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LITTLE G. T. FOR l_p-LATTICE SUMMING OPERATORS*

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Abstract. In this paper we introduce and study the l_p-lattice summing operators in the category of operator spaces which are the analogous of p-lattice summing operators in the commutative case. We study some interesting characterizations of this type of operators which generalize the results of Nielsen and Szulga and we show that $\Lambda_{l_\infty}(B(H), OH) \neq \Lambda_{l_2}(B(H), OH)$, in opposition to the commutative case.

Introduction. The notion of p-lattice summing was introduced and studied by Yanovskii in [24] for $p = 1$ and generalized by Nielsen and Szulga in [16, 22]. In this paper we extend this notion and some results to the theory of operator spaces (or the non-commutative case) which is recently studied by [1, 2, 3, 6, 7, 19, 20, 21].

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The paper is divided into four sections. In the first one, we recall some classical definitions and we give some preliminary facts such that: convexity and concavity, order bounded operators and the notion of p-summing operators.

In section two, we try to recover some definitions and results concerning the recent theory of operator spaces and we give some remarks about completely bounded operators.

We study in section three the notion of l_p-lattice summing operators $u : E \longrightarrow Y$ between an operator space E and a Banach lattice X, which extends the p-lattice summing operators. This generalization is a natural non-commutative analogous of the notion of p-lattice summing operators. We show some interesting characterizations of this type of operators. We also give briefly the connection between l_p-summing (as studied in [14]) and l_p-lattice summing operators for some special spaces.

In the final section, we show that

$$\pi_{l_p}\left(B\left(H\right),OH\right) \neq \Lambda_{l_2}\left(B\left(H\right),OH\right)$$

for all $2 < p < \infty$ in contrast to Banach space theory.
We finish this paper by mentioning that the little Grothendieck's theorem is not true for this notion.

1. Notation and preliminaries. For the background concerning ordered vector spaces and Banach lattices we refer to [13] and [25]. Let n be an integer. For a Banach lattice X and $1 \leq p \leq \infty$, we denote by $X\left(l_p^n\right)$ (the reader can consult [13, Part II. pp. 40–43]) the space of all sequences $x = (x_1, \ldots, x_n)$ of elements of X for which

$$\|x\|_{X\left(l_p^n\right)} = \left\|\left(\sum_1^n |x_i|^p\right)^{\frac{1}{p}}\right\| \text{ if } 1 \leq p < \infty$$

and

$$\|x\|_{X\left(l_\infty^n\right)} = \left\|\sup_{1\leq i\leq n} |x_i|\right\| \text{ if } p = \infty.$$

The space $X\left(l_p^n\right)$ is a Banach lattice equipped with the natural order

$$x \leq y \Longleftrightarrow \forall i, \quad x_i \leq y_i.$$

Let now X be a Banach space and $1 \leq p \leq \infty$. We denote by $l_p(X)$ (resp. $l_p^n(X)$) the space of all sequences (x_i) in X with the norm

$$\|(x_i)\|_{l_p(X)} = \left(\sum_1^\infty \|x_i\|^p\right)^{\frac{1}{p}}$$

$$(\text{resp. } \left\|(x_i)_{1\leq i\leq n}\right\|_{l_p^n(X)} = \left(\sum_1^n \|x_i\|^p)^{\frac{1}{p}}\right).$$

and by $l_p^\omega(X)$ (resp. $l_p^{n\ \omega}(X)$) the space of all sequences (x_i) in X with the norm

$$\|(x_n)\|_{l_p^\omega(X)} = \sup_{\|\xi\|_{X^\star}=1} \left(\sum_1^\infty |\langle x_i, \xi\rangle|^p)^{\frac{1}{p}}\right)$$

$$(\text{resp. } \|(x_n)\|_{l_p^{n\ \omega}(X)} = \sup_{\|\xi\|_{X^\star}=1} \left(\sum_1^n |\langle x_i, \xi\rangle|^p)^{\frac{1}{p}}\right).$$

We continue these preliminaries by recalling the definition of the p-convexity and p-concavity.

Definition 1.1. *Let E be an arbitrary Banach space, X a Banach lattice and let $1 \leq p \leq \infty$.*

(i) A linear operator $u : E \longrightarrow X$ is called p-convex if there is a constant C such that, for all n in $\mathbb{N}$ the operators

$$\begin{array}{lcl} l_p^n(E) & \longrightarrow & X\left(l_p^n\right) \\ (x_1, \ldots, x_n) & \longmapsto & (u(x_1), \ldots, u(x_n)) \end{array}$$

are uniformly bounded by C.

(ii) A linear operator $u : X \longrightarrow E$ is called p-concave if there is a constant C such that, for all n in $\mathbb{N}$ the operators

$$\begin{array}{lcl} X\left(l_p^n\right) & \longrightarrow & l_p^n(E) \\ (x_1, \ldots, x_n) & \longmapsto & (u(x_1), \ldots, u(x_n)) \end{array}$$

are uniformly bounded by C.
The smallest constant C for which this holds is denoted by $C^p(u)$ and $C_p(u)$ respectively.
A Banach lattice X is p-convex (resp. p-concave) if id_X is p-convex (resp. p-concave).

Remark 1.2. Any linear p-convex (resp. p-concave) operator u is bounded and $\|u\| \leq C^p(u)$ (resp. $\|u\| \leq C_p(u)$).
Every Banach lattice is 1-convex and ∞-concave. The p-convexity and p-concavity for $1 \leq p \leq \infty$ are decreasing and increasing in p, respectively see [13, Part II. 1.d.5]. For example, L_p for $1 \leq p < \infty$ is p-convex and p-concave, and $C^p(L_p) = C_p(L_p) = 1$.
Suppose now that X is a complete Banach lattice. An operator $u \in B(E, X)$ is called order bounded (see [15, 8]) if $u(B_X)$ is an order bounded subset of X. In this case, we put

$$M(u) \quad = \quad \left\| \sup_{x \in B_X} |u(x)| \right\|.$$

We can show that (see [22] or [11]) M is a norm on $M(E, X)$, the space of all order bounded maps from E to X.
If $w : X \longrightarrow Y$ (Y a complete Banach lattice) is a positive operator (i.e., $w(x) \geq 0$, for all x in X^+), then wu is order bounded.

The following simple remark will be needed in the sequel. For more precision see for example [8, Lemma 1.1].

Remark 1.3. Let n be an integer and X be a Banach lattice. Let $v : l_{p^*}^n \longrightarrow X$ such that $v(e_i) = x_i$ $(1 \leq p \leq \infty)$. We have

$$M(v) = \|(x_i)_{1 \leq i \leq n}\|_{X(l_p^n)} = \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \right\|. \tag{1.1}$$

If $X = C(K)$, $M(v) = \|v\|$.

We give now the *p-lattice summing* $(1 \leq p \leq \infty)$ notion for operators from a Banach space with values in a Banach lattice. It was first studied by Yanovskii in [24] for $p = 1$ and by Nielsen and Szulga in [16] and [22] for $p > 1$.

Definition 1.4. *Let $1 \leq p \leq \infty$. Let X be a Banach space, Y a Banach lattice and let $u : X \longrightarrow Y$ be a linear operator. We will say that u is "p-lattice summing" if there is a positive constant C such that for every n in $\mathbb{N}$ and $(x_1, \ldots, x_n)$ in E, we have*

$$\|(u(x_i))\|_{X(l_p^n)} \leq C \|(x_i)\|_{l_p^n\ \omega(E)}.$$

(If $p = \infty$ the sums should be replaced by sup).

We write $\lambda_p(u)$ for the smallest constant C satisfying the above inequality.

We will denote by $\Lambda_p(E, X)$ the space of all p-lattice summing operators, which is a Banach space if we consider as the norm $\lambda_p(.)$.

Remark 1.5. Let $u : E \longrightarrow X$ be a linear operator from a Banach space E into a complete Banach lattice X. Then,

$$u \in \Lambda_\infty (E, X) \Longleftrightarrow u \text{ is } \infty\text{-convex } \Longleftrightarrow u \in M (E, X)$$

and

$$\lambda_\infty (u) = C^\infty (u) = M (u) . \tag{1.2}$$

2. An introduction to operator spaces. If H is a Hilbert space, we let $B(H)$ denote the space of all bounded operators on H and for every n in $\mathbb{N}$ we let M_n denote the space of all $n \times n$-matrices of complex numbers, i.e., $M_n = B(l_2^n)$. If X is a subspace of some $B(H)$ and $n \in \mathbb{N}$, then $M_n (X)$ denotes the space the space of all $n \times n$-matrices with X-valued entries which we in the natural manner consider as a subspace of $B(l_2^n(X))$. An *operator space* X is a norm closed subspace of some $B(H)$ equipped with the distinguised matrix norm inherited by the spaces $M_n (X)$, $n \in \mathbb{N}$.

In [19], Pisier constructed the operator Hilbert space OH (i.e., the unique space verifying $\overline{OH^\star} = OH$ completely isometrically as in the case of Banach spaces, because there are Hilbert spaces in this category which are not completely isometric) and generalized in [20] (also Junge [9]) the notion of p-summing operators to the non-commutative case.

Let H be a Hilbert space. We denote by $S_p (H)$ $(1 \leq p < \infty)$ the Banach space of all compact operators $u : H \longrightarrow H$ such that $Tr(|u|^p) < \infty$, equipped with the norm

$$\|u\|_{S_p(H)} = (Tr(|u|^p))^{\frac{1}{p}}.$$

$H = l_2$ (resp. l_2^n), we denote simply $S_p (l_2)$ by S_p (resp. $S_p (l_2^n)$ by S_p^n). We denote also by $S_\infty (H)$ (resp. S_∞) the Banach space of all compact operators equipped with the norm induced by $B(H)$ (resp. $B(l_2)$) ($S_\infty^n = B(l_2^n)$). Recall that if $\dfrac{1}{p} = \dfrac{1}{q} + \dfrac{1}{r}$ $(1 \leq p, q, r < \infty)$, then $u \in B_{S_p(H)}$ if and only if there are $u_1 \in B_{S_q(H)}$, $u_2 \in B_{S_r(H)}$ such that $u = u_1 u_2$, where $B_{S_p(H)}$ is the closed unit ball of $S_p (H)$.

Let now X be a vector space. If for each $n \in \mathbb{N}$, there is a norm $\|\cdot\|_n$ on $M_n (X)$, the family of norms $\{\|\cdot\|_n\}_{n\geq 1}$ is called an L_p-matricial structure on X for $1 \leq p \leq \infty$ if: for all a, b in $M_n (\mathbb{C}) = B (l_2^n)$; $x \in M_n (X)$ and $y \in M_m (X)$ we have

$$(i) \quad \|axb\|_n \leq \|a\|_{M_n(\mathbb{C})} \|x\|_n \|b\|_{M_n(\mathbb{C})}$$

$$(ii) \quad \|x \oplus y\|_{n+m} = \begin{cases} (\|x\|_n^p + \|y\|_m^p)^{\frac{1}{p}} & \text{if } p \text{ is finite} \\ \max \{\|x\|_n , \|y\|_m\} & \text{if } p \text{ is infinite,} \end{cases}$$

where

$$\|x \oplus y\|_{n+m} = \left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m}.$$

We say that X is L_p-matricially normed if it is equipped with an L_p-matricial structure (which we suppose complete). Ruan proved in [21] and simplified (with Effros) in [7] an important theorem which is an abstract matrix norm characterization of operator spaces. This theorem states that: for any L_∞-matricial structure on a vector space X, there is a Hilbert space H and an embedding of X into $B(H)$ such that for all $n \geq 1$, the norm $\|\cdot\|_n$ on $M_n(X)$ coincides with the norm induced by the space $B(l_2^n(H))$. In other words, he has given an abstract characterization of operator spaces. Also in [12] we have proved that, if X is p-matricially normed with $p = 1$, then there is an operator structure on X such that $M_n(X) = S_1^n[X]$, where $S_1^n[X]$ is the finite dimensional version of $S_1[X] = S_1 \widehat{\otimes} X$, the projective tensor product of S_1 by X which is introduced in [3, 6] and [6]. For $p \neq 1$, the problem is open.

Definition 2.1. *Let H, K be Hilbert spaces. Let $X \subset B(H)$ and $Y \subset B(K)$ be two operator spaces. A linear map $u : X \longrightarrow Y$ is completely bounded (in short c.b.) if the maps*

$$\begin{array}{rccc} u_n & : & M_n(X) & \longrightarrow & M_n(Y) \\ & & (x_{ij})_{1 \leq i,j \leq n} & \longmapsto & (u(x_{ij}))_{1 \leq i,j \leq n} \end{array}$$

are uniformly bounded when $n \longrightarrow +\infty$, i.e. $\sup_{n \geq 1} \|u_n\| < +\infty$.

In this case we put, $\|u\|_{cb} = \sup_{n \geq 1} \|u_n\|$ ($\Longrightarrow \|u\| \leq \|u\|_{cb}$) and we denote by $cb(X, Y)$ the Banach space of all *c.b.* maps from X into Y which is also an operator space ($M_n(cb(X, Y)) = cb(X, M_n(Y))$, see [3, 6]. If we denote by $X \otimes_{\min} Y$ the subspace of $B(H \otimes_2 K)$ with induced norm, it is well known by [17] that

$$\|u\|_{cb} = \left\| I_{B(l_2)} \otimes u \right\|_{B(l_2) \otimes_{\min} X \longrightarrow B(l_2) \otimes_{\min} Y}.$$

We continue our introduction by mentioning briefly some properties concerning completely bounded operators. Consider $Y \subset A$ (a commutative C^*-algebra) $\subset B(H)$ and let X be an arbitrary operator space. Then,

$$B(X, Y) = cb(X, Y)$$

and

$$\|u\| = \|u\|_{cb}. \tag{2.1}$$

Because $M_n \otimes_{\min} Y \equiv M_n \otimes_\epsilon Y$ isometrically ($M_n \otimes_\epsilon Y$ is the injective tensor product of M_n by Y in the commutative case), see for example [18, p. 69, Corollary 3.18].
We recall that by [19, Proposition 1.5, p. 18] OH is homogeneous, namely, every bounded linear operator $u : OH \longrightarrow OH$ is automatically $c.b$ and

$$\|u\| = \|u\|_{cb}. \tag{2.2}$$

Note also by Corollary 2.4 in [19] that S_2 is completely isometric to $OH \times OH$. We denote by OH_n the n-dimensional version of the Hilbert operator space OH. If now S_2^N ($N \in \mathbb{N}$) is equipped with the operator space structure OH_{N^2}, then for any linear map $u : S_2^N \longrightarrow OH_{N^2}$ we have by homogeneity of OH

$$\|u\| = \|u\|_{cb}. \tag{2.3}$$

Let now $X \subset B(H)$. We have by Pisier [20, p. 32]

$$l_\infty(X) = l_\infty \otimes_{\min} X = B(l_1, X).$$

We can show that for all n in $\mathbb{N}$ and $1 \leq p \leq \infty$

$$\|v\|_{cb} = \sup_{a,b \in B^+_{S_{2p}(H)}} \Big(\sum_1^n \|ax_ib\|^p_{S_p(H)}\Big)^{\frac{1}{p}} = \Big\|\sum_1^n e_j \otimes x_j\Big\|_{l_p^n \otimes_{\min} X} \tag{2.4}$$

if p is finite and

$$\|v\|_{cb} = \Big\|\sum_1^n e_j \otimes x_j\Big\|_{l_\infty^n \otimes_{\min} X} = \Big\|\sum_1^n e_j \otimes x_j\Big\|_{l_\infty^n \otimes_\epsilon X} = \|v\| \tag{2.5}$$

if $p = \infty$. Where $v : l_{p^*}^n \longrightarrow X$ such that $v(e_i) = x_i$.

3. l_p-lattice summing operators.

We now give the *l_p-lattice summing* ($1 \leq p \leq \infty$) notion for operators from an operator space with values in a Banach lattice as an adaptation of the non-commutative case see [9, 14] to p-lattice summing as used in [22, 24] and we characterize them. The non-commutative version can be introduced as follows.

Definition 3.1. *Let $1 \leq p \leq \infty$. Let $E \subset B(H)$ be an operator space, X be a complete Banach lattice and $u : E \longrightarrow X$ be a linear operator. We will* *say that u is "l_p-lattice summing" if there is a positive constant C such that for every n in $\mathbb{N}$ the mappings*

$$\begin{array}{rccc} U_n: & l_p^n \otimes_{\min} E & \longrightarrow & X\left(l_p^n\right) \\ & \sum_1^n e_i \otimes x_i & \longmapsto & (u(x_1), \ldots, u(x_n)) \end{array}$$

are uniformly bounded by C (i.e. $\|U_n\|_{l_p^n \otimes_{\min} X \longrightarrow X(l_p^n)} \leq C$).

We denote by $\lambda_{l_p}(u) = \sup_n \|U_n\|_{l_p^n \otimes_{\min} X \longrightarrow X\left(l_p^n\right)}$.

We will denote by $\Lambda_{l_p}(E, X)$ the space of all l_p-lattice summing operators and we equip it with the norm $\lambda_{l_p}(\cdot)$ for which it becomes a Banach space.

We will need by (2.4) and (2.5) the following reformulation of the above definition.

The operator u is l_p-lattice summing and $\lambda_{l_p}(u) \leq C$, if and only if, for every n in $\mathbb{N}$ and every linear operator $v : l_{p^*}^n \longrightarrow E$ we have

$$\left\| \left(\sum_1^n |uv(e_i)|^p \right)^{\frac{1}{p}} \right\| \leq C \|v\|_{cb} \tag{3.1}$$

for $p < \infty$. For the case $p = \infty$, the sum should be replaced by sup. The space $l_{p^*}^n$ is equipped with its natural operator space structure, see [20, Chapter 2].

From this equivalence we obtain the following remark.

Remark 3.2.

1. p-lattice summing $\Longrightarrow$ l_p-lattice summing and $\lambda_{l_p}(u) \leq \lambda_p(u)$.

2. Let $E \subset A$ (a commutative C^*-algebra) $\subset B(H)$ and let X be an arbitrary Banach lattice. Then by (2.1) and (3.1), we have

$$\Lambda_{l_p}(E, X) = \Lambda_p(E, X).$$

3. If $E = OH$ we have, $\Lambda_{l_2}(E, X) = \Lambda_2(E, X)$ and $\lambda_{l_2}(u) = \lambda_2(u)$ because $l_2(I)$ is by [20, Proposition 2.1, p. 32] completely isometric to $OH(I)$ for any index set I.

Recalling now the definition of l_p-summing operator as studied in [14]. An operator u between an operator space $E \subset B(H)$ and a Banach space X is l_p-summing if there is a constant C such that for all n in $\mathbb{N}$ and all finite sequence $(x_i)_{1 \leq i \leq n}$ in X, we have

$$\left(\sum_1^n \|u(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{a,b \in B^+_{S_{2p}(H)}} \left(\sum_{n=1}^n \|a x_i b\|^p_{S_p(H)} \right)^{\frac{1}{p}}.$$

In other words, u transform weakly l_p-summable sequences in the non-commutative case into strongly l_p-summable sequences in the commutative case.

We denote by $\pi_{l_p}(u)$ the smallest constant C for which this holds and by $\Pi_{l_p}(E,X)$ the space of all l_p-summing operators with the norm $\pi_{l_p}(\cdot)$ which becomes a Banach space. We have

4. $X,\ p-\text{concave} \implies (\lambda_{l_p}(u) \implies \pi_{l_p}(u)),$
$X,\ p-\text{convex} \implies (\pi_{l_p}(u) \implies \lambda_{l_p}(u)),$
$X = L_p \implies (\pi_{l_p}(u) = \lambda_{l_p}(u)).$

Remark 3.3. We have from (2.3) that the operator u is l_p-lattice summing and $\lambda_{l_p}(u) \leq C$ if and only if

$$\left\| \left(\sum_1^n |u(x_i)|^p \right)^{\frac{1}{p}} \right\| \leq C \sup_{a,b \in B^+_{S_{2p}(H)}} \left(\sum_{n=1}^n \|a x_i b\|^p_{S_p(H)} \right)^{\frac{1}{p}} \tag{3.2}$$

if p is finite and if p is infinite we have by (2.5)

$$\Lambda_{l_\infty}(E,X) = \Lambda_\infty(E,X)$$

and

$$\lambda_{l_\infty}(u) = \lambda_\infty(u). \tag{3.3}$$

Let now $u : X \longrightarrow Y$ be a bounded linear operator between Banach lattices X, Y. We say that u is p-regular $(1 \leq p \leq \infty)$ if there is a positive constant C such that for all finite sequence $(x_i) \subset X$, we have

$$\left\| \left(\sum |T(x_i)|^p \right)^{\frac{1}{p}} \right\| \leq C \left\| \left(\sum |(x_i)|^p \right)^{\frac{1}{p}} \right\|.$$

The best possible constant will be denoted by $\rho_p(u)$.

We will denote by $\rho_p(X,Y)$ the space of all p-regular operators and we equip it with the norm $\rho_p(\cdot)$ for which it becomes a Banach space.

Recall that by Krivine [10] (see also [13, Part.II.1.f.14 and 1.d.9]) every linear operator is 2-regular and every positive operator is p-regular for $1 \leq p \leq \infty$. If $p = 2$, $\rho_p(w) = K_G \|w\|$ (K_G is the universal Grothendieck constant) and $\rho_p(w) = \|w\|$ if $p \neq 2$.

Proposition 3.4. *If $X = C(K)$, we have*

u is p-regular if and only if u is l_p-lattice summing

and

$$\rho_p(u) = \lambda_{l_p}(u).$$

Proof. By (2.1), it is easy to see that: u is p-regular if and only if for every n in $\mathbb{N}$ and $v : l_{p^*}^n \longrightarrow X$, we have

$$\left\| \left(\sum_1^n |uv(e_i)|^p \right)^{\frac{1}{p}} \right\| \leq \rho_p(u) M(v).$$

By Remark 1.3, we have $M(v) = \|v\| = \|v\|_{cb}$ (Remark 3.2.2) and we conclude by the proof, because $\|v\| = \|v(e_i)\|_{l_p^n w}$. □

Remark 3.5. (i) Clearly the class of l_p-lattice summing operators is not an ideal in Pietsch's sense but it is an ideal on left. Indeed, consider u, E, X as in Definition 3.1. Let E_0 be an operator space and let $u_0 : E_0 \longrightarrow E$ be a completely bounded operator. Then by (3.1), we have

$$\left\| \left(\sum_1^n |uu_0 v(e_i)|^p \right)^{\frac{1}{p}} \right\| \leq \lambda_{l_p}(u) \|u_0 v\|_{cb} \leq \lambda_{l_p}(u) \|u_0\|_{cb} \|v\|_{cb}.$$

Hence, uu_0 is l_p-lattice summing and $\lambda_{l_p}(uu_0) \leq \lambda_{l_p}(u) \|u_0\|_{cb}$.

(ii) On the other hand, if $w : X \longrightarrow Y$ is a bounded linear operator between Banach lattices X, Y such that w is p-regular (as defined above) for $1 \leq p \leq \infty$, then wu is l_p-lattice summing and $\lambda_{l_p}(wu) \leq \lambda_{l_p}(u) \rho_p(w)$. Indeed, always by (3.1) we have

$$\left\| \left(\sum_1^n |wuv(e_i)|^p \right)^{\frac{1}{p}} \right\| \leq \rho_p(w) \left\| \left(\sum_1^n |uv(e_i)|^p \right)^{\frac{1}{p}} \right\| \leq \lambda_{l_p}(u) \rho_p(w) \|v\|_{cb}.$$

We now give some characterizations for the l_p-lattice summing operators. The technique of proofs depend vigorously on ideas in [16], but slightly different from the original. Because we will be working with the finite dimensional l_p^n instead of L_p.

Theorem 3.6. *Let $1 \leq p \leq \infty$. Let E be any operator space and X be a complete Banach lattice. Then, the following properties of a positive constant C and a linear map $u : E \longrightarrow X$ are equivalent:*

(*i*) $u \in \Lambda_{l_p}(E, X)$ *and* $\lambda_{l_p}(u) \leq C$.
(*ii*) *For any* n *in* $\mathbb{N}$ *and any* $v : l_{p^*}^n \longrightarrow E$ *such that* $\|v\|_{cb} \leq 1$, *we have*

$$\lambda_{l_\infty}(uv) \leq C.$$

In this case we have

$$\lambda_{l_p}(u) = \sup\left\{M(uv) : n \in \mathbb{N}, v \in B_{cb\left(l_{p^*}^n, E\right)}\right\}.$$

P r o o f. Suppose in the first that p is finite. Consider $u \in \Lambda_{l_p}(E, X)$ and v in $B_{cb\left(l_{p^*}^n, E\right)}$. We have

$$uv(x) = uv\left(\sum_1^n \lambda_i e_i\right) = \sum_1^n \lambda_i uv(e_i)$$

and therefore

$$\begin{aligned} |uv(x)| &= \left|\sum_1^n \lambda_i uv(e_i)\right| \\ \text{(by Hölder's inequality)} &\leq \left(\sum_1^n |\lambda_i|^{p^*}\right)^{\frac{1}{p^*}} \left(\sum_1^n |uv(e_i)|^p\right)^{\frac{1}{p}} \\ &\leq \|x\| \left(\sum_1^n |uv(e_i)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\sup_{x \in B_{l_{p^*}^n}} |uv(x)| \leq \left(\sum_1^n |uv(e_i)|^p\right)^{\frac{1}{p}}.$$

Taking the norm on both sides

$$\begin{aligned} \left\|\sup_{x \in B_{l_{p^*}^n}} |uv(x)|\right\| &\leq \left\|\left(\sum_1^n |uv(e_i)|^p\right)^{\frac{1}{p}}\right\| \\ &\leq \lambda_{l_p}(u) \|v\|_{cb}. \end{aligned}$$

This implies by using (3.1) that

$$\lambda_{l_\infty}(uv) \leq \lambda_{l_p}(u) \leq C.$$

Conversely, let n be an integer in $\mathbb{N}$ and v be an operator in $B_{cb\left(l^n_{p^*},E\right)}$ such that $v(e_i) = x_i$. We have

$$\begin{aligned} \left\| \left(\sum_1^n |u(x_i)|^p \right)^{\frac{1}{p}} \right\| &= \left\| \left(\sum_1^n |uv(e_i)|^p \right)^{\frac{1}{p}} \right\| \\ \text{(by (1.1))} &\leq l(uv) \\ \text{(by (1.2) and (3.3))} &\leq \lambda_{l_\infty}(uv) \\ &\leq C. \end{aligned}$$

This implies $\lambda_{l_p}(u) \leq C$.

The case $p = \infty$. $i \Longrightarrow ii$ is trivial by Remark 3.5.i. Conversely, Let $x_1, \ldots, x_n$ in E, $\epsilon > 0$ and E_0 be the subspace of E spanned by $x_1, \ldots, x_n$. Consider the following diagram

$$l_1^N \xrightarrow{q} E_0 \xrightarrow{i} E \xrightarrow{u} X$$

where q is the canonical surjection from l_1^N (N is suitably chosen) into E_0. By (2.5), $\|q\|_{cb} = \|q\| \leq 1 + \epsilon$. Then we can take $v = iq$. Hence $uv \in \Lambda_{l_\infty}\left(l_1^N, X\right)$ implies that $u \in \Lambda_{l_\infty}(E, X)$ which concludes the proof. □

Corollary 3.7. *Consider p, E, X and u as in the above theorem. We have*

$$\Lambda_{l_\infty}(E, X) \subset \Lambda_{l_p}(E, X) \subset \Lambda_{l_2}(E, X)$$

and

$$\lambda_{l_2}(u) \leq \lambda_{l_p}(u) \leq \lambda_{l_\infty}(u).$$

Proof. The second inequality is a simple consequence of Remark 3.5.ii. Concerning the first and before embarking on the proof, let us recall some facts about l_2^n and its embedding into L_p^N. Let $D = \{-1, +1\}^{\mathbb{N}}$ equipped with its normalized uniform measure μ and its Borel σ-algebra $\mathcal{B}$. We denote by $\varepsilon_i : D \longrightarrow \{-1, +1\}$ the i-th coordinate and let $\mathcal{B}_n$ be the σ-algebra on D generated by the first n-coordinates. $L_p(D,\mathcal{B}_n,\mu)$ is isometric to $L_p^{2^n}$ (where L_p^N is the space $\mathbb{R}^N$ (or $\mathbb{C}^N$) equipped with the norm $\left\| \{\alpha_i\}_{1 \leq i \leq N} \right\|_{L_p^N} = \left(\frac{1}{N} \sum_1^N |\alpha_i|^p \right)^{\frac{1}{p}}$ if p is finite and we take $\max_{1 \leq i \leq N} |\alpha_i|$ if p is infinite.

From some classical inequalities of Kintchine, we have: for each p there are positive constants A_p and B_p such that

$$A_p\left(\sum_1^n |\alpha_i|^2\right)^{\frac{1}{2}} \le \left\|\sum_1^n \alpha_i\varepsilon_i\right\|_{L_p(D,\mathcal{B}_n,\mu)} \le B_p\left(\sum_1^n |\alpha_i|^2\right)^{\frac{1}{2}}.$$

We denote by R_p^n the closed linear subspace of $L_p^{2^n}$ of the functions $\{\varepsilon_i, 1 \le i \le n\}$. By the above inequalities, R_p^n is isomorphic to l_2^n. Let $r_p^n : R_p^n \longrightarrow l_2^n$ be this isomorphism. We can take $\left\|r_p^n\right\|_{cb} \le 1$. We know by [20, p. 109] that there is a c.b. projection $P_p : L_p^{2^n} \longrightarrow R_p^n$.

Consider now the following diagram

$$L_p^{2^n} \xrightarrow{P_p} R_p^n \xrightarrow{r_p^n} l_2^n \xrightarrow{v} E \xrightarrow{u} X.$$

If $u \in \Lambda_{l_p}(E,X)$ then $uvr_p^nP_p \in \Lambda_{l_\infty}(E,X)$. As $r_p^nP_p$ is surjective then $uv \in \Lambda_{l_\infty}(l_2^n, X)$ and therefore $u \in \Lambda_{l_2}(E,X)$. $\square$

Corollary 3.8. *If we replace E by OH in Corollary 3.7, then*

$$\Lambda_{l_\infty}(OH,X) = \Lambda_{l_2}(OH,X).$$

P r o o f. Consider u in $\Lambda_{l_2}(OH,X)$. Let n be in $\mathbb{N}$ and $i : OH_n \longrightarrow OH$ be the canonical injection. We have $ui \in \Lambda_{l_\infty}(OH_n, X)$ and by Remark 3.5.i, $u \in \Lambda_{l_\infty}(OH,X)$. $\square$

Before stating the next result, it will be convenient to recall here the following definition: we say that a bounded linear operator u between Banach spaces X, Y is integral and we write $u \in I(X,Y)$ if it admits a factorization

$$X \xrightarrow{\alpha} C(K) \xrightarrow{id} L_1(K,\mu) \xrightarrow{\beta} Y$$

where μ is a probability measure on a compact K and α, β are bounded linear operators. The integral norm of u is the infimum of all possible values of $\|\alpha\|\,\|\beta\|$ in the previous diagram. The integral operators $I(X,Y)$ with norm $\boldsymbol{i}(u)$ form a Banach operator ideal.

Proposition 3.9. *Let u be a linear operator from an operator space E into a complete Banach lattice X. Then, the following conditions of a positive constant C are equivalent:*

(*i*) $u \in \Lambda_{l_\infty}(E,X)$ *and* $\lambda_{l_\infty}(u) \le C$.

(*ii*) *For every n in $\mathbb{N}$ and every positive $w : X \longrightarrow l_1^n$, $\|w\| \leq 1$ then wu is integral and $\boldsymbol{i}(wu) \leq C$.*

Moreover,

$$\lambda_{l_\infty}(u) = \sup\left\{\boldsymbol{i}(wu), n \in \mathbb{N} \quad and\ w\ in\ B^+_{B(X,l_1^n)}\right\}.$$

P r o o f. $(i) \Longrightarrow (ii)$. By (3.3) we consider u in $\Lambda_\infty(E, X)$. Let n be in $\mathbb{N}$ and let $w : X \longrightarrow l_1^n$ be a positive operator. By Remark 1.5, we have wu order bounded from E into l_1^n and hence integral by [11, Proposition 3.1] because l_1^n is 1-concave with $C_1(l_1^n) = 1$ or by [5, Theorem 5.19, p. 104] we have

$$\begin{array}{lll} \boldsymbol{i}(wu) & = & M(wu) \\ \text{(by (1.3))} & = & \lambda_\infty(wu) \\ \text{(Remark 3.5.ii)} & \leq & \|w\|\,\lambda_\infty(u). \end{array}$$

$(ii) \Longrightarrow (i)$. Consider $x_1, \ldots, x_n$ in B_E. Let $y = \sup\{|u(x_i)|, 1 \leq i \leq n\}$. Let $L(y)$ be an abstract L_1-space generated by y (see [8, p. 221]) and let $w : X \longrightarrow L(y)$ the natural map (which is positive) such that $\|w(y)\| = \|y\|$. Let $M = \{|wu(x_i)|, 1 \leq i \leq n\} \cup \{w(y)\}$ which is finite and $\epsilon > 0$. We know by [5, Lemma 3.3] that there exists an N in $\mathbb{N}$, a finite rank projection p in $B(L(y))$, where $p(L(y))$ is isomertrically isomorphic to l_1^N with $N = \dim(p(L(y)))$, and $\|p\| = 1$ such that, for all i

$$\|pwu(x_i) - wu(x_i)\| < \epsilon$$

and

$$\|pw(y) - w(y)\| < \epsilon$$

Consider the following diagram

$$E \xrightarrow{u} X \xrightarrow{w} L(y) \xrightarrow{p} l_1^N.$$

Thus, we have that

$$\begin{aligned}
&\left\|\sup_{1\leq i\leq n}|u(x_i)|\right\| = \|y\| = \|w(y)\| = \left\|w\left(\sup_{1\leq i\leq n}|u(x_i)|\right)\right\| \\
&\leq (1+\epsilon)\|pw(y)\| = (1+\epsilon)\left\|pw\left(\sup_{1\leq i\leq n}|u(x_i)|\right)\right\| \\
&\leq (1+\epsilon)\left\|\left(\sup_{1\leq i\leq n}|pwu(x_i)|\right)\right\| \\
&\leq (1+\epsilon)\lambda_\infty(pwu) \\
&\leq (1+\epsilon)\boldsymbol{i}(pwu) \\
&\leq (1+\epsilon)C,
\end{aligned}$$

as desired. □

The next theorem translates the tie between the l_p-summing and l_p-lattice summing operators for $p=1$ or $p=2$.

Theorem 3.10. *Let $p\in\{1,2\}$. Let u be a linear operator from an operator space E into a complete Banach lattice X. The following properties are equivalent:*

(*i*) $u\in\Lambda_{l_p}(E,X)$.

(*ii*) *For every $n\in\mathbb{N}$ and every p-regular $w: X\longrightarrow l_1^n$ then, wu is l_p-summing.*

Moreover

$$\lambda_{l_p}(u) = \sup\left\{\pi_{l_p}(wu),\ n\in\mathbb{N},\ w: X\longrightarrow l_1^n \text{ ap-regular},\ \|w\|=1\right\}.$$

P r o o f. Let $1\leq p<\infty$. Consider u in $\Lambda_{l_p}(E,X)$, n in $\mathbb{N}$ and w a p-regular operator from X into l_1^n. As l_1^n is p-concave (Remark 1.2) we have

$$\begin{aligned}
\left(\sum_1^n\|wu(x_i)\|^p\right)^{\frac{1}{p}} &\leq \left\|\left(\sum_1^n|wu(x_i)|^p\right)^{\frac{1}{p}}\right\| \\
\text{(Remark 3.5.ii)}\qquad &\leq \|w\|\left\|\left(\sum_1^n|u(x_i)|^p\right)^{\frac{1}{p}}\right\| \\
&\leq \|w\|\,\lambda_{l_p}(u)\sup_{a,b\in B^+_{S_{2p}(H)}}\left(\sum_{n=1}^n\|ax_ib\|^p_{S_p(H)}\right)^{\frac{1}{p}}.
\end{aligned}$$

This implies that wu is l_p-summing and $\pi_{l_p}(wu) \leq \|w\| \lambda_{l_p}(u)$. Assume now that $p = 1$ or 2 and that $wu \in \pi_{l_p}(E, l_1^n)$ for every n in $\mathbb{N}$ and every p-regular $w : X \longrightarrow l_1^n$. Let $v \in cb\left(l_{p^*}^n, E\right)$. We have by [14, Remark 2.1, p. 702] that $wuv \in \pi_{l_p}\left(l_{p^*}^n, l_1^n\right)$ and hence by Remark 3.2.2 for $p^* = \infty$ and by Remark 3.2.3 for $p^* = 2$, we have $wuv \in \pi_p\left(l_{p^*}^n, l_1^n\right)$. This implies that wuv is integral. By Proposition 3.9, we have uv in $\Lambda_\infty\left(l_{p^*}^n, X\right)$ and by Theorem 3.6, we conclude that u is in $\pi_{l_p}(E, X)$. □

Remark 3.11. We can replace l_1^n in the first implication by any p-concave space Y and this by [4, Corollary 7] because every p-regular operator is p-concave.

4. Comparison with the commutative case.

Let us concentrate on the case where $E = B(H)$ and $X = OH$. The main result of this section is to prove that $\pi_{l_p}(B(H), OH) \neq \Lambda_{l_2}(B(H), OH)$, for all $2 < p < \infty$ unlike the commutative case [16, Theorem 1.5], where it is shown that $\pi_p(E, X) \subset \Lambda_2(E, X)$ for all p, $2 < p < \infty$.

Proposition 4.1. *Consider $2 < p < \infty$. Then,*

$$\pi_{l_p}(B(H), OH) \neq \Lambda_{l_2}(B(H), OH).$$

P r o o f. Suppose that $\pi_{l_p}(B(H), OH) \subset \Lambda_{l_2}(B(H), OH)$. Let u be in $\pi_{l_p}(B(H), OH)$, we have by Remark 3.2.4 that u is in $\pi_{l_2}(B(H), OH)$. This implies by Proposition 2.5 in [14] that $\pi_{l_p}(B(H), OH) = \pi_{l_2}(B(H), OH)$ which is impossible by [14, Theorem 4.1]. □

Finally, we end this work by the following theorem which is the principal result of this paper.

Theorem 4.2. *We have*

$$\Lambda_{l_\infty}(B(H), OH) \neq \Lambda_{l_2}(B(H), OH).$$

P r o o f. Suppose that $\Lambda_{l_2}(B(H), OH) \subset \Lambda_{l_\infty}(B(H), OH)$ (the converse is given by Corollary 3.7). Let u be in $\Lambda_{l_2}(B(H), OH)$. This implies also by Corollary 3.7 that $u \in \Lambda_{l_p}(B(H), OH)$. Thus, we have by Remark 1.2 and Remark 3.2.4 that $u \in \pi_{l_p}(B(H), OH)$. In this case we are in contradiction with Proposition 4.1. □

Remark 4.3. We can say that the little Grothendieck's theorem is not valid in the case of l_p-lattice summing operators.

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