1}$ and assume that $0 < d^+(A) < 1$. Then $A \notin \mathcal{D}_M$ for any M infinite subset of $\mathbb{N}$.

Proof. If $A \notin \mathcal{D}$, then the conclusion follows immediately from Remarks 2(3). So we can assume that $d(A) = t > 0$. Let M be an infinite subset of $\mathbb{N}$. Then Proposition 3, implies $d_M^{-}(\alpha) \le t \le d_M^{+}(\alpha)$.

Clearly there is an infinite subset N of M which is contained either in A or in $\mathbb{N} \setminus A$. We assume that $N \subseteq A$ (the other case is similar, since $\mathbb{N} \setminus A$ is of the same form as A). We can also assume that every interval of A contains at most one point of N.

Let $N = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ and $p_n \in \Delta_{k_n}$, $n \in \mathbb{N}$. Let also $m_n = \min \Delta_{k_n}$, $M_n = \max \Delta_{k_n}$, $n \in \mathbb{N}$ and

$$\Lambda = \{M_n - p_n : n \in \mathbb{N}\}.$$

We consider the cases:

(I) Λ is a finite set.

One can easily see that $\dfrac{\mid A \cap [p_n, p_n + \mid \Delta_{k_n+1} \mid -1] \mid}{\mid \Delta_{k_n+1} \mid} \to_n 0$ and therefore $d^-_M(A) = 0$.

(II) Λ is infinite.

Then we have $\dfrac{\mid A \cap [p_n, M_n] \mid}{M_n - p_n + 1} \to_n 1$ and the sequence $(M_n - p_n)_{n\in\mathbb{N}}$ has a strictly increasing subsequence, so $d^+_M(A) = 1$.

In any case $A \notin \mathcal{D}_M$ (see Proposition 3, Theorem 1). $\square$

Remarks 4.

(1) We note that apart from properties (1) and (2) above we can assume that $(\Delta_n)_{n\in\mathbb{N}}$ also satisfies property (3):

$$|\Delta_1| \leq |\Delta_2| \leq \cdots \leq |\Delta_n| \leq \cdots . \tag{3}$$

It then easily follows that $0 < d^+(A) < 1$, so the previous Proposition holds if we assume (1), (2) and (3) without any other assumption.

(2) If a subset A of $\mathbb{N}$ contains arbitrarily large intervals and has arbitrarily large gaps (as in the case of sets A considered in Proposition 5) then obviously $d^+_{\mathbb{N}}(A) = 1$ and $d^-_{\mathbb{N}}(A) = 0$; therefore the sequence $\alpha = \chi_A$ is not almost convergent.

(3) Note that the sets in the examples 1 and 2 we have already given are infinite unions of intervals of $\mathbb{N}$ satisfying properties (1) and (2). Furthermore both of them satisfy the hypothesis of the following Corollary with $d_M(A) = 1$.

Corollary 4. *Let $A \subseteq \mathbb{N}$ be an infinite union of intervals of $\mathbb{N}$ with the properties (1) and (2) of the previous Proposition. If $A \in \mathcal{D}_M$ for some infinite subset M of $\mathbb{N}$ then $d_M(A) \in \{0, 1\}$.*

We give two concrete examples of subsets of $\mathbb{N}$ satisfying the hypothesis of Proposition 5:

Examples. (1) There is a set $A \in \mathcal{D}$ such that $A \notin \mathcal{D}_M$, for every infinite subset M of $\mathbb{N}$, i.e. $\bigcup\{\mathcal{D}_M : M \subseteq \mathbb{N},\ M \text{ infinite }\} \subsetneq \mathcal{D}$.

Let $A = \bigcup_{n=1}^{\infty} \left[n^2, n^2 + n\right)$. Then we have $A \in \mathcal{D}$ and $d(A) = \dfrac{1}{2}$. Indeed, it suffices to consider the sequence of positive integers $N_k = k^2, k \geq 1$ with $\dfrac{N_{k+1}}{N_k} \to_k 1$ and check that $\dfrac{1}{N_k}\displaystyle\sum_{n=1}^{N_k} \chi_A(n) = \dfrac{\mid A \cap [1, k^2] \mid}{k^2} \to \dfrac{1}{2}$. Now it is easy to verify properties (1), (2) and (3) above, hence either Proposition 5 or Corollary 4 implies the result.

(2) There is a subset A of $\mathbb{N}$ satisfying properties (1), (2) and (3) without having density. Let $A = \bigcup_{n=0}^{\infty} \left[2^{2n}, 2^{2n+1}\right)$, then (obviously A satisfies the desired properties and) it is easy to show that $d^-(A) = \frac{1}{3} < \frac{2}{3} = d^+(A)$.

Let $\alpha \in \ell^\infty(\mathbb{N})$. If there is a subsequence (k_n) of the positive integers such that the sequence $f_{k_n}^{\alpha}$ converges weakly in $\ell^\infty(\mathbb{N})$, then the Mean Ergodic Theorem (see e.g. [12, p. 26]) implies that α is almost convergent (see p. 13). We give an example showing that the same is <u>not</u> true for M-almost convergence.

Example 3. We define inductively subsequences $(m_n), (k_n)$ of the positive integers satisfying

$$\begin{array}{ll} m_1 = 1, & k_1 = 2 = 2^{m_1} \\ m_2 = 2^{k_1}, & k_2 = 2^{m_2} \\ \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ m_n = 2^{k_{n-1}}, & k_n = 2^{m_n} \end{array}$$

Let $A = \bigcup_{n=1}^{\infty} [m_{n+1} + k_n, m_{n+1} + 2k_n)$ and $\alpha = \chi_A$. The set A has density 0, since $d^+(A) \leq \lim_n \dfrac{k_1 + \cdots + k_n}{m_{n+1} + 2k_n} \leq \lim_n \dfrac{{k_n}^2}{2^{k_n} + 2k_n} = 0$.

Let $M = (m_n)_{n\in\mathbb{N}}$. Obviously $A \notin \mathcal{D}_M$, because otherwise we would have $d_M(A) = 0$ (see Proposition 3), but $f_{2k_n}^{\alpha}(m_{n+1}) = \frac{1}{2}$ $\forall n \in \mathbb{N}$, so α is not M-almost convergent. Nevertheless we will prove that $f_{k_n}^{\alpha}, n \geq 2$ converges uniformly on M to 0. Consider the following cases:

1. <u>$l > n$</u>.
   Then $f_{k_n}^{\alpha}(m_l) = \dfrac{\alpha_{m_l} + \cdots + \alpha_{m_l+k_n-1}}{k_n} = 0$ (the mean does not include points belonging to the next block of those which constitute A).

2. <u>$l \leq n$</u>.
   In this case it is easy to check that $f_{k_n}^{\alpha}(m_l) = \dfrac{\alpha_{m_l} + \cdots + \alpha_{m_l+k_n-1}}{k_n} \leq \dfrac{k_1 + \cdots + k_{n-1}}{k_n} \to_n 0$ and the proof is complete.

M. Jerison in [7, p. 87] notes that the maximal value of all Banach limits $\tau : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ with $\tau(\alpha) = \sup\{L(\alpha) : L \text{ is a Banach limit}\}$ is a sublinear functional satisfying

(∗) If $\varphi : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ is a linear functional with $\varphi(\alpha) \leq \tau(\alpha)\, \forall \alpha \in \ell^\infty(\mathbb{N})$ then φ is a Banach limit.

Moreover, let $p : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ be a sublinear functional with property (∗) (i.e. if $\phi : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ is a linear functional with $\phi \leq p$ then $\phi \in \mathcal{BL}$). Then the Hahn-Banach Theorem yields that $p(x) \leq \tau(x) \forall x \in \ell^\infty(\mathbb{N})$. So τ is the maximum sublinear functional satisfying (∗).

We already mentioned the expression given by L. Sucheston in [13] for the functional τ. By Proposition 2 we conclude that $d^+_{\mathbb{N}} \equiv \tau$. Following M. Jerison we define the maximal value of M-Banach limits as follows:

Let M be a subset of $\mathbb{N}$. We denote the set of M-Banach limits (i.e. those that preserve M-almost convergence) by $\mathcal{BL}_M = \{L \in \mathcal{BL} : \alpha \text{ is } M\text{-almost convergent to } x \Rightarrow L(\alpha) = x\}$. It is easily seen that $\mathcal{BL}_M$ is a convex and weak-∗ compact subset of the unit ball of $\mathcal{M}(\mathbb{N})$.

Definition 3. *Let $\tau_M : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ be the functional with $\tau_M(\alpha) = \sup_{L \in \mathcal{BL}_M} L(\alpha)$.*

The functional τ_M is sublinear and it (obviously) preserves M-almost convergence. One can easily verify (in exactly the same way the corresponding argument concerning the functional τ is proved, see [7, p. 87]*) that τ_M is the maximum sublinear functional satisfying*

(♦) *If $\varphi : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ is a linear functional with $\varphi(\alpha) \leq \tau_M(\alpha), \forall \alpha \in \ell^\infty(\mathbb{N})$, then $\varphi \in \mathcal{BL}_M$.*

If $L : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$ is a linear functional with $L(\alpha) \leq d^+_M(\alpha)\, \forall \alpha \in \ell^\infty(\mathbb{N})$ then $d^-_M(\alpha) \leq L(\alpha) \leq d^+_M(\alpha)\, \forall \alpha \in \ell^\infty(\mathbb{N})$ and this easily yields that $L \in \mathcal{BL}_M$. So we always have $d^+_M(\alpha) \leq \tau_M(\alpha)\, \forall \alpha \in \ell^\infty(\mathbb{N})$. The following example though shows that d^+_M does not in general coincide with τ_M.

Example 4. Let $M = \{n^2 + 1 : n \in \mathbb{N}\}$ and α be the characteristic function of the set $A = \bigcup_{n=1}^\infty [n^2+n+1, (n+1)^2)$. We prove that $\underline{d^+_M(\alpha) < \tau_M(\alpha)}$.

It is easy to see that $d^+_M(\alpha) = \dfrac{1}{2}$. Let now $(A_n)_{n\in\mathbb{N}}$ be the sequence

$$A_n = \frac{1}{n} \sum_{t \in [n^2+n+1, (n+1)^2)} \delta_t, n \in \mathbb{N}$$

(where δ_t is the Dirac measure supported by $\{t\}$). It is easy to check that $A = (A_n)$ is a strongly regular and positive (matrix) summation method (see [9, p. 216] and [11, p. 21]). Then any weak-$*$ cluster point L of (A_n) (in the space $\mathcal{M}(\mathbb{N})$) is a Banach limit (see [11, p. 22]) with $L(\alpha) = 1$ (since $A_n(\alpha) = 1\ \forall n \in \mathbb{N}$).

It suffices to show that (A_n) preserves M-almost convergence. Since then, the weak-$*$ compactness of the unit ball of $\mathcal{M}(\mathbb{N})$ yields a Banach limit $L \in \mathcal{BL}_M$ (weak-$*$ cluster point of (A_n)) with $L(\alpha) = 1 > d_M^+(\alpha) = \frac{1}{2}$.

So let $b \in \ell^\infty(\mathbb{N})$ M-almost converge to x. Since $d_M^+(b) = d_M^-(b) = x$ we get that for any sequence (t_n) in M and any subsequence (k_n) of $\mathbb{N}$ the relation $\dfrac{b_{t_n} + \cdots + b_{t_n+k_n-1}}{k_n} \to_n x$ holds. Hence we have

$$A_n(b) = \frac{b_{n^2+n+1} + \cdots + b_{n^2+2n}}{n} =$$

$$= 2\frac{b_{n^2+1} + \cdots + b_{n^2+2n}}{2n} - \frac{b_{n^2+1} + \cdots + b_{n^2+n}}{n} \to 2x - x = x$$

and this implies the result.

In the case of almost convergence there are expressions of the maximal value of Banach limits containing only the terms of the sequence $\alpha \in \ell^\infty(\mathbb{N})$ (e.g. the expression of L. Sucheston).

Question: Is there a similar expression of the maximal value of M-Banach limits in the general case of M-almost convergence?

3. Banach spaces generated by shift-sequences. We first prove two important assertions we referred to in the introduction:

1. If the set $W(\alpha) = \{T^n\alpha : n \in \mathbb{N}\}$ is a weakly relatively compact subset of $\ell^\infty(\mathbb{N})$, then the sequence α is almost convergent.

Suppose that the set $W(\alpha)$ is weakly relatively compact. Then the theorem of Krein [6, vol. I Theorem 80, p. 52] yields that the closed convex hull of $\overline{W(\alpha)}$ is weakly compact. So the sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k(\alpha)\right)_{n\in\mathbb{N}}$ has a weak cluster point and by the Mean Ergodic Theorem [12, p. 26] for the shift operator T, this sequence converges in norm, or equivalently the sequence α is almost convergent (see also [2, Lemma 3.9]).

Let ℓ^1 be the Banach space of absolutely summable real sequences with the obvious norm. We recall that a bounded sequence (x_n) in a Banach space

X is said to be equivalent in the supremum norm to the usual basis (e_n) of ℓ^1 if there is a $\delta > 0$ so that $\forall n \in \mathbb{N}$ and for every choice of real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ the relation

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \geq \delta \cdot \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|$$

holds.

2. If the sequence $\{T^n \alpha\}$ is equivalent (in the supremum norm) to the usual basis of ℓ^1, then the sequence α is not almost convergent.
Suppose that $\{T^n \alpha\}$ is equivalent to (e_n). If the sequence $\left(\frac{1}{n} \sum\limits_{k=0}^{n-1} T^k(\alpha) \right)_{n \in \mathbb{N}}$ converges in norm, then clearly the sequence $\left(\frac{1}{n} \sum\limits_{k=0}^{n-1} e_k \right)_{n \in \mathbb{N}}$ converges in norm to a vector $y \in \ell^1$. But $\left(\frac{1}{n} \sum\limits_{k=0}^{n-1} e_k \right)_{n \in \mathbb{N}}$ converges coordinatewise to 0, so $y = 0 \in \ell^1$.
On the other hand

$$\left\| \left(\frac{1}{n} \sum_{k=0}^{n-1} e_k \right)_{n \in \mathbb{N}} \right\|_1 = 1 \; \forall n \in \mathbb{N}$$

which leads to a contradiction (see also Remark 7(1)).

Notation: If $A \subseteq \mathbb{N}$ and $n \geq 0$ we shall write

$$A_n = \{k \in \mathbb{N} : k + n \in A\} \quad (= A - n).$$

We note that: (i) $(\mathbb{N} \setminus A)_n = \mathbb{N} \setminus A_n$ and (ii) if $\alpha = \chi_A$ then $T^n(\alpha) = \chi_{A_n}$.

We prove the following useful Lemma concerning subsets of $\mathbb{N}$ as those used in Proposition 5:

Lemma 1. *Let $(\Delta_n)_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into intervals of the positive integers such that:*

1. $\max \Delta_n + 1 = \min \Delta_{n+1}$

2. $| \Delta_n | \to_n \infty$.

Set $A = \bigcup_{j=0}^{\infty} \Delta_{2j+1}$ and let $u \in \beta\mathbb{N} \setminus \mathbb{N}$. Then there is $n_0 \geq 0$ such that either $\{n \geq 0 : A_n \in u\} = [n_0, \infty)$ or $\{n \geq 0 : A_n \in u\} = [0, n_0)$.

Proof. Assume that this does not hold. Then one of the following sets

$$\{n \geq 0 : A_n \in u\} \text{ and } \{n \geq 0 : A_n \notin u\}$$

is not a subinterval of $\{0, 1, \cdots\}$. Suppose for instance that $\{n \geq 0 : A_n \in u\}$ is not an interval (the other case is analogous), i.e. there exist $0 \leq k_1 < k_2 < k_3$ such that $A_{k_1} \in u, A_{k_3} \in u$ and $A_{k_2} \notin u$ (so $\mathbb{N} \setminus A_{k_2} \in u$). Hence the set $N = A_{k_1} \cap A_{k_3} \cap (\mathbb{N} \setminus A_{k_2}) \in u$ and this set is infinite, since $u \in \beta\mathbb{N} \setminus \mathbb{N}$. We have that $\forall l \in N$ there are $m_1 < m_2 < m_3$ with $m_1 \in A$, $m_2 \in \mathbb{N} \setminus A$, $m_3 \in A$ such that $m_1 - k_1 = m_2 - k_2 = m_3 - k_3 = l$. That is, for each $l \in N$ there are $m_1 < m_2 < m_3$ with $m_1 \in A$, $m_2 \in \mathbb{N} \setminus A$, $m_3 \in A$ such that

$$(**) \qquad m_1 = k_1 + l, m_2 = k_2 + l, m_3 = k_3 + l.$$

Since N is infinite, for every $n \in \mathbb{N}$ there are $m_1 < m_2 < m_3$ satisfying $(**)$ which are contained in the set $\bigcup_{j=n+1}^{\infty} \Delta_j$.
Choose $n_0 \in \mathbb{N}$ so that $|\Delta_n| \geq k_3 - k_1$ when $n \geq n_0$. It is obvious that the set $\bigcup_{j=n_0+1}^{\infty} \Delta_j$ cannot contain such points, since $m_3 - m_1 = k_3 - k_1$. We have a contradiction and the Lemma holds. $\square$

We recall that a sequence (x_n) in a Banach space X is called weakly Cauchy if the sequence $(x^*(x_n))$ converges for any bounded functional x^* of the dual space X^*. A sequence (x_n) in $\ell^\infty(\mathbb{N})$ is weakly Cauchy, if and only if it is bounded and the $\lim_n u(x_n)$ exists $\forall u \in \beta\mathbb{N}$ (see the Introduction and [4, Theorem 1, p. 66]).

Corollary 5. *Let A be the set considered in the previous Lemma and $\alpha = \chi_A$. Then for any subsequence of $T^n\alpha$, $n = 0, 1, \ldots$ there is a further subsequence which is weakly Cauchy. Therefore no subsequence of $T^n\alpha$, $n = 0, 1, \ldots$ is equivalent to the usual ℓ^1-basis (this also follows from the next Theorem).*

Proof. Since $\mathbb{N}$ is a countable set, for any subsequence of $T^n\alpha, n = 0, 1, \ldots$ we can find (by a diagonal process) a further subsequence converging pointwise on $\mathbb{N}$. The previous Lemma also ensures that the whole sequence converges pointwise on $\beta\mathbb{N} \setminus \mathbb{N}$ and the result follows. $\square$

Theorem 2. *Let A be a subset of $\mathbb{N}$ satisfying the assumptions of Lemma 1. If $\alpha = \chi_A \in \ell^\infty(\mathbb{N})$ and $E(\alpha)$ is the closed linear span of $W(\alpha) = \{T^n\alpha : n \geq 0\}$ in $\ell^\infty(\mathbb{N})$, then $E(\alpha)$ is isomorphic to a subspace of c_0.*

Note: For properties of sets $A \subseteq \mathbb{N}$ satisfying the assumptions of Lemma 1, see Remarks 4 and Examples 1.

Proof. For the shake of clarity we present the proof in the following steps:

1. Let K be the compact topological space obtained as a quotient of $\beta\mathbb{N}$ after considering the equivalence relation:

$$u \sim u' \Leftrightarrow \text{ for each } n \geq 0, \text{ we have } A_n \in u \text{ iff } A_n \in u'.$$

Write $[u] \in K$ to denote the equivalence class of each $u \in \beta\mathbb{N}$ and let $\pi : \beta\mathbb{N} \to K$ be the quotient mapping.

2. Consider the one-to-one mapping $f : K \to \{0,1\}^{\mathbb{N}\cup\{0\}}$ defined by

$$f([u]) = \chi_{\{n\geq 0: A_n \in u\}}.$$

Given $n \geq 0$, write $p_n : \{0,1\}^{\mathbb{N}\cup\{0\}} \to \mathbb{R}$ to denote the n-th coordinate projection; then $p_n \circ f \circ \pi = \chi_{\overline{A_n}}$ is continuous (since $\overline{A_n}$ is open and closed in $\beta\mathbb{N}$). Hence f is continuous and K is homeomorphic to the compact metric space $L = f(K) \subset \{0,1\}^{\mathbb{N}\cup\{0\}}$.

3. Identify the elements of L as follows:

   (a) Given $k \in \mathbb{N}$, k^* denotes the corresponding principal ultrafilter. Notice that $f([k^*]) = \chi_{\{n\geq 0: k\in A_n\}} = \chi_{A_k\cup B_k}$, where $B_k = \{0\}$ if $k \in A$ and $B_k = \emptyset$ if $k \notin A$.

   (b) Given $u \in \beta\mathbb{N}\setminus\mathbb{N}$, Lemma 1 ensures that there exists $n_0 \geq 0$ such that either $f([u]) = \chi_{[n_0,\infty)}$ or $f([u]) = \chi_{[0,n_0)}$.

4. Let X be a compact space. The Cantor derivative of X is defined (inductively) as follows: $X^{(0)} = X, X^{(1)} = X' =$ the set of limit points of X. If β is an ordinal, then $X^{(\beta)} = (X^{(\gamma)})'$ for $\beta = \gamma + 1$ and $X^{(\beta)} = \cap_{\gamma<\beta} X^{(\gamma)}$ if β is a limit ordinal.

   One can check that $L^{(3)} = \emptyset$ with the help of the following facts:

   (a) $\chi_{A_k\cup B_k}$ is an isolated point of L for every $k \in \mathbb{N}$.
   Suppose $\chi_{A_k\cup B_k}$ is the pointwise limit of a sequence $(\chi_{F_n})_{n\in\mathbb{N}}$ of elements of L with $F_n \neq F_m$ for $n \neq m$. Then we can assume that every F_n is of the form $A_{j_n} \cup B_{j_n}$ (because there are positive integers $m < l < t$ with $\chi_{A_k\cup B_k}(m) = \chi_{A_k\cup B_k}(t) = 1$ and $\chi_{A_k\cup B_k}(l) = 0$) and moreover (passing to a subsequence if necessary) we can assume that j_n is strictly increasing. Since $|\Delta_n| \to \infty$, there is an $n_0 \in \mathbb{N}$ such that the length of each interval of $A_{j_n} \cup B_{j_n}$, $n \geq n_0$ (except possibly from the first one) as well as the length of each interval of $\mathbb{N} \setminus (A_{j_n} \cup B_{j_n})$, $n \geq n_0$ (except possibly from the first one) is greater than t. Since $\chi_{A_{j_n}\cup B_{j_n}}(m) \to 1$ and $\chi_{A_{j_n}\cup B_{j_n}}(t) \to 1$, while $\chi_{A_{j_n}\cup B_{j_n}}(l) \to 0$ we have a contradiction.
   Thus we obtain that

$$L' = I = \{\chi_{[n,\infty)} : n \geq 0\} \cup \{\chi_{[0,n)} : n \geq 0\} \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ denotes the constant zero function.

(b) Given $n_0 \in \mathbb{N}$, let us consider the following open subsets of L:

$$U = \{h \in L : h(n_0) = 1,\ h(n) = 0 \text{ for all } 0 \leq n < n_0\},$$

$$V = \{h \in L : h(n_0) = 0,\ h(n) = 1 \text{ for all } 0 \leq n < n_0\}.$$

Then $U \cap I = \{\chi_{[n_0,\infty)}\}$ and $V \cap I = \{\chi_{[0,n_0)}\}$. It is now easy to check that $L^{(2)} = \{\mathbf{0}, \mathbf{1}\}$ ($\mathbf{1}$ is the function with constant value 1).

5. Since $L^{(3)} = K^{(3)} = \emptyset$, a result of Bessaga and Pelczynski (see [6, ex. 32, p. 253]) ensures that $C(K)$ is isomorphic to c_0.

6. The family $\{\chi_{A_n} : n \geq 0\} = \{T^n\alpha : n \geq 0\} \subset E(\alpha)$ is linearly independent (use the fact that $|\Delta_n| \to \infty$ to obtain that when $n_1 < n_2 < \cdots < n_k$, there are successive intervals of integers contained in $A_{n_k}, A_{n_{k-1}} \cap A_{n_k}, \cdots, \cap_{i=2}^{k} A_{n_i}, \cap_{i=1}^{k} A_{n_i}, \cap_{i=1}^{k-1} A_{n_i}, \cap_{i=1}^{k-2} A_{n_i}, \cdots, A_{n_1}$ respectively).
So we can define a linear mapping on the linear span of $\{\chi_{A_n}\}$,

$$\varphi : \operatorname{span}\{\chi_{A_n} : n \geq 0\} \to C(K)$$

such that $\varphi(\chi_{A_n}) = p_n \circ f$ for every $n \geq 0$. Since φ preserves the norm, it can be extended to a linear mapping on $E(\alpha)$ that also preserves the norm. Thus $E(\alpha)$ is isometric to a subspace of $C(K)$. $\square$

Remark 5. Let $E(\alpha)$ be the space of the previous Theorem. Then one can prove that the sequence $(\chi_{A_n})_{n=0}^{\infty}$ is a fundamental system (i.e. for every $i \geq 0$ we have $\chi_{A_i} \notin \overline{\operatorname{span}\{\chi_{A_n} : n \neq i\}}$, where $\chi_{A_n} = T^n(\chi_A), n \geq 0$).

Theorem 3. *There is a set $A \subseteq \mathbb{N}$ with the following properties:*

1. *A is an infinite disjoint union of arithmetic progressions of $\mathbb{N}$.*

2. *For every $\varepsilon > 0$ there are $D, E \subseteq \mathbb{N}$ with D, E finite unions of arithmetic progressions, $D \subseteq A \subseteq E$ and $d(E \setminus D) < \varepsilon$.*

3. *$\chi_{A_n}, n = 0, 1, \ldots$ has a subsequence equivalent (in the supremum norm) to the usual ℓ^1 basis. (Recall that $A_n = \{k \in \mathbb{N} : k + n \in A\}$).*

4. *$\chi_{A_n}, n = 0, 1, \ldots$ has a non trivial weakly Cauchy subsequence.*

<u>Note:</u> The subsets of $\mathbb{N}$ fulfilling condition (2) constitute an algebra which was defined and studied by R. C. Buck in [3].

P r o o f. We consider a zero-one sequence $b = (b_n)_{n\in\mathbb{N}}$ ($b_n = 0$ or $b_n = 1$ for $n \in \mathbb{N}$), such that the sequence $T^n(b), n = 0, 1, \dots$ is dense in the Cantor space $\{0,1\}^{\mathbb{N}}$, for instance $b = (b_n)_{n\in\mathbb{N}}$ could be the sequence of digits of a number which is normal to the base 2 (see Remark 8 and the rest of this section for properties of such sequences).

Consider now the dyadic tree Δ consisting of the dyadic arithmetic progressions (APs) in $\mathbb{N}$. The n-th level of this tree ($n \geq 0$) contains the APs $\{2^n N + i, N = 0, 1, \dots\}$, $i = 1, 2, \dots, 2^n$ which we shall denote by $\{2^n N + i\}$. We shall denote the AP $\{2^n N + 2^n\}$ by $\{2^n N\}(= \{2^n N : N \geq 1\})$. The immediate successors of $\{2^n N + i\}$ are the APs $\{2^{n+1} N + i\}$ and $\{2^{n+1} N + 2^n + i\}$. We strongly recommend that the reader draws a sketch of this dyadic tree. Observe that the shift operator T moves each dyadic AP to another AP of the same level and given $n \in \mathbb{N}$, $T^{2^{n-1}}$ moves each AP of the n-th level to its pair knot.

Let $(P_n)_{n\geq 1}$ be the sequence of disjoint APs with $P_1 = \{2N\}$ and $P_n = \{2^{2n-1}N + (1 + 2^2 + \dots + 2^{2n-4})\}$, for $n \geq 2$. Let also $(Q_n)_{n\geq 1}$ be the sequence of pair knots of P_n, i.e. $Q_n = \{2^{2n-1}N + (1 + 2^2 + \dots + 2^{2n-4}) + 2^{2n-2}\}$. We note (for later use) that all Q_n belong to the same branch of the tree Δ.

We first look at the sequence of APs (p_j^1):$\{8N + 4\}, \{32N + 16\}, \{128N + 64\}, \dots, \{2^{2j+1}N + 2^{2j}\} \dots$. The set A contains the j-th AP $\{2^{2j+1}N + 2^{2j}\}$ if and only if the j-th digit b_j of the sequence b is equal to 1. The APs chosen in the $n = 1$ step are contained in $P_1 = \{2N : N = 1, 2, \dots\}$ of the first level of the tree.

For the $n = 2$ step we choose APs inside $P_2 = \{8N+1\}$ of the third level of the tree. Consider now the sequence $(p_j^2) : \{32N+17\}, \{128N+65\}, \dots, \{2^{2j+3}N + 2^{2j+2} + 1\}, \dots$ and let A contain the j-th AP if and only if the j-th digit of the sequence $T^2 b$ is equal to 1.

In the next step we choose APs inside $P_3 = \{32N + 5\}$ of the fifth level of the tree. Now the APs are chosen from the sequence $(p_j^3) : \{128N + 69\}, \{512N + 261\}, \dots, \{2^{2j+5}N + 2^{2j+4} + 5\}, \dots$ according to the digits of $T^4 b$.

It is obvious that this process can continue (inductively). In the n-th step we choose APs inside P_n (where $P_1 = \{2N\}$ and $P_n = \{2^{2n-1}N + (1 + 2^2 + 2^4 + \dots + 2^{2n-4})\}$ for $n \geq 2$) of the $(2n-1)$-level of the tree, then we take its pair knot $Q_n = \{2^{2n-1}N + (1+2^2+2^4+\dots+2^{2n-4})+2^{2n-2}\}$ and go down two levels in order to perform the $(n+1)$-step of the construction according to the digits of $T^{2n}b$. We observe that the construction is done in such a way that, given $n \geq 1$, every AP chosen from the $(n + 1)$-step and forward is contained in the AP Q_n belonging to the $(2n-1)$-level of the tree. Due to this observation the set A (which is obviously an infinite disjoint union of APs) satisfies condition (2). We note that, for $j, n \geq 1$, we have the expression $p_j^n = \{2^{2j+(2n-1)}N + 2^{2j+(2n-1)-1} + k_n\}$, where $k_1 = 0$ and for $n \geq 2$, $k_n = 1 + 2^2 + \cdots + 2^{2n-4}$ is the first term of the AP P_n inside which the choices of the n-th step are made.

We shall need the following:

Definition. *Let Ω be a nonempty set. A sequence $(K_n, L_n)_{n=1}^{\infty}$ of pairs of subsets of Ω is called* independent*, if K_n, L_n are disjoint $\forall n \in \mathbb{N}$ and for any choice F_1, F_2 of finite and disjoint subsets of $\mathbb{N}$ we have*

$$\bigcap_{n \in F_1} K_n \cap \bigcap_{n \in F_2} L_n \neq \emptyset.$$

We now prove that (A_n) satisfies (3). Let $\alpha = \chi_A$.

Claim. *The sequence $\alpha, T\alpha, T^5\alpha, \ldots, T^{k_n}\alpha, \ldots,$ is equivalent in the supremum norm to the usual ℓ^1 basis.*

Proof of the Claim. Let $C = \mathbb{N} \setminus A$. We will then show that the sequence $(T^{k_n}A, T^{k_n}C)_{n \in \mathbb{N}}$ is independent. Let $1 \cdot T^{k_n}A = T^{k_n}A$ and $0 \cdot T^{k_n}A = T^{k_n}C$. Let also $i_1, i_2, \ldots, i_\lambda \in \{0, 1\}$ be an arbitrary choice. We need to prove that

$$\bigcap_{p=1}^{\lambda} i_p T^{k_p} A \neq \emptyset.$$

Since the sequence $T^n b$, $n = 0, 1, \ldots$ is dense in $\{0,1\}^{\mathbb{N}}$, given a $\lambda \in \mathbb{N}$ and an arbitrary choice $i_1, i_2, \ldots, i_\lambda$ with $i_p \in \{0, 1\}$, $p = 1, 2, \ldots, \lambda$, for any $k \in \mathbb{N}$ there is a $j \in \mathbb{N}$, $j \geq k$ such that

$$b_j = i_1, b_{j+1} = i_2, \cdots, b_{j+\lambda-1} = i_\lambda \Leftrightarrow$$

$$b(j) = i_1, Tb(j) = i_2, \cdots, T^{\lambda-1}b(j) = i_\lambda.$$

Pick $j \geq \lambda$ such that the above equations are satisfied. The construction of A now gives that on the $(2j+1)$-level of the dyadic tree Δ, the AP

$$\{2^{2j+1}N + 2^{2j}\} \subseteq \bigcap_{p=1}^{\lambda} i_p T^{k_p} A \neq \emptyset.$$

Now since $(T^{k_n}A, T^{k_n}C)_{n \in \mathbb{N}}$ is independent, a well known result of Rosenthal for ℓ^1-embedding (see [4, Proposition 3, p. 207]) gives the result. Note that the choice of APs in each sequence (p_j^n) according to the digits of a zero-one sequence $b = (b_k)_{k\in\mathbb{N}}$ with $T^k b, k = 0, 1, \dots$ dense in $\{0,1\}^{\mathbb{N}}$ was the key to obtain an independent sequence.

Now we shall show that the sequence $f_n = T^{2^{2n+1}}\alpha = \chi_{A_{2^{2n+1}}}$, $n \geq 1$ is weakly Cauchy, but not weakly convergent in $\ell^\infty(\mathbb{N})$. We shall use all of the following easily verified facts:

Fact I: For every ultrafilter $u \in \beta\mathbb{N}$ there exists only one branch $b_u = \{\Pi_0 < \Pi_1 < \cdots < \Pi_n < \cdots\}$ of Δ so that $\Pi_n \in u$ for all $n \geq 0$.

Fact II: If $n \geq 0$ and $P = \{2^n N + a\}$ is any AP of (the n-th level of) the tree Δ, then we have:

1. for every $m \geq n$, $T^{2^m}(P) = P$ and
2. for every $m \geq n$ and each $P' \in \Delta_P = \{Q \in \Delta : P \leq Q\} \Rightarrow T^{2^m}(P') \in \Delta_P$.

Fact III: The branch containing (Q_n) is of the form:

$$Q'_0 = \mathbb{N} < Q_1 = \{2N+1\} < Q'_1 = \{4N+1\} < Q_2 = \{8N+5\} <$$

$$Q'_2 = \{16N+5\} < \cdots < Q_n < Q'_n < \cdots$$

Note that $Q'_n = \{2^{2n}N + (1 + 2^2 + \cdots + 2^{2n-2})\} \supseteq P_{n+1}, P_{n+2}, \dots, P_{n+k}, \dots$.

Let $u \in \beta\mathbb{N}$ and also let b_u be the branch of Δ corresponding to u according to Fact I. One has to check the following cases:

1. The branch b_u contains an AP of the ones that constitute A on the $(2k+1)$-level of the tree Δ. Clearly $u(A_{2^{2n+1}}) = 1$ for $n \geq k$, as the sets A and $A_{2^{2n+1}}$ have the same APs in the first $2k+1$ levels (see Fact II).
2. The branch b_u does not contain any AP of the ones that constitute A. Then we have exactly three cases:
   (a) Assume that there is a $k \in \mathbb{N}$, $k \geq 1$ so that on the $(2k+1)$-level of Δ b_u contains an AP disjoint from A (thus $A \notin u$). Then by using Fact II and the fact that $A_{2^{2n+1}}$ has the same APs with A in the first $2n+1$ levels, we conclude that $u(A_{2^{2n+1}}) = 0$ for $n \geq k$.
   (b) Let $l \geq 1$ and assume that the branch b_u contains P_l and also the pair knot of each p_j^l, $j \geq 1$. Assume for simplicity that $l = 1$, i.e. $P_l = \{2N\}$ and $b_u = \{\{\mathbb{N}\} < \{2N\} < \cdots < \{2^n N\} < \cdots\}$. Let $n \geq 1$.

Then the AP $\{2^{2n+2}N\}$ belongs to u and has empty intersection with every AP contained in $A_{2^{2n+1}}$ (using again Fact II). So the set $A_{2^{2n+1}}$ does not belong to u and thus $u(A_{2^{2n+1}}) = 0$. The general case is proved similarly.

(c) The case remaining is when b_u is the branch $\{\{\mathbb{N}\} < \{2N+1\} < \{4N+1\} < \cdots < \{2^{2n+1}N + (1+2^2+\cdots+2^{2n})\} < \{2^{2n+2}N + (1+2^2+\cdots+2^{2n})\} < \cdots\}$, that is the branch containing the sequence (Q_n). We note that each AP belonging to this branch intersects A. Then one can check (using Fact III) that b_u contains an AP disjoint from $A_{2^{2n+1}}$ for every $n \geq 1$ (this AP is the left successor Q'_{n+1} of Q_{n+1}). Hence $u(A_{2^{2n+1}}) = 0, \forall n \geq 1$.

So far we have proved that $f_n = T^{2^{2n+1}}\alpha$, $n \geq 1$ is a weakly Cauchy sequence in $\ell^\infty(\mathbb{N})$, that is, for every $u \in \beta\mathbb{N}$ the limit $f(u) = \lim_{n\to\infty} f_n(u)$ exists in $\mathbb{R}$. It remains to prove that (f_n) is not weakly convergent.

We first note that

$$f(m) = \lim_{n\to\infty} f_n(m) = \chi_A(m) = \alpha(m), \forall m \in \mathbb{N}. \tag{1}$$

Now it is easy to see that there is a $u_0 \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $A \in u_o$ and $\{2^n N\} \in u_0, \forall n \geq 0$. Therefore

$$f(u_0) = \lim_{n\to\infty} u_0(A_{2^{2n+1}}) = 0 \neq 1 = u_0(A). \tag{2}$$

It follows from (1) and (2) that u_0 is a discontinuity point of f and the proof is complete.

Recall that we left out the even levels of the tree when we chose APs in A. This way the branch $\{\{2N\} < \{4N\} < \cdots < \{2^n N\} < \cdots\}$ contains an AP disjoint from $A_{2^{2n+1}}$ for every $n \in \mathbb{N}$ and hence the sequence $T^{2^{2n+1}}\alpha, n \in \mathbb{N}$ cannot converge weakly to α. $\square$

Remarks 6.

(1) Let B be the set consisting of the first term of each progression of those chosen in the Theorem (whose union is the set A). One can check that $d^+_{\mathbb{N}}(B) = 0$ and the sequence $T^n B$, $n \geq 0$ has an independent subsequence.

(2) Let $G = \bigcup_{n\in\mathbb{N}} P_n$ (where P_n are the progressions defined in Theorem 3). Then it can be proved that the sequence $(T^n \chi_G)$ contains no ℓ^1-subsequence, it contains a non trivial weakly Cauchy subsequence and the space $E(\alpha)$, $\alpha = \chi_G$ contains ℓ^1 (isomorphicaly).

(3) We recall that a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ is called almost periodic if $\forall \varepsilon > 0$ the set $\{s \in \mathbb{N} : |f(n+s) - f(n)| \leq \varepsilon \ \forall n \in \mathbb{N}\}$ is a syndetic subset of $\mathbb{N}$, i.e. its gaps are bounded. It then follows that the set $\{T^k f : k = 0, 1, \dots\}$ is a norm relatively compact subset of $\ell^\infty(\mathbb{N})$, thus in particular the sequence $\alpha_n = f(n), n \geq 1$ is almost convergent (see [2], Lemma 3.5 and also assertion 1 in the beginning of section 3).

Consider now the Van der Corput sequence $\varphi : \mathbb{N} \longrightarrow [0,1]$ with $\varphi(N) = \dfrac{\alpha_0}{2} + \dfrac{\alpha_1}{2^2} + \cdots + \dfrac{\alpha_n}{2^{n+1}}$ (where $N = \alpha_0 + \alpha_1 \cdot 2 + \cdots + \alpha_n \cdot 2^n$, $\alpha_i = 0$ or 1, $i = 1, 2, \dots, n-1$ and $\alpha_n = 1$ is the unique dyadic representation of N), see also [9, p. 127]. φ maps each dyadic progression $\{2^k N + l\}$ (where $l = 1, 2, \dots, 2^k$) on the set of dyadic rational numbers contained in an interval of the form $\left[\dfrac{i}{2^k}, \dfrac{i+1}{2^k}\right)$ (where $i = 0, 1, \dots, 2^k - 1$). Using this, it is easy to prove that φ is an almost periodic function. Hence the set $W(\varphi) = \{T^n \varphi, \ n = 0, 1, \dots\}$ is a norm relatively compact subset of $\ell^\infty(\mathbb{N})$. The space $E(\varphi)$ though, the closed linear span of $W(\varphi)$ in $\ell^\infty(\mathbb{N})$, contains the Rademacher-like sequence $f_n = 2[(T^{2^{n-1}}\varphi - \varphi) + (T^{2^{n-1}+1}\varphi - T\varphi) + \cdots + (T^{2^n-1}\varphi - T^{2^{n-1}-1}\varphi)]$ which clearly is equivalent in the supremum norm to the usual ℓ^1-basis. This result can also be deduced by general considerations of Harmonic Analysis (see [8, p. 155–168]).

Definition 4. *Let X be a compact Hausdorff space and μ be a regular Borel probability measure on X. A sequence (x_n) in X is called μ-uniformly distributed (μ-u.d.) if the sequence $(f(x_n))$ is Cesaro summable to the value $\int_X f d\mu$ for every continuous function $f : X \longrightarrow \mathbb{R}$* [9, Definition 1.1, p. 171].

Notation: Let $\emptyset \neq M \subseteq \mathbb{N}$. By $T^n\alpha/M$ we denote the sequence $T^n\alpha$ restricted on the M-coordinates. From now on K stands for a compact subset of $\mathbb{R}$ with at least two points and M for a nonempty subset of $\mathbb{N}$.

Proposition 6. *Let K be a compact subset of $\mathbb{R}$ (with at least two points) and $M = \{m_0 < m_1 < \cdots < m_n < \cdots\}$ a nonempty (finite or infinite) subset of $\mathbb{N}$. If $\alpha \in K^{\mathbb{N}}$ and the sequence $T^n\alpha/M$, $n \in \mathbb{N}$ is dense in the space K^M, then the sequence $T^{m_i}\alpha$, $0 \leq i < |M|$, in $\ell^\infty(\mathbb{N})$ is equivalent to the usual basis of $\ell^1(|M|)$.*

Proof. Let $C = \max K$ and $c = \min K$ (so $c < C$). We set $\delta = \dfrac{C-c}{4} > 0$, $r = \dfrac{C+c}{2}$ and $A_n = \{p \in \mathbb{N} : g_n(p) \geq r + \delta\}, B_n = \{p \in \mathbb{N} : g_n(p) \leq r\}$ for every $n \in \mathbb{N}$, where $g_n = T^n\alpha$, $n \in \mathbb{N}$.

Claim. $(A_{m_i}, B_{m_i})_{0 \leq i < |M|}$ *is independent.*

Let $\varepsilon_i = 0$ *or 1,* $i = 0, 1, \cdots, k$*. We will show that*

$$\bigcap_{i=0}^{k} \varepsilon_i A_{m_i} \neq \emptyset,$$

where $1 \cdot A_i = A_i, 0 \cdot A_i = B_i, i = 0, 1, \cdots, k$.

Since the sequence $T^n\alpha/M$*,* $n \in \mathbb{N}$ *is dense in* K^M*, there is a* $p \in \mathbb{N}$ *such that for* $i = 0, 1, \cdots, k$ *we have*

$$\alpha_{p+m_i} \in [c, c+\delta)\,, \text{ if } \varepsilon_i = 0 \text{ and } \alpha_{p+m_i} \in (C-\delta, C]\,, \text{ if } \varepsilon_i = 1$$

therefore

$$g_{m_i}(p) = \alpha_{p+m_i} \leq r, \text{ if } \varepsilon_i = 0 \text{ and } g_{m_i}(p) = \alpha_{p+m_i} \geq r+\delta, \text{ if } \varepsilon_i = 1.$$

Obviously

$$p \in \bigcap_{i=0}^{k} \varepsilon_i A_{m_i}$$

and the Claim holds.

Now the result follows from the result of Rosenthal for ℓ^1 embedding (see also the proof of Theorem 3.(3)). $\square$

Remarks 7.

(1) In the special case where $M = \mathbb{N}$, if $T^n\alpha, n \in \mathbb{N}$ is dense in $K^{\mathbb{N}}$, we easily obtain that $d^-_{\mathbb{N}}(\alpha) = \min K < \max K = d^+_{\mathbb{N}}(\alpha)$ and thus α is not almost convergent, see Theorem 1.

(2) Requiring that the sequence α itself is dense in K is weaker than what we assumed in the previous Proposition. There is an example of a sequence $\alpha = (\alpha_n) \subseteq [0,1]$ which is dense in $[0,1]$, almost convergent (i.e. the sequence f^α_n, $n = 1, 2, \ldots$ converges in norm) and moreover every subsequence of $T^n\alpha$, $n = 0, 1, \ldots$ has a norm convergent subsequence.

It suffices to consider a sequence which is uniformly distributed in $[0,1]$ and at the same time is almost periodic on $\mathbb{N}$. The Van der Corput sequence satisfies both requirements (see Remarks 6(3) and Theorem 3.5 p. 127 of [9]).

The next Corollary is immediate:

Corollary 6. *Let* $K = \{0,1\}$*,* $\alpha \in \{0,1\}^{\mathbb{N}}$ *and* M *be a nonempty (finite or infinite) subset of* $\mathbb{N}$*. Let also* $A_n = \{k \in \mathbb{N} : T^n\alpha(k) = 1\}$ *and* $B_n = \{k \in \mathbb{N} : T^n\alpha(k) = 0\}$ $\forall n \in \mathbb{N}$*. Then the sequence* $T^n\alpha/M, n \in \mathbb{N}$ *is dense in* $\{0,1\}^M$ *if and only if the sequence* $(A_m, B_m)_{m\in M}$ *is independent.*

In the sequel we denote by D_M the set of sequences $\alpha \in K^{\mathbb{N}}$ for which the sequence $T^n\alpha/M$, $n = 0, 1, \dots$ is dense in K^M. It is easy to see that $D_{\mathbb{N}} \subseteq D_M$, $\forall M \subseteq \mathbb{N}$, $M \neq \emptyset$.

We will now show that $D_{\mathbb{N}}$ is a "big" subset of $K^{\mathbb{N}}$ (from the measure-theoretic point of view). For this purpose we consider a strictly positive and regular Borel probability measure μ on K. μ_∞ denotes the product measure on (the compact metric space) $K^{\mathbb{N}}$.

Definition 5. *A sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq K$ is called μ-completely uniformly distributed in K, if the sequence $(T^n\alpha)_{n\geq 0}$ is μ_∞-u.d. in $K^{\mathbb{N}}$ (see* [9, Definition 3.3, p. 204]*).*

It is well known that the following hold for a μ-completely u.d. sequence:

1. In particular α is μ-u.d. in K.
2. Since μ is strictly positive, μ_∞ is also strictly positive and hence $(T^n\alpha)_{n\geq 0}$ is a dense subsequence of $K^{\mathbb{N}}$. (It is easy to see that if ν is a strictly positive measure on the space X and $(x_n) \subseteq X$ is ν-u.d. in X, then (x_n) is dense in X).

The following notion was introduced by P. C. Baayen and G. Helmberg in [1, p. 264]:

Definition 6. *Let X be a compact space and μ be a regular Borel probability measure on X. A sequence $(x_n) \subseteq X$ is called almost μ-well distributed if there exists an infinite subset M of $\mathbb{N}$ so that the sequence $(f(x_n))$ is M-almost convergent to the value $\int_X f\,d\mu$ for any continuous function $f : X \longrightarrow \mathbb{R}$. If we want to refer to a particular subset M of $\mathbb{N}$ we shall call (x_n) almost μ-well distributed-M (if $M = \mathbb{N}$ then (x_n) is called μ-well distributed).*

The following result is known (see [1, pp. 265–266 and Theorem 12], also [9, Theorem 3.13, p. 204])

Theorem 4. *μ_∞-almost every sequence $\alpha \in K^{\mathbb{N}}$ satisfies that it is a μ-completely u.d. sequence in K and given $\delta \in (0,1)$, there is an infinite subset M of $\mathbb{N}$ with density $d(M) \geq 1-\delta$, such that α is almost μ-well distributed-M in K.*

This result implies:

Corollary 7. *If $D_{\mathbb{N}}$ is the set of sequences $\alpha \in K^{\mathbb{N}}$ for which the sequence $T^n\alpha$, $n = 0, 1, \dots$ is dense in $K^{\mathbb{N}}$, then $\mu_\infty(D_{\mathbb{N}}) = 1$ (hence $D_{\mathbb{N}}$ is dense in $K^{\mathbb{N}}$).*

P r o o f. Let S be the set of μ-completely u.d. sequences in K. Then Theorem 4 implies $\mu_\infty(S) = 1$. Property (2) of μ-completely u.d. sequences above gives that $S \subseteq D$ and the conclusion follows. $\square$

Theorem 5. *μ_∞-almost every sequence $\alpha \in K^{\mathbb{N}}$ satisfies:*

1. *The sequence $T^n\alpha, n \geq 0$ is equivalent in the supremum norm to the usual ℓ^1-basis and (hence) α is not almost convergent.*

2. *For any $\delta \in (0,1)$ there exists $M \subseteq \mathbb{N}$ with density $d(M) \geq 1-\delta$ such that the sequence α is almost μ-well distributed-M (in particular α is M-almost convergent).*

P r o o f. The proof is immediate consequence of Proposition 6, Remarks 7 (1) and Theorem 4. $\square$

Let $b \geq 2$ be a positive integer and $K = \{0, 1, \cdots, b-1\}$ (the discrete space consisting of b points), endowed with the measure $\mu(A) = \dfrac{|A|}{b}$ for $A \subseteq K$. Let $x = \sum\limits_{n=1}^{\infty} \dfrac{\alpha_n}{b^n}$ be the b-adic representation of $x \in [0,1)$. It is known that x is a normal number to the base b (for the definition see [9, p. 69]) if and only if the sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of its digits is μ-completely u.d. in K (see [9, Example 3.10, p. 206]).

We immediately obtain the following:

Corollary 8. *For every normal number $x = \sum\limits_{n=1}^{\infty} \dfrac{\alpha_n}{b^n}$ in its b-adic representation, if $\alpha = (\alpha_1, \dots, \alpha_n, \dots)$ (is the sequence of its digits), then we have:*

1. *The sequence $T^n\alpha$, $n \in \mathbb{N}$ is equivalent in the supremum norm to the usual basis of ℓ^1 and (hence) α is not almost convergent.*

2. *For any $\delta \in (0,1)$ there exists $M \subseteq \mathbb{N}$ with density $d(M) \geq 1-\delta$ such that the sequence α is almost μ-well distributed-M (in particular α is M-almost convergent).*

By $\mathcal{NA}_M$ we denote the set of sequences $\alpha \in K^{\mathbb{N}}$ which are <u>not</u> M-almost convergent. Remark 7 (1), implies that $D_{\mathbb{N}} \subseteq \mathcal{NA}_{\mathbb{N}}$, hence from Corollary 7 we get $\mu_\infty(\mathcal{NA}_{\mathbb{N}}) = 1$ (for any strictly positive and regular Borel probability measure μ on K).

We now show that the set $\mathcal{NA}_M$ (for any nonempty subset M of $\mathbb{N}$) is "big" in the sense of category. Let $\mathcal{NA}_1$ be the set of sequences $\alpha \in K^{\mathbb{N}}$ which are not Cesaro summable. Since clearly $\mathcal{NA}_1 \subseteq \mathcal{NA}_M \subseteq \mathcal{NA}_{\mathbb{N}}$ for every nonempty subset M of $\mathbb{N}$, it is enough to show that $\mathcal{NA}_1$ is "big" in the sense of category.

Proposition 7. *The set $\mathcal{NA}_1$ contains a dense G_δ subset of $K^{\mathbb{N}}$.*

Proof. Let $c = \min K$, $C = \max K$ $(c < C)$. It suffices to show that the set

$$U = \{\alpha \in K^{\mathbb{N}} : d^+(\alpha) = C, d^-(\alpha) = c\}$$

is a dense G_δ subset of $K^{\mathbb{N}}$.

Obviously $U = V \cap W$, with

$$V = \{\alpha \in K^{\mathbb{N}} : d^-(\alpha) = c\} \text{ and } W = \{\alpha \in K^{\mathbb{N}} : d^+(\alpha) = C\}.$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We set

$$V_\varepsilon = \{\alpha \in K^{\mathbb{N}} : d^-(\alpha) \leq c + \varepsilon\}$$

and

$$V_\varepsilon(n) = \{\alpha \in K^{\mathbb{N}} : \exists k \geq n \text{ with } \frac{1}{k}\sum_{i=1}^{k} \alpha_i < c + \varepsilon + \frac{1}{n}\}.$$

It is easy to check that $V_\varepsilon(n)$ is an open and dense subset of $K^{\mathbb{N}}$ and so the set $V_\varepsilon = \bigcap_{n=1}^{\infty} V_\varepsilon(n)$ is from Baire's Theorem a dense G_δ subset of $K^{\mathbb{N}}$. Hence $V = \bigcap_{n=1}^{\infty} V_{\frac{1}{n}}$ is dense and G_δ in $K^{\mathbb{N}}$. In a similar way we prove that W has the same property. $\square$

We conclude with some open questions concerning Banach spaces of the form $E(\alpha)$, $\alpha \in \ell^\infty(\mathbb{N})$:

1. Assume that the Banach space $E(\alpha)$ does not contain ℓ^1. Does then $E(\alpha)$ have a separable dual?

2. Assume that $E(\alpha)$ contains neither ℓ^1 nor c_0. Is then α an almost convergent sequence? In particular, does there exist an $\alpha \in \ell^\infty(\mathbb{N})$, such that $E(\alpha)$ is an infinite dimensional Banach space not containing the spaces ℓ^1 and c_0?

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University of Athens
Department of Mathematics
15784 Athens, Greece
e-mail: `smercour@math.uoa.gr`

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