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ON THE CLASSIFICATION OF UNITARY LOWEST WEIGHT REPRESENTATIONS OF CONFORMAL CURRENT ALGEBRAS

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To the memory of my teacher
Yaroslav Tagamlitzki

1. Introduction. In searching for the extreme elements of a partially ordered set K the good half of the work consists in devising a decomposition of a generic positive element of K into a sum (or a convex linear combination) of positive elements [1]. The purpose of this note is to point out the use of such a decomposition in a context, in which K is not a convex set.

The problem is concerned with the classification of the unitary lowest weight representations of an infinite dimensional Lie algebra VG, which appears in conformal quantum field theory models on the circle. In the rest of this introduction I shall define the algebra and say some words about its physical interpretation (for more detail see, e. g. [2]).

Let G be a simple compact Lie group and let d_G be its Lie algebra. To each element $X \in d_G$ we make correspond a sequence of elements X_n, $n \in Z$ of an infinite dimensional "affine Kac-Moody Lie algebra" $\widehat{d_G}$ in such a way that the following commutation relations hold:

$$[X_n, Y_m] = [X, Y]_{n+m} + kn(X, Y)\delta_{n,-m}$$

where (X, Y) is a suitably normalized invariant inner product (the positive "Killing form") on d_G, and k is a "central charge"

$$[k, X_n] = 0.$$

Thus, $\widehat{d_G}$ appears as a Z-graded infinite Lie algebra which contains d_G as a subalgebra (if we identify X with X_0).

The Virasoro algebra Vir is a central extension of the algebra of first-order differential operators that generate the diffeomorphisms of the circle. It is another Z-graded algebra with generators L_n and c satisfying

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \tag{1.3}$$

$$[c, L_n] = 0. \tag{1.4}$$

The conformal current algebra VG is defined as the semidirect sum of Vir with $\widehat{d_G}$, in which the commutation relations (1.1-1.4) are supplemented by

$$[X_m, L_n] = mX_{n+m}. \tag{1.5}$$

The physical context in which such an algebra appears is an 1-dimensional conformal quantum field theory on the circle with an internal symmetry group G. The fields of such a theory are defined in general as operator-valued distributions on a Hilbert space $\mathscr{H}$ (satisfying certain requirements). In our case they include the stress energy tensor

(1.6) $$T(z)=\Sigma_n L_n z^{-n-2},\ L_n^*=L_{-n}(n\in Z)$$

and the current

(1.7) $$X(z)=\Sigma_n X_n z^{-n-1},\ X_n^*=X_{-n},$$

(X belongs to a d_G dimensional space, where $d_G=\dim G$). The states of the theory include the vacuum vector $|0\rangle\in\mathscr{H}$ satisfying $\langle 0|0\rangle=1$ and

(1.8) $$L_n|0\rangle=0 \text{ for } n\geq -1,\ X_m|0\rangle=0 \text{ for } m\geq 0.$$

We make further the following irreducibility assumption: if a bounded operator B in $\mathscr{H}$ commutes with all the fields of the theory, then it is a multiple of the identity in $\mathscr{H}$. In particular, the central charges c and k act as multiplication of real numbers.

Corollaries. 1. The positivity of the inner product in $\mathscr{H}$ and the hermiticity properties in (1.6) and (1.7) imply that c and k are positive

(1.9) $$0<\|L_{-2}|0\rangle\|^2=\langle 0|L_2L_{-2}|0\rangle=\frac{c}{2}$$

(1.10) $$0<\|X_{-1}|0\rangle\|^2=\langle 0|X_1X_{-1}|0\rangle=k(X,X).$$

2. Eqs. (1.6-1.8) allow to find all correlation functions of the stress-energy tensor and the current:

(1.11) $$\langle 0|T(z_1)T(z_2)|0\rangle=\frac{c}{2}z_{12}^{-4},\ z_{12}=z_1-z_2.$$

(1.12) $$\langle 0|X(z_1)Y(z_2)|0\rangle=\frac{k}{z_{12}^2}(X,Y)$$

(1.13) $$\langle 0|X(z_1)Y(z_2)T(z_3)|0\rangle=k(X,Y)z_{13}^{-2}z_{23}^{-2} \text{ etc.}$$

The operator L_0 is interpreted physically as the energy operator. Energy positivity singles out lowest weight (LW) representations of VG. Any LW unitary irreducible representation (UIR) of VG is characterized by a LW vector $|\Psi\rangle$ such that

(1.14) $$L_n|\Psi\rangle=0=X_n|\Psi\rangle \text{for } n=1,2,\ldots (L_0-\Delta)|\Psi\rangle=0$$

(which is usually also assumed to be a highest weight eigenvector of a Cartan subalgebra of d_G).

The classification of LWUIRs of VG has only been completed recently [3], following developments in [4—7] among others. We shall review in Sec. 2 a central point of this derivation, related to a decomposition of $T(z)$.

2. LWUIRs of VG. Proposition 1. Given any unitary LW representation of d_G, it can be extended to a LW representation of VG by setting

(2.1) $$L_{2n}^G=\frac{1}{k+C_2}\{\vec{Q}_n^2+2\sum_{n=1}^{\infty}\vec{Q}_{n-m}\vec{Q}_{n+m}\},\ L_{2n+1}^G=\frac{2}{k+C_2}\sum_{l=0}^{\infty}\vec{Q}_{n-m}\vec{Q}_{n+1+m},$$

where $\vec{Q}=(Q_1,\ldots,Q^{dG})$ is an orthonormal basis of d_G so that

(2.2) $$(Q^a,Q^b)=\frac{1}{2}\delta_{ab},\ [Q_n^a,Q_m^b]=if_{abc}Q_{n+m}^c+\frac{kn}{2}\delta_{n,-m}\delta_{ab}$$

and G_2 is the eigenvalue of the second order Casimir operator for the adjoint representation of d_G

(2.3) $$f_{sta}f_{stb}=C_2\delta_{ab}.$$

The Virasoro central charge associated with L_n^G is

(2.4) $$c_G(=2[L_2^G, L_{-2}^G]-8L_0^G)=\frac{k}{k+C_2}\,d_G.$$

The proof (2.6) involves a straightforward (albeit somewhat lengthy) computation using (2.1-2.3) (One should first verify $[Q_m^a, L_n^G]=m\,Q_{m+n}^a$ and then (1.3), (2.4)).

Given a LWUIR of d_G, the construction of Proposition 1 gives a minimal representation of VG in the sense of the following result.

Proposition 2. Given an arbitrary LWUIR of VG the operators

(2.5) $$l_n=L_n-L_n^G$$

give rise to a LWUIR of Vir with central charge

(2.6) $$c_l=c-c_G\geq 0.$$

Proof. It follows from (1.5) and from Proposition 1 that

(2.7) $$[Q_m^a, l_n]=0=[L_m^G, l_n].$$

The rest is straightforward. The inequality (2.6) is a consequence of the assumed unitarity of the representation under consideration.

Propositions 1 and 2 reduce the study of the LWUIRs of VG to the known classification of such representations for d_G and Vir (2.4-2.7).

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