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CAUCHY SEQUENCES AND COMPLETENESS IN QUASI-METRIC SPACES

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1. Introduction. A quasi-metric space (X, d) is a set equipped with a quasi-metric d, i. e. with a non-negative function d defined on $X\times X$ and satisfying the following two conditions:

a) $d(x, y)=0$, if and only if $x=y$;

b) $d(x, z)\leq d(x, y)+d(y, z)$ for $x, y, z\in X$.

When in addition the condition of symmetry $d(x, y)=d(y, x)$ is fulfilled, then d is a metric. Sometimes, when no ambiguity is possible, the space (X, d) is denoted simply by X. In particular, when $Y\subset X$, the subspace $(Y, d_{|Y})$ of the space (X, d) is usually denoted by Y.

Quasi-metric spaces represent a generalization of metric spaces and, on the other hand, are a special kind of quasi-uniform spaces.

Every quasi-metric space (X, d) can be considered as a topological space on which the topology is introduced by taking for every $x\in X$ the collection $\{B_r(x)\,|\,r>0\}$ for a base of the neighbourhood filter of the point x. Here, the ball $B_r(x)$ is defined as follows

$$B_r(x)=\{y\in X\,|\,d(x, y)<r\}.$$

According to this convention, a sequence $\{x_n\}$ in X is convergent to a point $x\in X$, which is denoted by $x_n\to x$, when $\lim d(x, x_n)=0$.

The interest in quasi-metric spaces is justified partly by the fact that every topological space is quasi-uniformizable and each T_1-quasi-uniformity with countable base is quasi-metrizable.

A basic problem in the theory of quasi-metric spaces is to define in a convenient manner the notion of completeness and to propose some construction of completion of a quasi-metric space. This construction should, of course, lead in the metric case (i. e. when (X, d) is a metric space) to the usual metric completion. The analogous problems arises evidently in a more general aspect in the theory of quasi-uniform spaces.

The notion of completeness has to be certainly founded on a concept of Cauchy sequence, which generalizes the concept of convergent sequence. Then, a space is complete, if every Cauchy sequence in it is convergent. A completion of a quasi-metric space (X, d) is a complete quasi-metric space (X^*, d^*) in which (X, d) can be quasi-isometrically embedded as a dense subspace. The standard method of constructing the completion of a metric space is based on an equivalence relation between Cauchy sequences. It is very natural to desire that this method is preserved for quasi-metric spaces.

It is clear that in the problems about completeness and completions only those quasi-metric spaces are worth to be consedered in which the topology is a Hausdorff one. The last assumption is equivalent to the following condition imposed on the quasi-metric d: if $\lim d(x, x_n)=0$ and $\lim d(y, x_n)=0$, then $x=y$ (i. e. it is impossible to have $x_n\to x$ and $x_n\to y$ for $x\neq y$). The quasi-metric d satisfying this condition will be called further T_2-quasi-metric.

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So the problem is posed as follows. A notion of Cauchy sequence in any T_2-quasi-metric space (X, d) has to be defined in such a manner that the following requirements are fulfilled:

(i) every convergent sequence is a Cauchy sequence;

(ii) every subsequence of a Cauchy sequence is a Cauchy sequence;

(iii) in the metric case (i. e. when (X, d) is a metric space) the Cauchy sequences are the usual ones.

Further, an equivalence relation between Cauchy sequences has to be inroduced so that:

(iv) every Cauchy sequence is equivalent to each its subsequence;

(v) for each $x \in X$ the collection $\varphi(x)$ of the sequences convergent to x is an equivalence class of Cauchy sequences;

(vi) the collection X^* of all equivalence classes of Cauchy sequences leads to a T_2-completion (X^*, d^*) of the space (X, d).

This means that:

(vi-a) a T_2-quasi-metric d^* can be introduced in X^* which extends d in the sense that $d^*(\varphi(x'), \varphi(x''))=d(x', x'')$, i. e. that the mapping $\varphi: (X, d) \to (X^*, d^*)$ is a quasi-metric embedding;

(vi-b) the space (X^*, d^*) is complete, i. e. every Cauchy sequence in $X^*, d^*)$ is convergent;

(vi-c) $\varphi(X)$ is dense in X^* (what is surely fulfilled, if for every $\xi \in X^*$ $\{x_k\} \in \xi$ implies $\varphi(x_k) \to \xi$).

At last, the following two requirements have to be satisfied:

(vii) if $(X, d_X) \subset (Y, d_Y)$, then $(X^*, d^*_X) \subset (Y^*, d^*_Y)$, where the inclusions are understood as quasi-metric embeddings and the second one is an extension of the former;

(viii) in the metric case (i. e. when d is a metric) (X^*, d^*) coincides with the usual metric completion of the metric space (X, d).

It is easy to observe that the upper program is impossible to be fulfilled. More precisely, the following simple example shows that the requirements (vii) and (viii) contradict each other in the general case.

Example I. Let $X=(0, 1]$, $Y=(0, 1]$ and $d_X(x, y)=|x-y|$ for $x, y \in X$,

$$d_Y(x, y)=\begin{cases} |x-y|, & \text{if } x, y \in Y \text{ and } y \neq 0, \text{ or } x=y=0, \\ 1, & \text{if } y=0 \text{ and } 0<x\leq 1. \end{cases}$$

Then, (X, d_X) is a metric space and (Y, d_Y) is a T_2-quasi-metric one. Suppose (X^*, d^*_X) and (Y^*, d^*_Y) are T_2-quasi-metric completions of (X, d_X) and (Y, d_Y) respectively, satisfying (vii) and (viii). Then,

$$X^*=[0, 1] \text{ and } d^*_X(x, y)=|x-y| \text{ for } x, y \in X^*.$$

But (X, d^*_X) and $(Y \setminus \{0\}, d_Y)$ (where d_Y is the corresponding restriction which is a metric) are obviously isometric so that $(X, d_X) \subset (Y, d_Y)$. On the other hand, we have $\frac{1}{n} \to 0$ in (X^*, d^*_X) and $\frac{1}{n} \to 0$ in (Y, d_Y). Therefore, from $(X^*, d^*_X) \subset (Y^*, d^*_Y)$ and $(Y, d_Y) \subset (Y^*, d^*_Y)$ (the inclusions understood in the sense of (vii)) it follows (since (Y^*, d^*_Y) is a T_2-space) that (X^*, d^*_X) and (Y, d_Y) are quasi-isometric, i. e. that d^*_X and d_Y coincide — a contradiction.

So the requirements (vii) and (viii) cannot be satisfied simultaneously by T_2-completions in the general case and the problem arises to find some special class of T_2-quasi-metric spaces for which the outlined program is possible to be done.

At the same time it is very desirable that the notion of Cauchy sequence satisfies the requirements (i), (ii) and (iii) in the most general case (even for non-T_2-quasi-metric spaces).

The usual definition of Cauchy sequence for metric spaces can be formulated as follows:

(1) The sequence $\{x_n\}$ is called Cauchy sequence, if for every natural number k there exists such a N_k that $d(x_m, x_n)<\frac{1}{k}$ for all $m, n>N_k$.

It is easy to see that this definition is not suitable for quasi-metric spaces, because it does not satisfy in the general case the most basic requirement (i).

In order to express the property of a filter in a quasi-uniform space to contain "arbitrarily small" sets, the authors of [4] gave the following definition (adopted also in [2] and [1]):

A filter $\mathscr{F}$ in a quasi-uniform space $(X, \mathscr{U})$ is called Cauchy filter, if for each $U\in\mathscr{U}$ there is a point $x=x_U$ in X, such that $U(x)\in\mathscr{F}$. (Here, as usually $U(x)=\{y\in X \mid (x, y)\in U\}$).

In the case of a quasi-metric space (X, d), the above definition turns into the following one:

(2) A sequence $\{x_n\}$ is called Cauchy sequence, if for every natural number k there are a $y_k\in X$ and a N_k, such that $d(y_k, x_n)<\frac{1}{k}$, whenever $n>N_k$.

The definition (2) evidently satisfies the requirements (i), (ii) and (iii). But, as far as it is known to the author, no convenient notion of equivalence relation between Cauchy sequences in the sense of it was introduced up to now*.

There is another, non-formal objection to the definition (2). It could be understood in connection with the space (R, d_S) in the following example.

E x a m p l e 1. Let R be the real line equipped with the quasimetric

$$d_S(x, y)=\begin{cases} y-x & \text{if } x\leq y, \\ 1 & \text{if } x>y. \end{cases}$$

The topology of the quasi-metric space (R, d_S), known as topology of the Sorgenfrey line, has for each $x\in R$ the collection $\{[x, x+r) \mid r>0\}$ for a base at the point x. This topology, as known, is non-metrizable.

Now, one observes that, for instance, the sequence $\{-\frac{1}{n}\}$, although non-convergent, is a Cauchy sequence in the sense of (2). But in view of the special character of the topology on the space (R, d_S), it seems very inconvenient to regard this sequence as a potentially convergent one, i. e. as such one that could be made convergent by completing the space (R, d_S).

Just in order to abolish this objection, another concept of Cauchy sequence is proposed here, which also enables us to realize the program outlined above under some relatively simple additional assumption about the quasi-metric.

All the statements in the sequel are given here without proofs.

2. Cauchy sequences. We start by the following:

D e f i n i t i o n 1. *A sequence $\{x_n\}$ in the quasi-metric space (X, d) is called* C a u c h y s e q u e n c e, *if for every natural number k there exist a $y_k\in X$ and a N_k, such that $d(y_m, x_n)<\frac{1}{k}$, whenever $m, n>N_k$.*

* In [3] a number of different definitions of Cauchy sequences in a quasi-metric space are proposed. But only one of the them, namely that which coincides with the above definition (2), satisfies the requirement (i).

Further, when $\{x'_m\}$ and $\{x''_n\}$ are two sequences in a quasi-metric space (X, d), we will write shortly

$$\lim_{m,n} d(x'_m, x''_n) = r,$$

if for every $\varepsilon > 0$ there is such a N_ε that $|d(x'_m, x''_n) - r| < \varepsilon$, whenever $m, n > N_k$.

In particular, the requirement of Definition I could be written as

$$\lim_{m,n} (y_m, x_n) = 0$$

(while the respective requirement in the definition (1) of Cauchy sequence for metric spaces is written as $\lim_{m,n} d(x_m, x_n) = 0$).

In order to be concise, every sequence $\{y_m\}$ which satisfies the condition of Definition I with reference to a Cauchy sequence $\{x_n\}$, i. e. for which $\lim_{m,n} d(y_m, x_n) = 0$ holds, will be called co-sequence to $\{x_n\}$.

Throughout the sequel the term Cauchy sequence will be used in the sense of Definition I.

The following three propositions show that the requirements (i), (ii) and (iii) from Section 1 are now fulfilled.

Proposition 1. *Every convergent sequence in a quasi-metric space is a Cauchy sequence.*

Proposition 2. *Every subsequence of a Cauchy sequence is a Cauchy sequence too.*

Proposition 3. *For metric spaces Definition I is equivalent to the usual definition (1) of Cauchy sequence.*

Now we adopt the standard:

Definition II. *A quasi-metric space is called complete, if every Cauchy sequence in it is convergent.*

One verifies that, for instance, the quasi-metric space (R, d_S) from Example II is a complete one. Here is another example of such a space.

Example III. Let us remind that a real function f defined on an interval $[a, b]$ on the real line is called upper semi-continuous, if for every $x \in [a, b]$ and every $\varepsilon > 0$ there is a (depending on x and ε) $\delta > 0$, such that $f(y) < f(x) + \varepsilon$, whenever $|x - y| < \delta$.

Denote by $C^+([a, b])$ the collection of all upper semi-continuous functions defined on $[a, b]$ and provide it with the following quasi-metric

$$d_{C^+}(f, g) = \begin{cases} \min\{\sup\{g(x) - f(x) \mid x \in [a, b]\}, 1\} & \text{if } f \leq g, \\ 1, \text{ if } f(x) > g(x) & \text{for some } x \in [a, b], \end{cases}$$

where $f \leq g$ means that $f(x) \leq g(x)$ for each $x \in [a, b]$.

Then one checks that the quasi-metric space $(C^+([a, b]), d_{C^+})$ is complete*.

Remark I. It should be noted that a complete subspace of a quasi-metric space may fail to be closed. So the interval (0, 1) in the space (R, d_S) from Example II is complete but not closed.

Two quasi-metrics d' and d'' introduced in the same set X will be called equivalent to each other, if they induce the same quasi-uniformity on X, i. e. if for every $\varepsilon > 0$ there is such a $\delta > 0$ that $d'(x, y) < \delta$ implies $d''(x, y) < \varepsilon$ and $d''(x, y) < \delta$ implies $d'(x, y) < \varepsilon$.

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* One more example of a complete quasi-metric space is obtained when we consider the collection of all continuous real functions defined on $[a, b]$ and provide it with the above quasi-metric d_{C^+}.

As known, the quasi-metric d in every quasi-metric space (X, d) is equivalent to a bounded one, and even to a bounded by 1 one, i. e. to such a quasi-metric d' that $d'(x, y) \leq 1$ for all $x, y \in X$.

On the other hand, when $\{(X_i, d_i)\}$ is a finite or countable family of bounded quasi-metric spaces and $d_i(x, y) \leq b_i$ for $x, y \in X_i$, then a bounded quasi-metric d in the Cartesian product $\Pi_i X_i$ is defined by the equality

$$d(x, y) = \Sigma_i \frac{1}{2^i b_i} d_i(x_i, y_i),$$

where $x = \{x_i\}$ and $y = \{y_i\}$ are two points of $\Pi_i X_i$. The quasi-metric d induces on $\Pi_i X_i$ the product of the quasi-uniformities induced on X_i by d_i. One lets usually $(\Pi_i X_i, d) = \Pi_i (X_i, d_i)$.

One proves easily the following:

Theorem 1. If $\{(X_i, d_i)\}$ is a finite or countable family of complete bounded quasi-metric spaces, then the product space $\Pi_i (X_i, d_i)$ is complete too.

As usually, we use the following:

Definition III. *A completion of a quasi-metric space (X, d) is a complete quasi-metric space (X^*, d^*) in which (X, d) can be quasi-isometrically embedded as a dense subspace.*

Remark II. A quasi-metric space could have more than one completions. Moreover, a complete quasi-metric space could be densely embedded in another one. For instance, the interval $(0, 1)$, regarded as a subspace of the space (R, d_S) from Example II, is complete and is dense in the complete subspace $[0, 1)$ of the same space.

3. Completion. Now we turn to the construction of a completion of a given quasi-metric space. As shown in Section 1, in order to fulfil the requirements (vii) and (viii), we are obliged to restrict ourselves to considering some suitably defined subclass of the class of all T_2-quasi-metric spaces.

So we consider further only those quasi-metric spaces whose quasi-metric d satisfies the following supplementary condition:

$(\mathscr{B})$ if $\{x'_n\}$ and $\{x''_m\}$ are two sequences in (X, d) and $x', x'' \in X$, then from

$$d(x', x'_n) \leq r' \text{ for } n = 1, 2, \ldots, \quad d(x''_m, x'') \leq r'' \text{ for } m = 1, 2, \ldots$$

and

$$\lim_{m, n} d(x''_m, x'_n) = 0$$

it follows that

$$d(x', x'') \leq r' + r''.$$

Every quasi-metric d satisfying the condition $(\mathscr{B})$ will be called balanced quasi-metric or $\mathscr{B}$-quasi-metric, the quasi-metric spaces with balanced quasi-metrics will be called balanced quasi-metric spaces or $\mathscr{B}$-quasi-metric spaces and their class will be denoted by $\mathscr{B}$.

Let us note that each metric d satisfies obviously the condition $(\mathscr{B})$. Therefore, the class of all metric spaces is a subclass of the class $\mathscr{B}$. On the other hand, there exist non-metrizable quasi-metric spaces belonging to the class $\mathscr{B}$. Such are the space (R, d_S) from Example II and the space $(C^+([a, b]), d_{C^+})$ from Example III.

The balanced quasi-metrics possess a number of useful properties. Some of them are formulated in the following three lemmas:

Lemma 1. *For a $\mathscr{B}$-quasi-metric d we have:*

if $\lim d(x, x_n)=0$ *and* $d(y, x_n) \leq r$ *for* $n=1, 2, \ldots$, *then* $d(y, x) \leq r$;
if $\lim d(x_n, x)=0$ *and* $d(x_n, y) \leq r$ *for* $n=1, 2, \ldots$, *then* $d(x, y) \leq r$.

Lemma 2. *Any* $\mathcal{B}$*-quasi-metric* d *has the properties:*
if $\lim d(x, x_n)=0$ *and* $\lim d(y, x_n)=0$, *then* $x=y$;
if $\lim d(x_n, x)=0$ *and* $\lim d(x_n, y)=0$, *then* $x=y$.
Consequently, every $\mathcal{B}$*-quasi-metric is a* T_2*-quasi-metric.*

Lemma 3. *For a* $\mathcal{B}$*-quasi-metric* d *we have:*
if $\lim d(x_n, x)=0$ *and* $\lim d(y, y_m)=0$, *then*

$$\lim_{m, n} d(x_n, y_m)=d(x, y).$$

Let us introduce now the following:

Definition IV. *Two Cauchy sequences* $\{x'_n\}$ *and* $\{x''_m\}$ *in a* $\mathcal{B}$*-quasi-metric space* (X, d) *are called equivalent to each other, if each co-sequence to* $\{x'_n\}$ *is a co-sequence to* $\{x''_m\}$ *and vice versa.*

Throughout the sequel of this Section, we assume that a quasi-metric space (X, d) of the class $\mathcal{B}$ is given. This assumption, as a rule, will not be mentioned every time explicitly.

Lemma 4. *It* $\{x'_n\}$ *and* $\{x''_m\}$ *are two Cauchy sequences with a common co-sequence and* $x'_n \to x$, *then* $x''_m \to x$.

Hence, one obtains immediately

Proposition 4. *It* $\{x'_n\}$ *and* $\{x''_m\}$ *are two equivalent Cauchy sequences and* $x'_n \to x$, *then* $x''_m \to x$.

Further, we have the very useful

Proposition 5. *If two Cauchy sequences have common co-sequences, then they are equivalent to each other.*

This Proposition allows us to speak about co-sequences to an equivalence class of Cauchy sequences (instead of co-sequences to a Cauchy sequence).

From Proposition 5 follows immediately

Proposition 6. *Every Cauchy sequence is equivalent to each its subsequence.*

Propositions 4 and 6 yield

Proposition 7. *The collection of all sequences convergent to a point* $x \in X$ *is an equivalence class of Cauchy sequences.*

So the requirements (iv) and (v) from Section 1 are fulfilled.

Now, denote by X^* the collection of all equivalence classes of Cauchy sequences in the given $\mathcal{B}$-quasi-metric space (X, d). So

$X^*=\{\xi \mid \xi$ is an equivalence class of Cauchy sequences in $(X, d)\}$.

If for every $x \in X$ we let

$$\varphi(x)=\{\{x_n\} \mid x_n \to x\},$$

then, according to Proposition 7, $\varphi(x) \in X^*$. Thereby, a mapping

$$\varphi: X \to X^*$$

is defined.

We will provide X^* with a quasi-metric. For this purpose one establishes the important

Proposition 8. *Let* $\xi', \xi'' \in X^*$. *If* $\{y'_m\}$ *is a co-sequence to the class* ξ' *and* $\{x''_n\}$ *is a Cauchy sequence of the class* ξ'', *then there exists the limit*

$\lim\limits_{m,n} d(y'_m, x''_n)$. *The value of this limit depends only on* ξ' *and* ξ'' *and does not depend on the choice of the sequences* $\{y'_m\}$ *and* $\{x''_n\}$.

This Proposition justifies the following

Definition V. *Let* $\xi', \xi'' \in X^*$, $\{y'_m\}$ *be a co-sequence to the class* ξ' *and* $\{x''_n\}$ *be a Cauchy sequence of the class* ξ''. *Then we let*

$$d^*(\xi', \xi'') = \lim_{m,n} d(y'_m, x''_n).$$

Further, one proves successively

Proposition 9. d^* *is a quasi-metric. The quasi-metric* d^* *satisfies the condition* $(\mathscr{B})$.

Proposition 10. *For* $x, y \in X$ *we have* $d^*(\varphi(x), \varphi(y)) = d(x, y)$. *Consequently, the mapping* φ *is a quasi-metric embedding.*

Proposition 11. *For every* $\xi \in X^*$ *we have*:

a) $\{x_n\} \in \xi$, *if and only if* $\lim d^*(\xi, \varphi(x_n)) = 0$;

b) $\{y_m\}$ *is a co-sequence to the class* ξ, *if and only if* $\lim d^*(\varphi(y_m), \xi) = 0$.

Proposition 12. *The quasi-metric space* (X^*, d^*) *is complete.*

Propositions 9, 10, 11 (assertion a) and 12 show that the requirements (vi-a), (vi-b) and (vi-c) from Section 1 are fulfilled and one obtains

Theorem 2. *Every quasi-metric space* (X, d) *of the class* $\mathscr{B}$ *has a completion* (X^*, d^*) *constructed in a standard manner. The space* (X^*, d^*) *also belongs to the class* $\mathscr{B}$.

The completion (X^*, d^*) will be called the standard completion of the quasi-metric space.

Proposition 13. *When* (X, d) *is a metric space, then its standard quasi-metric completion* (X^*, d^*) *coincides with the usual metric completion of* (X, d).

Proposition 14. *Let* (X, d_X) *and* (Y, d_Y) *be two* $\mathscr{B}$*-quasi-metric spaces. If* (Y, d_Y) *is complete and* $(X, d_X) \subset (Y, d_Y)$, *then* $(X^*, d^*) \subset (Y, d_Y)$, *where* (X^*, d^*) *is the standard completion of* (X, d) *and the inclusions are understood as quasi-metric embeddings, the second one being an extension of the former. In particular, for every* $\mathscr{B}$*-quasi-metric completion* (X', d') *of* (X, d) *we have* $(X^*, d^*_X) \subset (X', d')$.

Corollary. *If* (X, d_X) *and* (Y, d_Y) *are* $\mathscr{B}$*-quasi-metric spaces and* $(X, d_X) \subset (Y, d_Y)$, *then for their standard quasi-metric completions we have* $(X^*, d^*_X) \subset (Y^*, d^*_Y)$.

Thus, the requirements (vii) and (viii) from Section 1 are also fulfilled and, thereby, the program sketched in the beginning is carried out.

By means of Theorem 2 one establishes in addition the following

Theorem 3. *If* $\{(X_i, d_i)\}$ *is a finite or countable family of bounded* $\mathscr{B}$*-quasi-metric spaces, then the product* $\Pi_i (X_i, d_i)$ *belongs also to the class* $\mathscr{B}$.

Finally, we have also

Proposition 15. *If* $\{(X_i, d_i)\}$ *is a finite or countable family of bounded* $\mathscr{B}$*-quasi-metric spaces and, for each i,* (X^*_i, d^*_i) *is the corresponding standard completion, then the product* $\Pi_i (X^*_i, d^*_i)$ *is quasi-isometric to the standard completion of the product* $\Pi_i (X_i, d_i)$.

4. Extension of quasi-uniformly continuous mappings. A mapping $f: (X, d_X) \to (Y, d_Y)$, where (X, d_X) and (Y, d_Y) are quasi-metric spaces, is called quasi-uniformly continuous, if for every $\varepsilon > 0$ there is such a $\delta > 0$ that $d_Y(f(x'), f(x'')) < \varepsilon$, whenever $d_X(x', x'') < \delta$.

L e m m a 5. *Let* (X, d_X) *and* (Y, d_Y) *be (arbitrary) quasi-metric spaces and* $f: (X, d_X) \to (Y, d_Y)$ *be a quasi-uniformly continuous mapping. Then the image* $\{f(x_n)\}$ *of every Cauchy sequence* $\{x_n\}$ *in* (X, d_X) *is a Cauchy sequence in* (Y, d_Y). *If* $\{x'_m\}$ *is a co-sequence to* $\{x_n\}$, *then* $\{f(x'_m)\}$ *is a co-sequence to* $\{f(x_n)\}$.

T h e o r e m 4. *Let* (X, d_X) *and* (Y, d_Y) *be two* $\mathscr{B}$*-quasi-metric spaces,* (X^*, d_X^*) *and* (Y^*, d_Y^*) *be their standard completions. Then every quasi-uniformly continuous mapping* $f: (X, d_X) \to (Y, d_Y)$ *has a (unique) quasi-uniformly continuous extension* $f^*: (X^*, d_X^*) \to (Y^*, d_Y^*)$ *(in the sense that* $f^* \circ \varphi_X = \varphi_Y \circ f$, *where* $\varphi_X: X \to X^*$ *and* $\varphi_Y: Y \to Y^*$ *are the corresponding standard embeddings).*

C o r o l l a r y. *Let* (X, d_X) *and* (Y, d_Y) *be* $\mathscr{B}$*-quasi-metric spaces and* (Y, d_Y) *be complete. Then every quasi-uniformly continuous mapping* $f: (X, d_X) \to (Y, d_Y)$ *has a quasi-uniformly continuous extension* $f^*: (X^*, d_X^*) \to (Y, d_Y)$ *over the standard completion* (X^*, d_X^*) *of* (X, d_X).

5. Conjugate completion. As known, to any quasi-metric d introduced in a set X, a conjugate quasi-metric d^- is related, which is defined by the equality

$$d^-(x, y) = d(y, x).$$

The following two lemmas are obvious.

L e m m a 6. *If* $(X, d) \in \mathscr{B}$, *then* $(X, d^-) \in \mathscr{B}$.

L e m m a 7. *Let* $(X, d) \in \mathscr{B}$. *Then* $\{x_n\}$ *is a Cauchy sequence in the space* (X, d) *with* $\{y_m\}$ *as a co-sequence, if and only if* $\{y_m\}$ *is a Cauchy sequence in the space* (X, d^-), *with* $\{x_n\}$ *as a co-sequence.*

One easily proves

T h e o r e m 5. *Let* $(X, d) \in \mathscr{B}$. *Then*:

a) (X, d) *is complete, if and only if* (X, d^-) *is complete;*

b) *for the standart completions* (X^*, d^*) *and* $(X^*, (d^-)^*)$ *of the conjugate spaces* (X, d) *and* (X, d^-) *we have*

$$(X^*, (d^-)^*) = (X^*, (d^*)^-)$$

(where the equality is understood as a quasi-isometry).

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