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ON THE PYTHAGOREAN THEOREM AND THE TRIANGLE INEQUALITY

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1. Introduction. The Pythagorean proposition states that

$$(P_0P_1)^2+(P_0P_2)^2=(P_1P_2)^2, \tag{1}$$

if and only if the vectors P_0P_1 and P_0P_2 are orthogonal. If $P_0P_1,\dots,P_0P_3$ are orthogonal, then the areas of the faces of $P_0P_1P_2P_3$ satisfy the following equation:

$$(P_0P_1P_2)^2+(P_0P_1P_3)^2+(P_0P_2P_3)^2=(P_1P_2P_3)^2; \tag{2}$$

however, examples show that (2) may hold even if $P_0P_1,\dots,P_0P_3$ are not mutually orthogonal. The triangle inequality states that $(P_1P_2)\leqq(P_0P_1)+(P_0P_2)$, and that the equality holds, if and only if P_0 is a point in the segment P_1P_2. Similarly,

$$(P_1P_2P_3)\leqq(P_0P_1P_2)+(P_0P_1P_3)+(P_0P_2P_3), \tag{3}$$

and the equality holds, if and only if P_0 is a point in the triangle $P_1P_2P_3$. There are generalizations of all these results for the m-simplex in R^n, and this note uses known theorems, especially theorems on determinants, to establish them.

2. The 3-simplex. Let P_k: $(x_k^1,\dots,x_k^n)$, $k=0, 1,\dots, 3$, be points in R^n, and let v_k, with components $(x_k^1-x_0^1,\dots,x_k^n-x_0^n)$, be the vectors from P_0 to P_1, P_2, P_3. Let (v_i, v_j) denote the inner product of v_i and v_j, and let $(P_1P_2P_3)$ denote the area of the face $P_1P_2P_3$ of $P_0\dots P_3$.

Theorem 1. *If $P_0\dots P_3$ is the simplex just described, then*

$$(P_1P_2P_3)^2=\frac{1}{(2!)^2}\left[\begin{vmatrix}(v_2, v_2) & (v_2, v_3)\\ (v_3, v_2) & (v_3, v_3)\end{vmatrix}+\begin{vmatrix}(v_1, v_1) & (v_1, v_3)\\ (v_3, v_1) & (v_3, v_3)\end{vmatrix}+\begin{vmatrix}(v_1, v_1) & (v_1, v_2)\\ (v_2, v_1) & (v_2, v_2)\end{vmatrix}\right] \tag{4}$$
$$+\frac{2}{(2!)^2}\left[(-1)^{1+2}\begin{vmatrix}(v_2, v_1) & (v_2, v_3)\\ (v_3, v_1) & (v_3, v_3)\end{vmatrix}+(-1)^{1+3}\begin{vmatrix}(v_2, v_1) & (v_2, v_2)\\ (v_3, v_1) & (v_3, v_2)\end{vmatrix}\right.$$
$$\left.+(-1)^{2+3}\begin{vmatrix}(v_1, v_1) & (v_1, v_2)\\ (v_3, v_1) & (v_3, v_2)\end{vmatrix}\right].$$

Proof. The proof will be given first for a simplex $P_0,\dots,P_3$ in R^3. The methods are completely general, however, and they can be used to prove the theorem in R^n. Let the points be P_k: (x_k, y_k, z_k), $k=0, 1, 2, 3$. Then [1, p. 167, 171]

$$(P_1P_2P_3)^2=\frac{1}{(2!)^2}\left[\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2+\begin{vmatrix}x_1 & z_1 & 1\\ x_2 & z_2 & 1\\ x_3 & z_3 & 1\end{vmatrix}^2+\begin{vmatrix}y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix}^2\right]. \tag{5}$$

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By an elementary property of determinants,

$$(6)\qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1-x_0 & y_1-y_0 & 1 \\ x_2-x_0 & y_2-y_0 & 1 \\ x_3-x_0 & y_3-y_0 & 1 \end{vmatrix}.$$

Expand the determinant on the right in (6) by minors of elements in the third column. Similar transformations of the other two determinants in (5) show that

$$(7)\qquad (P_1P_2P_3)^2=\frac{1}{(2!)}\left[\left\{\begin{vmatrix} x_2-x_0 & y_2-y_0 \\ x_3-x_0 & y_3-y_0 \end{vmatrix} - \begin{vmatrix} x_1-x_0 & y_1-y_0 \\ x_3-x_0 & y_3-y_0 \end{vmatrix}\right.\right.$$

$$\left.+\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ x_2-x_0 & y_2-y_0 \end{vmatrix}\right\}^2 + \left\{\begin{vmatrix} x_2-x_0 & z_2-z_0 \\ x_3-x_0 & z_3-z_0 \end{vmatrix} - \begin{vmatrix} x_1-x_0 & z_1-z_0 \\ x_3-x_0 & z_3-z_0 \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} x_1-x_0 & z_1-z_0 \\ x_2-x_0 & z_2-z_0 \end{vmatrix}\right\}^2 + \left\{\begin{vmatrix} y_2-y_0 & z_2-z_0 \\ y_3-y_0 & z_3-z_0 \end{vmatrix}\right.$$

$$\left.\left.-\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ y_3-y_0 & z_3-z_0 \end{vmatrix} + \begin{vmatrix} y_1-y_0 & z_1-z_0 \\ y_2-y_0 & z_2-z_0 \end{vmatrix}\right\}^2\right].$$

Square the expressions as indicated in (7) and collect the results in six braces. There are three similar expressions, the first of which is

$$(8)\qquad \frac{1}{(2!)^2}\left\{\begin{vmatrix} x_2-x_0 & y_2-y_0 \\ x_3-x_0 & y_3-y_0 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_0 & z_2-z_0 \\ x_3-x_0 & z_3-z_0 \end{vmatrix}^2 + \begin{vmatrix} y_2-y_0 & z_2-z_0 \\ y_3-y_0 & z_3-z_0 \end{vmatrix}^2\right\}.$$

There are three other similar expressions, the first of which is

$$(9)\qquad \frac{2(-1)^{1+2}}{(2!)^2}\left\{\begin{vmatrix} x_2-x_0 & y_2-y_0 \\ x_3-x_0 & y_3-y_0 \end{vmatrix}\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ x_3-x_0 & y_3-y_0 \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} x_2-x_0 & z_2-z_0 \\ x_3-x_0 & z_3-z_0 \end{vmatrix}\begin{vmatrix} x_1-x_0 & z_1-z_0 \\ x_3-x_0 & z_3-z_0 \end{vmatrix} + \begin{vmatrix} y_2-y_0 & z_2-z_0 \\ y_3-y_0 & z_3-z_0 \end{vmatrix}\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ y_3-y_0 & z_3-z_0 \end{vmatrix}\right\}.$$

Use the Binet-Cauchy multiplication theorem for determinants [1, pp. 589-591] to write (8) in the following form:

$$(10)\qquad \frac{1}{(2!)^2}\begin{vmatrix} (v_2, v_2) & (v_2, v_3) \\ (v_3, v_2) & (v_3, v_3) \end{vmatrix}.$$

There are similar determinants for the two expressions similar to (8). Use the Binet-Cauchy multiplication theorem for determinants again to represent (9) as follows

$$(11)\qquad \frac{2(-1)^{1+2}}{(2!)^2}\begin{vmatrix} (v_2, v_1) & (v_2, v_3) \\ (v_3, v_1) & (v_3, v_3) \end{vmatrix}.$$

There are similar determinants for the two expressions similar to (9). Equation (7) and the results indicated in (10) and (11) show that (4) is true, and the proof of Theorem 1 is complete for $P_0P_1P_2P_3$ in R^3.

The formula in (4) does not contain the dimension of the space in which P_0, P_1, P_2, P_3 are located. A review of the proof of the formula shows that it is valid in the form (4), if $P_0P_1P_2P_3$ is in R^n. □

Corollary 1. *If v_1, v_2, v_3 in Theorem 1 are mutually orthogonal vectors, then*

$$(P_1P_2P_3)^2=(P_0P_1P_2)^2+(P_0P_1P_3)^2+(P_0P_2P_3)^2. \tag{12}$$

Proof. If v_1, v_2, v_3 are mutually orthogonal, then

$$(v_1,\ v_2)=0,\ (v_1,\ v_3)=0,\ (v_2,\ v_3)=0. \tag{13}$$

Then, the last three determinants on the right in (4) are each equal to zero because each contains a row of zeros. Also [1, p. 167],

$$(P_0P_2P_3)^2=\frac{1}{(2\,!)^2}\begin{vmatrix}(v_2,\ v_2) & (v_2,\ v_3)\\ (v_3,\ v_2) & (v_3,\ v_3)\end{vmatrix}, \tag{14}$$

and the second and third determinants on the right in (4) have similar interpretations. Thus, if v_1, v_2, v_3 are mutually orthogonal, (4) is equivalent to (12). □

3. The m-simplex in R^n. The methods employed in Section 2 can be extended without change to treat a m-simplex in R^n. Let P_k: $(x_k^1, \dots, x_k^n)$, $k=0, 1, \dots, m$, be the vertices of a simplex $P_0P_1 \dots P_m$ in R^n, and let v_k be the vector whose components are $(x_k^1-x_0^1, \dots, x_k^n-x_0^n)$.

Theorem 2. *If $v_1, \dots, v_m$ are the vectors related to the simplex $P_0P_1 \dots P_m$ as just described, the volume $(P_1 \dots P_m)$ of $P_1 \dots P_m$ is given by the following formula*

$$(P_1 \dots P_m)^2=\frac{1}{[(m-1)!]^2}\sum_{i=1}^{n}\det\begin{bmatrix}v_1\\ \dots\\ \widehat{v_i}\\ \dots\\ v_m\end{bmatrix}\begin{bmatrix}v_1\\ \dots\\ \widehat{v_i}\\ \dots\\ v_m\end{bmatrix}^{\mathrm{T}} \tag{15}$$
$$+\frac{2}{[(m-1)!]^2}\sum(-1)^{i+j}\det\begin{bmatrix}v_1\\ \dots\\ \widehat{v_i}\\ \dots\\ v_m\end{bmatrix}\begin{bmatrix}v_1\\ \dots\\ \widehat{v_j}\\ \dots\\ v_m\end{bmatrix}^{\mathrm{T}}.$$

An explanation of the notation in (15) is necessary. The superscript T denotes the transpose of the matrix on which it is placed. A circumflex $\widehat{\ }$ over a symbol means that the symbol is omitted from the sequence in which it occurs. The second summation in (15) is extended over all i, j such that $1\leq i<j\leq m$. Finally,

$$\det\begin{bmatrix}v_1\\ \dots\\ \widehat{v_i}\\ \dots\\ v_m\end{bmatrix}\begin{bmatrix}v_1\\ \dots\\ \widehat{v_j}\\ \dots\\ v_m\end{bmatrix}^{\mathrm{T}}=\det\begin{bmatrix}(v_1,\ v_1) \cdots (v_1,\ \widehat{v_j}) \cdots (v_1,\ v_m)\\ \dots\dots\dots\dots\dots\dots\\ (\widehat{v_i},\ v_1) \cdots (\widehat{v_i},\ \widehat{v_j}) \cdots (\widehat{v_i},\ v_m)\\ \dots\dots\dots\dots\dots\dots\\ (v_m,\ v_1) \cdots (v_m,\ \widehat{v_j}) \cdots (v_m,\ v_m)\end{bmatrix}. \tag{16}$$

P r o o f o f T h e o r e m 2. The volume $(P_1 \dots P_m)$ is given by the following formula [1, Exercise 20.6, p. 171]

$$(17) \qquad (P_1 \dots P_m)^2 = \frac{1}{[(m-1)!]^2} \Sigma \begin{vmatrix} \overset{i}{x}{}_1^1 & \cdots & \overset{i}{x}{}_1^{m-1} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \overset{i}{x}{}_m^1 & \cdots & \overset{i}{x}{}_m^{m-1} & 1 \end{vmatrix}^2 .$$

The summation in (17) extends over all sets $\{i_1, \dots, i_{m-1}\}$ such that $1 \leq i_1 < i_2 \dots < i_{m-1} \leq n$. Multiply the last column of the determinant in (17) by $\overset{i}{x}{}_0^1$ and substract it from the first column: multiply the last column by $\overset{i}{x}{}_0^2$ and subtract it from the second column; and so forth. Then expand each determinant by minors of elements in the last column. By using the Binet-Cauchy multiplication theorem for determinants as in Section 2, the resulting expression can be transformed into the formula in (15). □

The formula in (15), in the special case in which $m=2$, is the Law of Cosines in trigonometry.

4. The Pythagorean theorem. Let $P_0P_1P_2$ be a triangle in R^n. The Pythagorean proposition states that

$$(18) \qquad (P_1P_2)^2 = (P_0P_1)^2 + (P_0P_2)^2,$$

if and only if the vectors P_0P_1 and P_0P_2 are orthogonal. The next theorem contains this theorem and its (partial) generalization for simplexes $P_0P_1 \dots P_m$ with $m > 2$.

T h e o r e m 3. *Let $P_0P_1, \dots, P_m$, $m \geq 2$, be the simplex in Section 3. If $v_1, \dots, v_m$ are mutually orthogonal, then*

$$(19) \qquad (P_1 \dots P_m)^2 = \sum_{i=1}^{m} (P_0P_1 \dots \widehat{P}_i \dots P_m)^2.$$

If $m=2$, then (19) holds only if v_1 and v_2 are orthogonal; but if $m>2$, then (19) holds in many cases in which $v_1, \dots v_m$ are not mutually orthogonal.

P r o o f. The statement in (19) will be provéd by showing that each determinant in the second summation in (15) is sero, if $v_1, \dots, v_m$ are mutually orthogonal. Since $1 \leq i < j \leq m$, then (16) shows that

$$(20) \qquad \det \begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_j \\ \cdots \\ v_m \end{bmatrix} \begin{bmatrix} v_1 \\ \cdots \\ v_i \\ \cdots \\ \widehat{v}_j \\ \cdots \\ v_m \end{bmatrix}^T .$$

contains the row

$$(21) \qquad (v_j, v_1), \dots, (v_j, v_i), \dots, (v_j, \widehat{v}_j), \dots, (v_j, v_m).$$

If $v_1, \dots, v_m$ are mutually orthogonal, then this row consists entirely of zeros and (20) equals zero. Thus, (19) is true, if $v_1, \dots, v_m$ are mutually orthogonal.

If $m=2$, the second summation in (15) contains the single term (v_1, v_2). Thus, (19) holds for $m=2$, if and only if v_1 and v_2 are orthogonal.

The proof of Theorem 3 will now be completed by constructing an example to show that, if $m>2$, then (19) may be true even if $v_1, \ldots, v_m$ are not mutually orthogonal. Let $P_0, \ldots, P_3$ be the following points:

$$P_0:\ (0,\ 0,\ 0),\ P_1:\ (1,\ 0,\ 0),\ P_2:\ (0,\ 1,\ 0),\ P_3:\ (x,\ y,\ z). \tag{22}$$

Assume that

$$xy-x-y=0. \tag{23}$$

Then

$$v_1=(1,\ 0,\ 0),\ v_2=(0,\ 1,\ 0),\ v_3=(x,\ y,\ z), \tag{24}$$

and a straightforward calculation shows that the second summation in (15) is

$$-\det\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_3 \end{bmatrix}^{\mathrm{T}}+\det\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^{\mathrm{T}}-\det\begin{bmatrix} v_1 \\ v_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^{\mathrm{T}}=xy-x-y=0. \tag{25}$$

The equation in (23) is satisfied, if $x=0$ and $y=0$, and in this case v_1, v_2, v_3 are mutually orthogonal. In all other cases, v_3 is not orthogonal to v_1 and v_2; nevertheless, the relation (19) holds for $P_0P_1 \ldots P_3$.

5. The triangle inequality. The following lemma is needed in the proof of the general case of the triangle inequality.

Lemma 1. *Let $P_0P_1 \ldots P_m$ be the simplex in Section 3. Then*

$$\text{abs. val.}\left\{\det\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_j \\ \cdots \\ v_m \end{bmatrix}^{\mathrm{T}}\right\} \tag{26}$$

$$\leq\left\{\det\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}^{\mathrm{T}}\right\}^{1/2}\left\{\det\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_j \\ \cdots \\ v_m \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ \widehat{v}_j \\ \cdots \\ v_m \end{bmatrix}^{\mathrm{T}}\right\}^{1/2}.$$

Proof. By the Binet-Cauchy multiplication theorem for determinants, the determinant on the left in (26) can be written as a sum of products of determinants [compare (8), ..., (11)]. Apply the Schwarz inequality [1, p. 606] to this sum of products. Then use the Binet-Cauchy multiplication theorem again, in order to state the result in the form shown in (26). □

Theorem 4. *Let $P_0P_1 \ldots P_m$, $m\geq 2$, be the simplex in Section 2. Then*

$$(P_1 \ldots P_m) \leq \sum_{i=1}^{m} (P_0P_1 \ldots \widehat{P}_i \ldots P_m). \tag{27}$$

Proof. By (15), $(P_i \ldots P_m)^2$ is equal to or less than the sum of the absolute values of all the terms on the right. Apply Lemma 1. It is known [1, Ex. 20.6, p. 171] that

$$(28)\qquad (P_0P_1\cdots\widehat{P}_i\cdots P_m)^2=\frac{1}{[(m-1)!]^2}\det\begin{bmatrix}v_1\\ \cdots\\ \widehat{v}_i\\ \cdots\\ v_m\end{bmatrix}\begin{bmatrix}v_1\\ \cdots\\ \widehat{v}_i\\ \cdots\\ v_m\end{bmatrix}^{\mathrm{T}}.$$

Thus, the inequality obtained from (15) can be written as follows

$$(29)\quad (P_1\cdots P_m)^2\leq\sum_{i=1}^{m}(P_0\cdots\widehat{P}_i\cdots P_m)^2+2\Sigma(P_0\cdots\widehat{P}_i\cdots P_m)(P_0\cdots\widehat{P}_j\cdots P_m).$$

The second sum on the right is extended over all i, j such that $1\leq i<j\leq m$. Thus,

$$(30)\qquad (P_1\cdots P_m)^2\leq\{\sum_{i=1}^{m}(P_0\cdots\widehat{P}_i\cdots P_m)\}^2,$$

and (30) is equivalent to (27). □

We now investigate conditions under which the equality holds in (27). The following lemma is needed.

Lemma 2. *Let P_k: $(x_k^1,\cdots x_k^n)$, $k=1,\cdots,m$, be points in R^n, $m-1\leq n$, and let*

$$(31)\qquad P_0=(\sum_{k=1}^{m}t_kx_k^1,\cdots,\sum_{k=1}^{m}t_kx_k^n),\quad \sum_{k=1}^{m}t_k=1.$$

Then

$$(32)\qquad \sum_{i=1}^{m}(P_0P_1\cdots\widehat{P}_i\cdots P_m)=(\sum_{k=1}^{m}|t_k|)(P_1\cdots P_m).$$

Furthermore, if P_0 is a point of the form (31), *then*

$$(33)\qquad (P_1\ldots P_m)=\sum_{i=1}^{m}(P_0P_1\cdots\widehat{P}_i\ldots P_m),$$

if and only if

$$(34)\qquad 0\leq t_k\leq 1,\ \sum_{k=1}^{m}t_k=1;$$

that is, (33) *holds, if and only if P_0 is a point in $P_1\cdots P_m$.*

Proof. Formula (17) shows that

$$(P_0P_1\cdots\widehat{P}_i\cdots P_m)^2=\frac{1}{[(m-1)!]^2}\Sigma\begin{vmatrix}x_0^{i_1}&\cdots&x_0^{i_{m-1}}&1\\ \cdot&\cdot&\cdot&\cdot\\ \widehat{x}_i^{i_1}&\cdots&\widehat{x}_i^{i_{m-1}}&1\\ \cdot&\cdot&\cdot&\cdot\\ x_m^{i_1}&\cdots&x_m^{i_{m-1}}&1\end{vmatrix}^2.$$

Multiply the row corresponding to P_k, $k=1, \ldots, \hat{i}, \ldots m$, by t_k and subtract it from the first row. The result shows that $(P_0P_1 \cdots \hat{P}_i \cdots P_m)=|t_i|(P_1 \cdots P_m)$, and (32) follows. Now (32) shows that

(35) $$\sum_{i=1}^{m} (P_0P_1 \cdots \hat{P}_i \cdots P_m)>(P_1 \cdots P_m),$$

unless $\Sigma_{k=1}^{m} |t_k|=1$, and this equation is true, if and only if (34) is satisfied. Now P_0 in (31) is in $P_1 \cdots P_m$, if and only if (34) holds. Thus, (35) holds if P_0 in (31) is not in $P_1 \cdots P_m$, and (33) holds if P_0 is in $P_1 \cdots P_m$. □

Theorem 5. *Let $P_1 \ldots P_m$ be a simplex in R^n, $m-1 \leq n$,* such that

(36) $$(P_1 \cdots P_m)>0.$$

If P_0 is in $P_1 \cdots P_m$, then

(37) $$(P_1 \cdots P_m) = \sum_{i=1}^{m} (P_0P_1 \cdots \hat{P}_i \cdots P_m);$$

if P_0 is not in $P_1 \cdots P_m$, then

(38) $$(P_1 \cdots P_m)< \sum_{i=1}^{m} (P_0P_1 \cdots \hat{P}_i \cdots P_m).$$

Proof. Lemma 2 has shown that (37) is true, if P_0 is in $P_1 \ldots P_m$, and that (38) is true, if P_0 is a point of the form shown in (31) but not in $P_1 \ldots P_m$. The proof of Theorem 5 can be completed by showing that (38) is true for all points P_0, which cannot be represented as shown in (31). The proof proceeds as follows: Let P_0: $(x_0^1, \ldots, x_0^n)$ be a point in R^n, which is not in the plane of $P_1 \ldots P_m$, and let H: $(h^1, \ldots, h^n)$ be the foot of the perpendicular from P_0 onto the plane of $P_1 \ldots P_m$. Then

(39) $$(HP_1 \cdots \hat{P}_1 \cdots P_m)< (P_0P_1 \cdots \hat{P}_i \cdots P_m), \quad i=1, \cdots, m,$$

(40) $$\sum_{i=1}^{m} (HP_1 \cdots \hat{P}_i \cdots P_m)< \sum_{i=0}^{m} (P_0P_1 \cdots \hat{P}_i \cdots P_m).$$

If H is in $P_1 \cdots P_m$, the sum on the left in (40) equals $(P_1 \cdots P_m)$ by Lemma 2, and (38) follows. If H is not in $P_1 \cdots P_m$, then

(41) $$(P_1 \cdots P_m)< \sum_{i=1}^{m} (HP_1 \cdots \hat{P}_i \cdots P_m)$$

again by Lemma 2, and (38) follows from (40) and (41). The proofs of these statements follow.

Let v_k: $(v_k^1, \cdots, v_k^n)$, $k=0, 2, \cdots, m$, be the vector with components $(x_k^1-x_1^1, \cdots, x_k^n-x_1^n)$. For every point $(x^1, \cdots, x^n)$ in the plane of $P_1 \cdots P_m$ there are numbers $t_2, \cdots, t_m$ in R such that

(42) $$(x^1, \cdots, x^n) = (x_1^1, \cdots, x_1^n) + \sum_{k=2}^{m} t_k v_k.$$

The altitude is a vector from $(x^1, \cdots, x^u)$ to $(x_1^1, \cdots, x_1^n)+v_0$; it is the vector

(43) $$v_0 = \sum_{k=2}^{m} t_k v_k.$$

The vector in (43) is an altitude from the plane of $P_1 \cdots P_m$ to P_0, if and only if it is orthogonal to each vector $v_2, \cdots, v_m$. Thus, if $t_2, \cdots, t_m$ satisfy the following equations, then $(x^1, \cdots, x^n)$ in (42) is H: $(h^1, \cdots, h^n)$, the foot of the altitude from P_0 to $P_1 \cdots P_m$:

(44) $$\begin{array}{l} (v_2, v_2)t_2 + \cdots + (v_m, v_2)t_m = (v_0, v_2), \\ \cdot, \\ (v_2, v_m)t_2 + \cdots + (v_m, v_m)t_m = (v_0, x_m). \end{array}$$

Hypothesis (36) shows that the determinant of the coefficients in (44) is not zero (compare (28); [1, p. 167-170]), thus (44) has a unique solution for $t_2, \cdots, t_m$. Henceforth, let $t_2, \cdots, t_m$ denote this solution. The altitude from H to P_0 has components $(x_0^1 - h^1, \cdots, x_0^n - h^n)$; denote it by w: $(w^1, \cdots, w^n)$. Now by (17),

(45) $$(P_0 P_1 \cdots \widehat{P}_i \cdots P_m)^2 = \frac{1}{[(m-2)!]^2} \sum \begin{vmatrix} x_0^{i_1} & \cdots & x_0^{i_{m-1}} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \widehat{x_i^{i_1}} & \cdots & \widehat{x_i^{i_{m-1}}} & \widehat{1} \\ \cdot & \cdot & \cdot & \cdot \\ x_m^{i_1} & \cdots & x_m^{i_{m-1}} & 1 \end{vmatrix}^2 .$$

Since $1 = 1 + 0$ and

(46) $$(x_0^1, \cdots, x_0^n) = (h^1 + w^1, \ldots, h^n + w^n),$$

the determinant in (45) equals

(47) $$\begin{vmatrix} h^{i_1} & \cdots & h^{i_{m-1}} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \widehat{x_i^{i_1}} & \cdots & \widehat{x_i^{i_{m-1}}} & \widehat{1} \\ \cdot & \cdot & \cdot & \cdot \\ x_m^{i_1} & \cdots & x_m^{i_{m-1}} & 1 \end{vmatrix} + \begin{vmatrix} w^{i_1} & \cdots & w^{i_{m-1}} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \widehat{x_i^{i_1}} & \cdots & \widehat{x_i^{i_{m-1}}} & \widehat{1} \\ \cdot & \cdot & \cdot & \cdot \\ x_m^{i_1} & \cdots & x_m^{i_{m-1}} & 1 \end{vmatrix}.$$

Replace the determinant in (45) by its value in (47), and square the terms as indicated. The sum of the squares of the first terms gives $(HP_1 \cdots \widehat{P}_i \cdots P_m)^2$ by (17) or (45). Except for a constant factor, the sum of the middle terms is a sum of products of the two determinants in (47). As a matter of notation, let u_k: $(u_k^1, \cdots, u_k^n)$ be the vector such that

$$(48) \qquad (u_k^1, \cdots, u_k^n) = (x_k^1 - h^1, \cdots, x_k^n - h^n), \quad k = 1, 2, \ldots, m.$$

In the first determinant in (47), subtract the first row from each of the other rows and expand by minors of elements in the last column; in the second determinant in (47), subtract the second row from each row which follows it and then expand by minors of elements in the last column. Thus, the sum of middle terms becomes, except for a constant multiplier, the following:

$$(49) \qquad \Sigma \begin{vmatrix} u_1^{i_1} & \cdots & u_1^{i_{m-1}} \\ u_2^{i_1} & \cdots & u_2^{i_{m-1}} \\ \cdot & \cdots & \cdot \\ \widehat{u}_i^{i_1} & \cdots & \widehat{u}_i^{i_{m-1}} \\ \cdot & \cdots & \cdot \\ u_m^{i_1} & \cdots & u_m^{i_{m-1}} \end{vmatrix} \begin{vmatrix} w^{i_1} & \cdots & w^{i_{m-1}} \\ v_2^{i_1} & \cdots & v_2^{i_{m-1}} \\ \cdot & \cdots & \cdot \\ \widehat{v}_i^{i_1} & \cdots & \widehat{v}_i^{i_{m-1}} \\ \cdot & \cdots & \cdot \\ v_m^{i_1} & \cdots & v_m^{i_{m-1}} \end{vmatrix} .$$

By the Binet-Cauchy multiplication theorem for determinants, the sum in (49) equals

$$(50) \qquad \det \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ \widehat{u}_i \\ \cdots \\ u_m \end{bmatrix} \begin{bmatrix} w \\ v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}^{T} .$$

Now $u_1, \ldots, u_m$ lie in the plane of $P_1 \ldots P_m$, and w is a normal to this plane. Thus, $(u_k, w) = 0$ for $k = 1, \ldots, m$ and the determinant in (50) is zero Finally, a similar analysis shows that the sum of squares of the second determinant in (47) is

$$(51) \qquad \det \begin{bmatrix} w \\ v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix} \begin{bmatrix} w \\ v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}^{T} .$$

Now $v_2, \ldots, v_m$ are in the plane of $P_1 \ldots P_m$, and w is normal to this plane; thus, $(v_k, w) = 0$ for $k = 2, \ldots, m$. Therefore, the determinant in (51) simplifies to

$$(52) \qquad (w, w)\det \begin{bmatrix} v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix} \begin{bmatrix} v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}^T .$$

Collect results, beginning with (45); the analysis has shown that

$$(53) \qquad (P_0P_1 \cdots \widehat{P}_i \cdots P_m)^2 = (HP_1 \cdots \widehat{P}_i \cdots P_m)^2 + \frac{(w, w)}{(m-1)^2}$$

$$\times \left\{ \frac{1}{[(m-2)!]^2} \det \begin{bmatrix} v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix} \begin{bmatrix} v_2 \\ \cdots \\ \widehat{v}_i \\ \cdots \\ v_m \end{bmatrix}^T \right\}$$

for $i=2, \ldots, m$. Now $(w, w)>0$ since, by hypothesis, P_0 is not in the plane of $P_1 \ldots P_m$. Also (compare (28); [1, p. 167-170]), the expression in the curly braces in (53) is the square of the measure (area, volume, etc.) of $P_1P_2 \cdots \widehat{P}_i \cdots P_m$. Now $(P_1P_2 \cdots P_m)$ equals the product of $1/(m-1)$, the length of the altitude from P_i to the plane of $P_1 \cdots \widehat{P}_i \cdots P_m$, and the square root of the expression in the braces in (53); therefore, the hypothesis in (36) that $(P_1 \cdots P_m)>0$ shows that the expression in the braces is positive. Thus, (53) shows that $(HP_1 \cdots \widehat{P}_i \cdots P_m) < (P_0P_1 \cdots \widehat{P}_i \cdots P_m)$ for $i=2, \ldots, m$. A similar analysis shows that the same inequality holds for $i=1$. Finally, (39), (40) and (41) show that (38) is true as stated, and the proof of Theorem 5 is complete. □

Another statement of the general triangle inequality is the following: If $P_1 \cdots P_m$ is a simplex in R^n such that $(P_1 \cdots P_m)>0$, then $\Sigma_{i=1}^m (P_0P_1 \cdots \widehat{P}_i \cdots P_m)$, considered as a function of P_0, takes on its minimum value at each point of $P_1 \cdots P_m$, and this minimum value is $(P_1 \cdots P_m)$.

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