PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

TOTAL PROGENY IN A SUBCRITICAL BRANCHING PROCESS WITH TWO TYPES OF IMMIGRATION

M. Slavtchova-Bojkova, P. Becker-Kern, K. V. Mitov[1]

We consider subcritical Bellman-Harris branching processes with two types of immigration - one appears whenever the process hits zero state and another one is in accordance of an independent renewal process. The law of large numbers (LLN) for the total progeny of these processes and Anscombe's type central limit theorem (CLT) for the total number of particles in the cycles completely finished by the moment t are obtained.

1. Introduction

In general, the total progeny is studied for different classes of branching processes in two settings. For Galton-Watson processes the sum of the particles in the first n generations was investigated by several authors. Pakes (1971) have considered the total progeny for the Galton-Watson branching processes and Kulkarni and Pakes (1983) for the Galton-Watson branching processes with immigration in the state zero.

For continuous-time branching processes, the total number of particles up to the instant t is the following continuous-time characteristic of the process, (e.g. for $Z(t)$),

$$\int_0^t Z(u)du, \quad t \geq 0,$$

which is analogous to the total number of particles up to the instant t. More comments and discussions on this characteristic can be found in Pakes (1972) or in Jagers (1975).

In the recent paper of Glynn and Whitt (2002) the problem is solved in a more general setting. They have obtained necessary and sufficient conditions for LLN and CLT for an integral of a delayed regenerative process. However, our method is quite different as the total progeny we are interested in could not be reduced directly to the problem involving random sums of independent identically distributed (iid) random variables (r.v.) and will comment this matter again later in Section 2.

The total progeny of our process has been partially investigated by Weiner (1991) in the critical case.

The rest of the paper is organized as follows.: in Section 2, we give basic definitions and introduce some auxiliary random quantities connected to our model. In Section 3 some preliminary results are proved in detail. In Section 4 the asymptotic behaviour of the moments of the numbers of life cycles that are completely finished respectively unfinished at the instant t is investigated. In Section 5 the WLLN and SLLN for the total progeny of the processes of interest and CLT for the total number of particles in the cycles completely finished by the instant t are obtained. Section 6 contains a version of the key renewal theorem which we apply essentially in the proofs of the previous results.

2.   Definitions and notations

Let $\{X(t)\}_{t\geq 0}$ be a population, wherein the individuals reproduce according to a Bellman-Harris branching process with immigration only in the state zero (BHIO) (generically denoted $Y(t), t \geq 0$) and in addition a random number of immigrants enters the population at the event times $\tau_0 \equiv 0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$ of a given renewal process. It is assumed that the inter arrival times $T_1 = \tau_1 - \tau_0 = \tau_1$, $T_2 = \tau_2 - \tau_1, \ldots$ are iid r.v. with cumulative distribution function (cdf) $G_0(t)$. The numbers of immigrants I_i are assumed to be iid r.v.'s with probability generating function (pgf) $f_0(s) = Es^{I_i}$, $|s| \leq 1$. Denote by

$$n(t) = \max\{n : \tau_n \leq t\}$$

the number of renewal events in the sequence $\tau_n, n = 1, 2, \ldots$ during the time interval $[0, t]$.

The BHIO process $\{Y(t)\}_{t\geq 0}$ is constructed by a sequence of iid classical Bellman-Harris branching processes (denoted generically by $Z(t), t \geq 0$). The process $Z(t), t \geq 0$ is governed by the life time θ of one particle with cdf $G(t)$, $t \geq$

0, the offspring of one particle ξ with pgf $h(s)$, the pgf $f(s)$ of the random number ν_i of immigrants in the state zero and the cdf $K(t)$ of the duration L_i of the stay in the state zero. The construction is as follows (see e.g. Mitov and Yanev (1985)): Let σ_i be the life period of the process $Z_i(t)$. Then the sequence $U_i = L_i + \sigma_i, \; i = 1, 2, \ldots$ defines

$$(1) \qquad\qquad S_0 = 0, \;\; S_n = S_{n-1} + U_n, n = 1, 2, \ldots$$

and

$$(2) \qquad\qquad N(t) = \max\{n : S_n \le t\}.$$

The BHIO process $Y(t)$ is defined by

$$Y(t) = Z_{N(t)+1}(t - S_{N(t)} - L_{N(t)+1})\mathbb{I}_{\{S_{N(t)}+L_{N(t)+1}\le t\}},$$

where $\mathbb{I}_A$ denotes the indicator of the event A.

Now the process $X(t)$ can be defined as follows (taking into account that $\tau_0 \equiv 0$ is the first renewal event when the I_0 independent BHIO processes start)

$$X(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} Y^{(i,k)}(t - \tau_i), \;\; t \ge 0,$$

where $Y^{(i,k)}(t), t \ge 0$ are independent copies of $Y(t)$.

The process $X(t)$ is studied by Weiner (1991) in the critical case, and by Yanev and Slavtchova-Bojkova (1994) in the non-critical cases. Multi type generalization of these results in non-critical cases was obtained by Slavtchova-Bojkova (1996). In Slavtchova-Bojkova (2002) for subcritical processes LLN was proved.

In the present paper we will consider the total number of particles for the process $X(t)$ in the subcritical case. Let us denote by $\zeta(t)$, the total number of particles which are born up to the moment t in the process $Z(t)$, and by ζ the total number of particles which are born in the process $Z(t)$ during its life period σ. It is well known that the r.v. ζ is proper in the sense that $P(\zeta < \infty) = 1$, provided that the process $Z(t)$ is not supercritical. Moreover the distribution of ζ has a pgf

$$(3) \qquad\qquad g(s) = f(g_1(s)),$$

where $g_1(s)$ is the pgf of the total number of particles in a Bellman-Harris branching process starting with one ancestor. The pgf $g_1(s)$ satisfies the functional equation (see e.g. Section 2.11, Jagers (1975), or Pakes (1971))

$$(4) \qquad\qquad g_1(s) = sh(g_1(s)).$$

Let us denote by $V(t)$ the total number of particles up to the moment t in the process $Y(t)$. Then

$$V(t) = \sum_{i=1}^{N(t)} \zeta_i + \zeta_{N(t)+1}(t - S_{N(t)} - L_{N(t)+1})\mathbb{I}_{\{S_{N(t)}+L_{N(t)+1}\leq t\}}.$$

Kulkarni and Pakes (1983) have studied the corresponding quantity to $V(t)$ for Galton-Watson branching processes. In the recent paper of Glynn and Whitt (2001) the problem is solved in a more general setting. They have obtained necessary and sufficient conditions for LLN and CLT for an integral of a delayed regenerative process, i.e. $\int_0^t Y(u)du$ in our notations.

Finally, denote by $W(t)$ the total number of particles in the process $X(t)$, i.e.

$$(5) \qquad\qquad W(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} V^{(i,k)}(t - \tau_i)$$

$$= \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{l=0}^{N^{(i,k)}(t-\tau_i)} \zeta_l^{(i,k)}$$

$$+ \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \zeta_{N^{(i,k)}(t-\tau_i)+1}^{(i,k)}(t - S_{N^{(i,k)}(t-\tau_i)}^{(i,k)} - L_{N^{(i,k)}(t-\tau_i)+1}^{(i,k)})$$

$$\times \; \mathbb{I}_{\{S_{N^{(i,k)}(t-\tau_i)}^{(i,k)}+L_{N^{(i,k)}(t-\tau_i)+1}^{(i,k)}\leq t\}},$$

where $V^{(i,k)}(t), t \geq 0$ are independent copies of $V(t), t \geq 0$.

From now on we will investigate the limiting behavior of the process $W(t)$, $t \geq 0$, assuming the following basic moment conditions:

1. For the processes $Z(t)$:

$$(6) \qquad\qquad 0 < A = \mathbb{E}\xi = h'(1) < 1, \;\; 0 < B = \mathbb{V}ar\xi < \infty,$$

$$(7) \qquad\qquad r_1 = \mathbb{E}\theta = \int_0^\infty xdG(x) < \infty, \;\; r_2 = \mathbb{V}ar\theta < \infty.$$

2. For the processes $Y(t)$:

$$(8) \qquad\qquad m_1 = \mathbb{E}\nu = f'(1) < \infty, \;\; 0 < m_2 = \mathbb{V}ar\nu < \infty,$$

$$(9) \qquad\qquad a_1 = \mathbb{E}L_i = \int_0^\infty xdK(x) < \infty, \;\; a_2 = \mathbb{V}arL_i < \infty.$$

3. For the characteristics of the sequence $\{\tau_n, n = 0, 1, 2, \dots \}$, we assume

(10) $\qquad c_1 = \mathbb{E}I_i = f_0'(1) < \infty, \quad c_2 = f_0''(1) < \infty, \quad c_3 = \mathbb{V}ar I_i < \infty,$

(11) $\qquad \mu_0 = \mathbb{E}\tau_1 = \displaystyle\int_0^\infty x dG_0(x) < \infty, \quad \beta_0 = \mathbb{V}ar\tau_1 < \infty.$

By differentiating equations (3) and (4) and setting $s = 1$ we get for the moments of the total number of particles ζ in the process $(Z(t), 0 \le t \le \sigma)$,

(12) $\qquad v_1 = \mathbb{E}\zeta = \dfrac{\mathbb{E}\nu}{1 - \mathbb{E}\xi} = \dfrac{m_1}{1 - A},$

(13) $\qquad v_2 = \mathbb{V}ar\zeta = \dfrac{\mathbb{E}(\nu)B}{(1 - A)^3} + \dfrac{\mathbb{V}ar(\nu)}{(1 - A)^2} = \dfrac{m_1 B}{(1 - A)^3} + \dfrac{m_2}{(1 - A)^2}.$

Under the conditions (6) and (7),

$$\mathbb{P}(\sigma > t) = \mathbb{P}(Z(t) > 0) \sim C \exp(\alpha t),$$

where C is a positive constant and α is a Malthusian parameter defined by

$$A \int_0^\infty e^{-\alpha t} dG(t) = 1.$$

We always assume that Malthusian parameter exists and in the subcritical case $\alpha < 0$. Hence σ has finite moments of all orders. Therefore, for the moments of $U_i = L_i + \sigma_i$ we get (see (9))

(14) $\qquad \mu_1 = \mathbb{E}U_i = \mathbb{E}L_i + \mathbb{E}\sigma_i < \infty, \quad \beta_1 = \mathbb{V}ar U_i < \infty.$

3.   Basic equations and inequalities

In this section we will introduce two important random quantities our approach is based on and will obtain integral equations for their pgf's.

The following inequalities are fulfilled almost surely:

(15) $$\sum_{i=1}^{N(t)} \zeta_i \le V(t) \le \sum_{i=1}^{N(t)+1} \zeta_i,$$

and

(16) $$\sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{j=1}^{N^{(i,k)}(t-\tau_i)} \zeta_j^{(i,k)} \le W(t) \le \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{j=1}^{N^{(i,k)}(t-\tau_i)+1} \zeta_j^{(i,k)}.$$

Denote by

$$(17) \qquad n^*(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} N^{(i,k)}(t - \tau_i)$$

the number of the cycles in all the renewal processes S_n governing the processes $Z^{i,k}(t - \tau_i)$, $t \geq 0$ which are completely finished up to the moment t and by

$$(18) \qquad n^{**}(t) = \sum_{i=0}^{n(t)} I_i$$

the number of BHIO processes starting at the moments $\tau_0, \tau_1, \dots, \tau_{n(t)}$ during the interval $[0, t]$. In other words, $n^{**}(t)$ is the number of the cycles which does not become extinct at the instant t.

If we enumerate the iid r.v.'s $\zeta_j^{(i,k)}$ by one index (in some order) then by (17), (18) and (16) one can obtain:

$$(19) \qquad \sum_{l=1}^{n^*(t)} \zeta_l \ \leq W(t) \leq \sum_{l=1}^{n^*(t)+n^{**}(t)} \zeta_l.$$

We will use these inequalities together with the definition (5) to investigate the limiting behaviour of $W(t)$.

Denote by $\Phi^*(t, s) = \mathbb{E}s^{n^*(t)}$, $\Phi^{**}(t, s) = \mathbb{E}s^{n^{**}(t)}$ and $\Psi(t, s) = \mathbb{E}s^{N(t)}$.

Lemma 1. _The pgf's $\Phi^*(t, s)$ and $\Phi^{**}(t, s)$ satisfy the following equations:_

$$(20) \qquad \Phi^*(t, s) = f_0(\Psi(t, s))[1 - G_0(t) + \int_0^t \Phi^*(t - u, s)dG_0(u)],$$

$$(21) \qquad \Phi^{**}(t, s) = f_0(s)[1 - G_0(t) + \int_0^t \Phi^{**}(t - u, s)dG_0(u)].$$

P r o o f. We will prove (20) only.

1) If $\tau_1 > t$ then $n^*(t) = \sum_{k=1}^{I_0} N^{(0,k)}(t)$ and in this case

$$\Phi^*(t, s) = \mathbb{E}s^{n^*(t)} = \mathbb{E}s^{\sum_{k=1}^{I_0} N^{(0,k)}(t)} = f_0(\Psi(t, s)).$$

2) If $\tau_1 = u \leq t$ then $n^*(t)$ has the same distribution as the sum $n^*(t - u) + \sum_{k=1}^{I_0} N^{(0,k)}(t)$ and the two terms in this sum are independent. Hence in this case

$$\Phi^*(t, s) = \mathbb{E}s^{n^*(t)} = f_0(\Psi(t, s))\mathbb{E}s^{n^*(t-u)} = f_0(\Psi(t, s))\Phi^*(t - u, s).$$

By the total probability formula we get the assertion.

The equation (21) follows by the same arguments and that's why we omit the proof. $\square$

4. Moments of $n^*(t)$ and $n^{**}(t)$

In this section we will use the renewal equations for the moments of the processes $n^*(t)$ and $n^{**}(t)$ to obtain their asymptotic behavior as $t \to \infty$.

Denote

$$M_1^*(t) = \mathbb{E}n^*(t), \quad M_2^*(t) = \mathbb{E}n^*(t)[n^*(t) - 1], \quad D^*(t) = \mathbb{V}ar(n^*(t)),$$

and

$$M_1^{**}(t) = \mathbb{E}n^{**}(t), \quad M_2^{**}(t) = \mathbb{E}n^{**}(t)[n^{**}(t) - 1], \quad D^{**}(t) = \mathbb{V}ar(n^{**}(t)).$$

Lemma 2. *The moments of $n^*(t)$ satisfy:*

$$(22) \qquad\qquad M_1^*(t) \sim \frac{c_1 t^2}{2\mu_0 \mu_1}, \quad t \to \infty,$$

$$(23) \qquad\qquad M_2^*(t) \sim \frac{c_1^2 t^4}{4\mu_0^2 \mu_1^2}, \quad t \to \infty,$$

$$(24) \qquad\qquad D^*(t) = o(t^4), \quad t \to \infty.$$

P r o o f. Differentiating (20) with respect to s we have

$$\frac{\partial \Phi^*(t, s)}{\partial s} = f_0'(\Psi(t, s))\Psi_s'(t, s)[1 - G_0(t) + \int_0^t \Phi^*(t - u, s)dG_0(u)]$$
$$+ \quad f_0(\Psi(t, s)) \int_0^t \frac{\partial \Phi^*(t - u, s)}{\partial s}dG_0(u)$$

and hence

$$\frac{\partial^2 \Phi^*(t, s)}{\partial s^2} = \left(f_0''(\Psi(t, s))[\Psi_s'(t, s)]^2 \right.$$
$$+ \quad f_0'(\Psi(t, s))\Psi_{ss}''(t, s) \right) \left[1 - G_0(t) + \int_0^t \Phi^*(t - u, s)dG_0(u) \right]$$
$$+ \quad 2f_0'(\Psi(t, s))\Psi_s'(t, s) \int_0^t \frac{\partial \Phi^*(t - u, s)}{\partial s}dG_0(u)$$
$$+ \quad f_0(\Psi(t, s)) \int_0^t \frac{\partial^2 \Phi^*(t - u, s)}{\partial s^2}dG_0(u).$$

Setting $s = 1$ in these equations we get

$$(25) \qquad M_1^*(t) = c_1 M_1^0(t) + \int_0^t M_1^*(t - u) dG_0(u)$$

and

$$M_2^*(t) = c_2 [M_1^0(t)]^2 + c_1 M_2^0(t) + 2c_1 M_1^0(t) \int_0^t M_1^*(t - u) dG_0(u)$$

$$+ \quad \int_0^t M_2^*(t - u) dG_0(u)$$

$$= \quad c_2 [M_1^0(t)]^2 + c_1 M_2^0(t) + 2c_1 M_1^0(t)[M_1^*(t) - c_1 M_1^0(t)]$$

$$(26) \qquad + \quad \int_0^t M_2^*(t - u) dG_0(u),$$

where $M_1^0(t) = \mathbb{E} N(t)$, $M_2^0(t) = \mathbb{E} N(t)[N(t) - 1]$.

Since (1), (2) and (14), it is well known that (see Feller (1971), Sect. 11.10, Problem 13 and Sect. 11.3, Theorem 1)

$$(27) \qquad M_1^0(t) = \frac{t}{\mu_1} + o(t), \quad t \to \infty$$

and

$$\mathbb{E}(N(t))^2 = \frac{t^2}{\mu_1^2} + \frac{\beta_1}{\mu_1^3} t + o(t), \quad t \to \infty.$$

From (25) and (27) by the key renewal theorem (Section 6) we get (22). Further we get

$$(28) \qquad M_2^0(t) = \mathbb{E}(N(t))^2 - M_1^0(t)$$

$$= \quad \frac{t^2}{\mu_1^2} + \frac{\beta_1}{\mu_1^3} t + o(t) - \left(\frac{t}{\mu_1} + o(t) \right) = \frac{t^2}{\mu_1^2} + o(t^2), \quad t \to \infty.$$

Since (22), (27) and (28) one obtains

$$c_2 [M_1^0(t)]^2 + c_1 M_2^0(t) + 2c_1 M_1^0(t)[M_1^*(t) - c_1 M_1^0(t)]$$

$$= \quad c_2 \left(\frac{t}{\mu_1} + o(t) \right)^2 + c_1 \left(\frac{t^2}{\mu_1^2} + o(t^2) \right)$$

$$+ \quad 2c_1 \left(\frac{t}{\mu_1} + o(t) \right) \left[\frac{c_1 t^2}{2\mu_0 \mu_1} + o(t^2) - c_1 \left(\frac{t}{\mu_1} + o(t) \right) \right]$$

$$= \quad \frac{c_1^2 t^3}{\mu_0 \mu_1^2} + o(t^3), \quad t \to \infty.$$

Applying the key-renewal theorem (see Section 6) to (26) we get (23). Finally from the representation

$$D^*(t) = M_2^*(t) + M_1^*(t) - [M_1^*(t)]^2,$$

(22) and (23) we get (24). □

Lemma 3. *The moments of* $n^{**}(t)$ *satisfy:*

(29)
$$M_1^{**}(t) = \frac{c_1}{\mu_0}t + o(t), \quad t \to \infty,$$

$$M_2^{**}(t) = \frac{c_1^2}{\mu_0^2}t^2 + o(t^2), \quad t \to \infty,$$

$$D^{**}(t) = o(t^2), \quad t \to \infty.$$

The proof is similar to the proof of the previous lemma and it is omitted.

Lemma 4. *The following limits take place:*

(30)
$$\frac{n^*(t)}{M_1^*(t)} \xrightarrow{p} 1, \quad t \to \infty$$

and

(31)
$$\frac{n^{**}(t)}{M_1^{**}(t)} \xrightarrow{p} 1, \quad t \to \infty.$$

P r o o f. Since (22) and (24) by the Chebyshev's inequality it follows that for any $\varepsilon > 0$

$$\mathbb{P}\left(\left|\frac{n^*(t)}{\mathbb{E}n^*(t)} - 1\right| > \varepsilon\right) \le \frac{1}{\varepsilon^2}\mathbb{V}ar\left(\frac{n^*(t)}{\mathbb{E}n^*(t)}\right)$$

$$= \frac{1}{\varepsilon^2}\frac{\mathbb{V}ar(n^*(t))}{[\mathbb{E}n^*(t)]^2} = \frac{1}{\varepsilon^2}\frac{D^*(t)}{[M_1^*(t)]^2} \to 0, \quad t \to \infty,$$

which proves (30). The proof of (31) is similar and we omit it. □

Lemma 5. *Under the conditions above*

(32)
$$\frac{n^{**}(t)}{n^*(t)} \xrightarrow{p} 0, \quad t \to \infty$$

and

(33)
$$\frac{n^{**}(t)}{\sqrt{n^*(t)}} \xrightarrow{p} \sqrt{\frac{2c_1\mu_1}{\mu_0}}, \quad t \to \infty.$$

P r o o f. One has

$$\frac{n^{**}(t)}{n^{*}(t)} = \frac{n^{**}(t)}{M_1^{**}(t)} \cdot \frac{M_1^{*}(t)}{n^{*}(t)} \cdot \frac{M_1^{**}(t)}{M_1^{*}(t)}$$

and hence (32) follows from (22), (29), (30) and (31). The proof of (33) is similar.
□

Lemma 6. *We even have a stronger convergence in* (32):

$$\frac{n^{**}(t)}{n^{*}(t)} \overset{a.s.}{\to} 0, \quad t \to \infty.$$

P r o o f. We have

$$\frac{n^{*}(2t)}{n^{**}(2t)}$$

$$= \frac{\sum_{i=0}^{n(2t)} \sum_{k=1}^{I_i} N^{(i,k)}(2t - \tau_i)}{\sum_{i=0}^{n(2t)} I_i}$$

$$\geq \frac{\sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} N^{(i,k)}(2t - \tau_{n(t)})}{\sum_{i=0}^{n(2t)} I_i}$$

$$= \frac{2t}{\sum_{i=0}^{n(2t)} I_i} \cdot \frac{1}{2} \sum_{i=0}^{t(n(t)/t)} \sum_{k=1}^{I_i} \frac{N^{(i,k)}(t)}{t}$$

$$\geq \frac{2t}{n(2t)\frac{\sum_{i=0}^{n(2t)} I_i}{n(2t)}} \frac{1}{2} \sum_{i=0}^{t(n(t)/t)} \frac{N^{(i,1)}(t)}{t}$$

and by the SLLN for the renewal processes as $t \to \infty$

$$\frac{2t}{n(2t)} \overset{a.s.}{\to} \mu_0, \quad \frac{\sum_{i=0}^{n(2t)} I_i}{n(2t)} \overset{a.s.}{\to} c_1, \quad \frac{N^{(i,1)}(t)}{t} \overset{a.s.}{\to} \frac{1}{\mu_1}, \quad t\frac{n(t)}{t} \overset{a.s.}{\to} \infty.$$

Hence

$$\frac{n^{*}(2t)}{n^{**}(2t)} \geq \frac{2t}{n(2t)\frac{\sum_{i=0}^{n(2t)} I_i}{n(2t)}} \frac{1}{2} \sum_{i=0}^{t(n(t)/t)} \frac{N^{(i,1)}(t)}{t} \overset{a.s.}{\to} \infty, \quad t \to \infty.$$

□

5. Main results

Theorem 1. *Under the conditions (6)-(11), as $t \to \infty$,*

$$(34) \qquad \frac{W(t)}{n^*(t)} \xrightarrow{p} v_1,$$

$$(35) \qquad \frac{W(t)}{n^*(t) + n^{**}(t)} \xrightarrow{p} v_1,$$

$$(36) \qquad \frac{W(t)}{t^2} \xrightarrow{p} \frac{c_1 v_1}{2\mu_0 \mu_1},$$

$$(37) \qquad \mathbb{E}W(t) \sim \frac{v_1 c_1 t^2}{2\mu_0 \mu_1}.$$

P r o o f. Since (19) we can write

$$(38) \qquad \frac{\sum_{l=1}^{n^*(t)} \zeta_l}{n^*(t)} \leq \frac{W(t)}{n^*(t)} \leq \frac{\sum_{l=1}^{n^*(t)} \zeta_l}{n^*(t)} + \frac{\sum_{p=1}^{n^{**}(t)} \zeta_{p+n^*(t)}}{n^*(t)}.$$

For the last quantity on the right hand side we have:

$$(39) \qquad \frac{\sum_{p=1}^{n^{**}(t)} \zeta_{p+n^*(t)}}{n^*(t)} = \frac{n^{**}(t)}{n^*(t)} \frac{\sum_{p=1}^{n^{**}(t)} \zeta_{p+n^*(t)}}{n^{**}(t)} \xrightarrow{p} 0, \quad t \to \infty,$$

by the LLN and (32). On the other hand by the LLN and (22)

$$(40) \qquad \frac{\sum_{l=1}^{n^*(t)} \zeta_l}{n^*(t)} \xrightarrow{p} \mathbb{E}\zeta = v_1, \quad t \to \infty.$$

Now (38)-(40) prove (34). The proof of (35) is similar. The proof of (36) follows from (34), (22) and (30). The proof of (37) follows from the inequality

$$v_1 \mathbb{E}n^*(t) \leq \mathbb{E}W(t) \leq v_1[\mathbb{E}n^*(t) + \mathbb{E}n^{**}(t)], \quad t \geq 0,$$

which is a direct consequence of (19), and the asymptotics (22) and (29). $\qquad \square$

Theorem 2. *Under the conditions (6)-(11),*

$$(41) \qquad \frac{\sum_{i=1}^{n^*(t)} \zeta_i - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \xrightarrow{d} N(0,1), \quad t \to \infty,$$

and

$$(42) \qquad \frac{\sum_{i=1}^{n^*(t)+n^{**}(t)} \zeta_i - v_1[n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \xrightarrow{d} N(0,1), \quad t \to \infty.$$

P r o o f. Since (30), (12) and (13) and the independence of ζ_i the conditions of the Anscombe central limit theorem are satisfied (Chow and Teicher (1978), p.139), which proves (41). Similarly, (42) follows from (31), (12), (13) and (32). $\qquad\square$

Theorem 3. *Under the conditions (6)-(11),*

$$\limsup_{t \to \infty} \mathbb{P}\left(\frac{W(t) - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \leq x \right) \leq \Phi(x),$$

$$\liminf_{t \to \infty} \mathbb{P}\left(\frac{W(t) - v_1[n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \leq x \right) \geq \Phi(x),$$

where $\Phi(x)$ is the standard normal distribution function.

P r o o f. From the inequalities (19) we get

$$P\left(\frac{\sum_{i=0}^{n^*(t)} \zeta_i - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \leq x \right) \geq P\left(\frac{W(t) - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \leq x \right)$$

and

$$P\left(\frac{\sum_{i=0}^{n^*(t)+n^{(**)}(t)} \zeta_i - v_1[n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \leq x \right)$$

$$\leq \quad P\left(\frac{W(t) - v_1[n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \leq x \right),$$

which, together with (41) and (42) proves the theorem.

As a concluding remark our conjecture is that CLT holds for the process $W(t)$ itself but we haven't still prove it. $\qquad\square$

6. A renewal theorem

The following version of the key renewal theorem is used in the paper.

Theorem 4. *Let $A(t) \sim t^a L(t), t \to \infty, a > 0$, be monotone increasing, where L is slowly varying at infinity. Let a cdf $F(t)$ be such that $F(t) = 0$ for $t \le 0$ and $0 < \mu = \int_0^\infty t dF(t) < \infty$. Then the solution of the renewal equation*

$$X(t) = A(t) + \int_0^t X(t - u) dF(u)$$

has the following asymptotics

$$X(t) = \frac{\int_0^t A(u) du}{\mu}(1 + o(1)) = \frac{t^{a+1} L(t)}{(a + 1)\mu}(1 + o(1)), t \to \infty.$$

P r o o f. Let us note first that $X(t)$ is also monotone increasing. Taking the Laplace transforms on both sides of the renewal equation we have

$$\hat{X}(\lambda) = \hat{A}(\lambda) + \hat{X}(\lambda)\hat{F}(\lambda),$$

where

$$\hat{X}(\lambda) = \int_0^\infty X(t) e^{-\lambda t} dt, \quad \hat{A}(\lambda) = \int_0^\infty A(t) e^{-\lambda t} dt, \quad \hat{F}(\lambda) = \int_0^\infty e^{-\lambda t} dF(t).$$

Now

$$\hat{X}(\lambda) = \frac{\hat{A}(\lambda)}{1 - \hat{F}(\lambda)} = \frac{\hat{A}(\lambda)}{\lambda} \frac{\lambda}{1 - \hat{F}(\lambda)}.$$

By the Tauberian theorem (Theorem 4, Section 13.5, Feller (1971)) one gets

$$\hat{A}(\lambda) \sim \Gamma(a + 1) L(1/\lambda) \lambda^{-a}, \lambda \to 0.$$

On the other hand,

$$\frac{\lambda}{1 - \hat{F}(\lambda)} \to \frac{1}{\mu}, \lambda \to 0.$$

Therefore,

$$\hat{X}(\lambda) \sim \frac{\Gamma(a + 1) L(1/\lambda) \lambda^{-(a+1)}}{\mu}, \lambda \to 0.$$

By the same Tauberian theorem (since $X(t)$ is monotone) it follows that

$$X(t) \sim \frac{t^{a+1} L(t)}{\mu(a + 1)}, \quad t \to \infty.$$

REFERENCES

[1] P. W. GLYNN, W. WHITT. Necessary conditions in limit theorems for cumulative processes, *Stoch. Proc. Appl.* **98** (2002), 199–209.

[2] W. FELLER. An Introduction to Probability Theory and its Application, vol. 2, 2nd edition, Wiley, New York, 1971.

[3] P. JAGERS. Branching Processes with Biological Applications, Wiley, New York, 1975.

[4] A. G. PAKES. Some limit theorems for the total progeny of a branching process, *Adv. Appl. Prob.* **3** (1971), 176–192.

[5] A. G. PAKES. A limit theorem for the integral of a critical age-dependent branching process, *Math. Biosci.* **13** (1972), 109–112.

[6] H. WEINER. Age dependent branching processes with two types of immigration, *J. Information Theory* **2** (1991), 207–218.

[7] M. SLAVTCHOVA-BOJKOVA, N. YANEV. Non-critical branching processes with two types of state-dependent immigration, *Comptes Rend. de l'Acad. Bulgar des Sci.* **47** (1994), 13–16.

[8] M. SLAVTCHOVA-BOJKOVA Multi-type age-dependent branching processes with state-dependent immigration, *In: Proc. of Athens Conf. on Applied Prob. and Time series* (1995), Edts: C.C.Heyde, Yu. Prohorov, R. Pyke and S. Rachev, Lecture Notes in Statistics, 114, Springer, Berlin, 1996, 192–205.

[9] M. SLAVTCHOVA-BOJKOVA. On the subcritical age-dependent branching processes with two types of immigration, *Proc. of 31-st Spring Conf. of the Union of Bulgarian Mathematicians, Borovetz* (2002), 187–191.

[10] M. V. KULKARNI, A. G. PAKES. Total progeny of a simple branching process with state dependent immigration, *J. Appl. Prob.* **20** (1983), 472–482.

[11] Y. S. CHOW, H. TEICHER. Probability Theory, Springer-Verlag, New York, 1978.

[12] K. MITOV, N. M. YANEV. Bellman-Harris branching processes with state-dependent immigration, *J. Appl. Prob.* **22** (1985), 757–765.

M. N. Slavtchova-Bojkova
Department of Probability and Statistics
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, BULGARIA
e-mail: `bojkova@math.bas.bg`

P.Becker-Kern
University of Dortmund
Fachbereich Mathematik
Vogelpothsweg 87, 44221 Dortmund, GERMANY
e-mail: `pbk@mathematik.uni-dortmund.de`

K. V. Mitov
Faculty of Aviation
National Military University 'V. Levski'
5856 D. Mitropolia, Pleven, BULGARIA
e-mail: `kmitov@af-acad.bg`