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# ПЛИСКА

## БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

# SOME COVERING PROPERTIES OF LOCALLY UNIVALENT FUNCTIONS

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**1. Introduction.** In this note we study some aspects of the covering properties of functions $f$ that are analytic and locally univalent in $U=\{|z|<1\}$, and at most $p$-valent in $U$ but not univalent in $U$.

For each $t\in]0,1[$ greater than the radius of univalence of such a $f$ there must exist two points $z_t$ and $z_t'$ on $\{|z|=t\}$, with $f(z_t)=f(z_t')(=w_t$, say) such that:

1) $f$ is univalent on the anticlockwise-described arc $C(t)$ of $\{|z|=t\}$ between $z_t$ and $z_t'$,

2) $z_t$ and $z_t'$ are the initial and terminal points respectively of the directed arc $C(t)$, and

3) $f(C(t))$ is described clockwise relative to its inside.

Then, $\Gamma(t)=f(C(t))$ is a closed Jordan curve, analytic except at $w_t$. The preimage $f^{-1}$ (Int $\Gamma(t)$) consists of a countable number of disjoint domains in $U$; let $D(t)$ denote that component which has $C(t)$ as part of its boundary. We call $D(t)$ the adhering domain to the generating arc $C(t)$. It was shown in [1] that $D(t)$ is simply-connected and goes to the boundary of $U$.

The question arises from [1(c), p. 97] as to whether

$$D(t)\subset\{|z|>t\} \tag{1}$$

for all $t$ larger than the radius of univalence of $f$.

Here we show that (1) is not true in general, and we ask some further questions about the domains $D(t)$.

In addition we give an example that shows that the conformality condition in the following result cannot be removed:

Theorem A (Theorem 2 of [1]). Let $w=f(z)=z+a_2z^2+\dots$ be analytic, locally univalent but not univalent in $U$, and strictly $p$-valent in $U$. Then, there exists some point $w_0$ in $\mathbf{C}_w$ such that $f(z)-w_0$ has at most $(p\text{-}2)$ zeros in $U$.

**2. Example 1.** We now construct a Riemann surface $\mathcal{R}$ that shows that (1) cannot hold for all sufficiently large $t$. $\mathcal{R}$ will be a modification of another Riemann surface $\mathcal{R}_{\varepsilon_1}$ that we construct first, using the following domains in the $w$-plane:

$G_1=\{\text{Re } w>-1\}$;

$G_2=\{\text{Re } w<-1,\ \text{Im } w<-1\}$;

$G_3=\{\text{Re } w<-2,\ -1<\text{Im } w<1\}$;

$G_4=\{\text{Re } w<-1,\ \text{Im } w>1\}$;

$G_5=G_1$; and

$G_6=$ the triangle in $\mathbf{C}_w$ with vertices $-1-\frac{7}{8}i, -1-\frac{5}{8}i$ and $-\frac{5}{4}-\frac{3}{4}i$.

Then, for each $\varepsilon \in [0,1]$, $\mathscr{R}_\varepsilon$ is the two-sheeted Riemann surface obtained by sewing $G_k$ to $G_{k+1}$, $1 \leq k \leq 5$, along their common boundary, and slitting $G_1$ along the line segment $L_\varepsilon = [-1-i, \varepsilon-1-i]$.

Let the function

$$f_\varepsilon(z) = \sum_{n=1}^{\infty} a_n(\varepsilon) z^n$$

map $U$ onto $\mathscr{R}_\varepsilon$ with $f_\varepsilon(0)$ lying on the portion $G_1$ of $\mathscr{R}_\varepsilon$.

For all sufficiently large $t \in ]0, 1[$, the level curve $f(\{|z|=t\})$ closely approximates $\partial\mathscr{R}_\varepsilon$, at least near to

$$S_\varepsilon = S \cup L_\varepsilon,$$

where $S$ is the square

$$S = \partial(\mathbf{C}_w - f_\varepsilon(U)).$$

Let $\exp(i\theta_1)$ and $\exp(i\theta_2)$ denote the points of $\partial U$ that are the preimages under $f_0$ of the point $w = -1-i$, arranged such that

$$f_0(e^{i\theta_1}) \in \partial G_3 \text{ and } f_0(e^{i\theta_2}) \in \partial G_5.$$

For $\varepsilon > 0$ and $t$ sufficiently close to 1, the distance between the level curve $f_\varepsilon(\{|z|=t\})$ and $S_\varepsilon$ is of the order of magnitude of $(1-t)$, except that where a corner of $S_\varepsilon$ is also a corner of $\partial\mathscr{R}_\varepsilon$, the level curve is pulled in towards the corner. This is because near such a corner, $w' = f_\varepsilon(e^{i\theta'})$ say, we have

$$f(z) - w' \simeq (z - e^{i\theta'})^{3/2} A(w')$$

for some $A(w')$ independent of $z$.

Clearly, it is then possible to choose a particular pair $(t_1, \varepsilon_1)$ with $t_1 \in ]0,1[$ sufficiently large and $\varepsilon_1 > 0$ sufficiently small, such that there exist two points $z_{t_1}$ and $z'_{t_1}$ on $\{|z|=t_1\}$ near to $\exp(i\theta_1)$ and $\exp(i\theta_2)$ respectively, with the following properties:

(a) $C(t_1) = (z_{t_1}, z'_{t_1})$ is a generating arc on $\{|z|=t_1\}$, and

(b) $f_{\varepsilon_1}^{-1}(-1)$ lies in the adhering domain $D(t_1)$ generated by the arc $C(t_1)$.

This follows from the Carathéodory Kernel Theorem and the fact that $-1$ belongs to $\mathscr{R}_\varepsilon$ for each $\varepsilon \geq 0$.

We have to choose $\varepsilon_1$ sufficiently small and $t_1$ sufficiently large for (b) to hold, and $\varepsilon_1$ and $t_1$ sufficiently large so that the level curve has a double point near $w = -1-i$; this can be done by choosing first $t_1$ and then $\varepsilon_1$. In (a), $z_{t_1}$ is chosen on $\{|z|=t_1\}$ such that $f(z_{t_1})$ is the 'last' double point on $f(\{|z| = t_1\})$ before the level curve sweeps round $S$ to intersect the line segment $]-\infty, -2[$.

Then the point $w = -1$ must lie on $\mathscr{R}_{\varepsilon_1}$, inside the image under $f_{\varepsilon_1}$ of the level curve $\{|z|=t_1\}$, so that $f_{\varepsilon_1}^{-1}(-1)$ lies inside $\{|z|<t_1\}$. It follows that

$$D_{t_1} \cap \{|z|>t_1\} \neq \emptyset.$$

Finally, the desired Riemann surface $\mathscr{R}$ is obtained from $\mathscr{R}_{\varepsilon_1}$ by slitting $\mathscr{R}_{\varepsilon_1}$ in $G_1$ along very small line segments $[-1-2_i^{-n}, \varepsilon_{n+1}-1-2_i^{-n}]$, $n = 1, 2, \ldots$, where $\varepsilon_n \downarrow \to 0$, and by attaching small triangles inside $S$ to $G_1$ midway between these slits. Similar arguments to those earlier applied inductively to the effect of each successive addition show that there exists a sequence $t_n \uparrow \to 1$ and a sequence of adhering domains $D(t_n)$ such that

$$D(t_n) \cap \{|z| < t_n\} \neq \emptyset .$$

**3. Remark.** It would be interesting to know if there exists a function $f$ analytic in $U$ and locally univalent in $U$, such that for some nested family of adhering domains, $D(t)$, we can have

$$D(t) \cap \{|z| < t\} \neq \emptyset$$

for all $t$ sufficiently close to 1, or even perhaps for all $t$ larger than the radius of univalence of $f$.

Also, the question arises as to whether, if $f$ is assumed to be strictly $p$-valent in $U$ with $f'(0)=1$, the number

$$T = \inf_f \{t : D(t) \cap \{|z| < t\} \neq \emptyset\}$$

is equal to $R_u$, the radius of univalence of the family of all such $f$, or whether $T > R_u$.

**4. Example 2.** We now construct a function $f$ with the following properties: $f$ is analytic and strictly $p$-valent in $U$, and the Riemann surface $\mathscr{R} = f(U)$ covers every point in the image plane al least $(p-1)$ times. This shows that the conformality condition in Theorem A cannot be removed.

Let $\mathscr{R}_1$ denote the image Riemann surface associated with the function

$$w = f_2(z) - 3 + i, \quad z \in U,$$

where $f_2$ is the function defined in Example 2 of [1, p. 99] with the choice

$$w_i = 1 + (i-1)/(p-2), \quad 1 \leq i \leq p-1.$$

Let $\mathscr{R}_2$ denote the Riemann surface associated with the function

$$w = z^2, \quad z \in U.$$

Now delete from $\mathscr{R}_1$ the copy of $\{|w| \leq 1\}$, whose interior lies in a single sheet of $\mathscr{R}_1$ and whose boundary meets $\partial\mathscr{R}_1$, and sew in its place a copy of $\mathscr{R}_2$ along $T = \{|w| = 1,\ w \neq i\}$; do this in such a way that adjacent points of $\partial\mathscr{R}_1$ on $T$ are sewn to adjacent points (on the same sheet) of $\mathscr{R}_2$. Denote by $\mathscr{R}_3$ the resulting Riemann surface.

Next, to $\mathscr{R}_3$ sew a copy of

$$\mathscr{R}_4 = \{\mathrm{Re}\, w > -3, 0 < \mathrm{Im}\, w < 2\} - [\{|w| \leq 1\} \cup \{\mathrm{Im}\, w \leq 1, \mathrm{Re}\, w \geq 0\}]$$

along the connected copy of

$$\{w = e^{i\theta} : \tfrac{1}{2}\pi \leq \theta \leq \pi\} \cup \{\mathrm{Im}\, w = 1, \mathrm{Re}\, w \geq 0\}$$

on $\mathscr{R}_3$. Denote by $\mathscr{R}$ the resulting Riemann surface.

Then $\mathscr{R}$ has the desired properties. (Note too that $f'$ has just one zero in $U$.)

Related questions will be discussed in [2].

## REFERENCES


1. D. A. Brannan, W. E. Kirwan. Some covering theorems for analytic functions. *J. London Math. Soc.*, **19**, 1979, 93-101.
2. D. A. Brannan, A. K. Lyzzaik. Some covering properties of locally univalent functions (to appear).


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