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ON THE EQUALITY OF SHARP AND GERM σ - FIELDS FOR GAUSSIAN PROCESSES AND FIELDS

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Let $\Phi = \{\phi(x) : x \in R^n\}$ be a Gaussian random process ($n = 1$) or field ($n > 1$) on R^n. For $S \subset R^n$ we investigate the relationship between the σ- field, $\mathcal{F}(\Phi, S) = \sigma\{\phi(x) : x \in S\}$ and the infinitesimal or germ σ-field $\overline{\mathcal{F}}(\Phi, S) = \cap_{\epsilon>0}\mathcal{F}(\Phi, S_\epsilon)$, where S_ϵ is an ϵ neighborhood of S. We show here that the equality $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ is equivalent to an approximation problem in the reproducing kernel Hilbert space associated with Φ. The method is then applied to identify the sets S for which the equalty $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ holds for the Ornstein - Uhlenbeck process, the classical Brownian motion, the Levy Browninan motion, the fractional Brownian motion, the the Bessel fields, and the solution of the stochastic heat equation.

1. Introduction

We consider a stochastically continuous mean zero real valued random field $\Phi = \{\phi(x) : x \in R^n\}$. Our focus in this paper is the relationship between the *sharp σ-field*

$$\mathcal{F}(\Phi, S) \stackrel{\text{def}}{=} \sigma\{\phi(x) : x \in S\},$$

and the *germ σ-field*

$$\overline{\mathcal{F}}(\Phi, S) \stackrel{\text{def}}{=} \cap_{\varepsilon>0}\mathcal{F}(\Phi, S_\varepsilon),$$

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with S_ε denoting the uniform neighborhood $\{x: \operatorname{dist}(x, S) < \varepsilon\}$ of S. Writing $\bar{S}$ for the closure of S, and equating σ-fields that differ only by null sets,

$$\mathcal{F}(\Phi, S) \subseteq \overline{\mathcal{F}}(\Phi, S) \quad \text{and} \quad \mathcal{F}(\Phi, S) = \mathcal{F}(\Phi, \bar{S}).$$

If $S = \Gamma \subset R^n$ is a surface that separates R^n into complementary open sets D_+ and D_-, the problem of equality between the sharp and germ σ fields for Φ is closely related to question of whether Φ satisfies the sharp Markov property at Γ.

The field Φ is said to satisfy the *germ field Markov property* at Γ if $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$ are conditionally independent given $\overline{\mathcal{F}}(\Phi, \Gamma)$. If Φ satisfies the more restrictive condition with $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$ are conditionally independent given $\mathcal{F}(\Phi, \Gamma)$, Φ satisfies the *sharp Markov property at* Γ (see e.g. Pitt (1971) and Rozanov (1982)).

In the typical cases of interest when Γ is the boundary of both sets D_+ and D_-, the relationship between the two Markov properties is relatively direct and elementary. Namely, in this case,

$$\overline{\mathcal{F}}(\Phi, \Gamma) \subseteq \overline{\mathcal{F}}(\Phi, D_+) \cap \overline{\mathcal{F}}(\Phi, D_-),$$

and, modulo null sets, $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$ can not be conditionally independent over any proper sub σ-field of $\overline{\mathcal{F}}(\Phi, \Gamma)$. We can thus state

Proposition 1. *Suppose that* Γ *is a closed set separating* R^n *into complementary open sets* D_+ *and* D_-*, with* $\Gamma = \bar{D}_- \cap \bar{D}_+$*. Then* $\Phi = \{\phi(x) : x \in R^n\}$ *satisfies the sharp Markov property at* Γ *iff* Φ *satisfies the germ field Markov property at* Γ *and the identity*

$$\overline{\mathcal{F}}(\Phi, \Gamma) = \mathcal{F}(\Phi, \Gamma) \tag{1}$$

holds.

These considerations raise the following fundamental question:

Question. *What conditions on a closed set* $S \subset R^n$ *imply that 1 holds?*

In Pitt and Robeva (1994, 2003) we showed that 1 is equivalent to an approximation property for the set S in the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Phi)$ associated with Φ. The same question for open sets is also of interest, as it gives conditions under which a slightly different definitions of the sharp Markov property (see e.g. Dalang and Walsh (1991) and Rozanov (1982)) coinsides with our definition above. The paper Pitt and Robeva (2003) presents a complete characterization of the sets $S \subset R^2$ that satisfy 1 for the so-called Bessel fields. In the same paper, for smooth curves $\Gamma \subset R^2$, we also give a decomposition of the germ σ- field $\overline{\mathcal{F}}(\Phi, \Gamma)$ into a tangential (sharp field) component and the σ- fields generated by the generalized normal derivatives of the Bessel field along Γ.

The reproducing kernel Hilbert space approach is, in principle, very general. However, in practice, it requires detailed structural knowledge of the space $\mathcal{H} = \mathcal{H}(\Phi)$, and such knowledge may not be generally available.

In this paper we apply the reproducing kernel Hilbert space approach to a range of random processess and fields and present conditions on $S \subset R^n$ for which 1 is satisfied. Although some of the arguments can in principle be extended, we will only discuss closed sets S here since in this case 1 is equivalent to the sharp Markov property.

The paper is organized as follows. In Section 2 we describe the reproducing kernel Hilbert spaces approach and show that 1 is equivalent to an approximation problem in this space. In Section 3, we consider the Whittle field and the family of Bessel fields and identify their reproducing kernel Hilber spaces with the classical Bessel potential spaces. We also present a characterization for the sets S for which 1 holds for the Bessel fields. These results are then used in Section 4 to investigate the same question for other random processes and fields including the Ornstein - Uhlenbeck process and the classical Brownian motion, the Lévy Browninan motion and the fractional Brownian motion, and a solution of the stochastic heat equation.

2. An Approximation Problem in the Reproducing Kernel Hilbert Space

Here we show that for a stochastically continuous real valued Gaussian random field Φ the equality $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ is equivalent to an approximation condition in the reproducing kernel Hilbert space associated with Φ (Theorem 1). A description of this approach can also be found in Pitt and Robeva (1994, 2003). We include it here for completeness.

Let $\Phi = \{\phi(x) : x \in R^2\}$ be a real valued mean zero Gaussian random field defined over a complete probability space (Ω, Σ, P) and let $\mathcal{F}(\Phi, S)$ and $\overline{\mathcal{F}}(\Phi, S)$ be the sharp and the germ σ-fields of Φ for a set S, as defined in the Introduction. Associated with these σ-fields, are two subspaces of the Hilbert space $L^2(P) = L^2(\Omega, \Sigma, P)$:

1) $H(\Phi, S) \stackrel{\text{def}}{=} \overline{sp}\{\phi(x), x \in S\}_{L^2}$ - the closed linear subspace of $L^2(P)$ obtained as the closed linear span of $\{\phi(x), x \in S\}$ in $L^2(P)$; and

2) $\overline{H}(\Phi, S) \stackrel{\text{def}}{=} \bigcap H(\Phi, S_\epsilon)$, with the intersection taken over all ϵ neighbor-

hoods S_ϵ of $\overline{S}$.

The assumption that Φ is Gaussian implies [see e.g. Rosanov (1982), p.41] that

$$\mathcal{F}(\Phi, S) = \sigma(H(\Phi, S)), \quad \overline{\mathcal{F}}(\Phi, S) = \sigma(\overline{H}(\Phi, S)),$$

and thus

$$\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S) \quad \text{if and only if} \quad H(\Phi, S) = \overline{H}(\Phi, S). \tag{2}$$

In terms of the reproducing kernel Hilbert space associated with Φ, this can be rephrased as follows.

Let $\mathcal{H}(\Phi, S)$ be the space of functions on R^2 given by

$$\mathcal{H}(\Phi, S) = \{u(x) \overset{\text{def}}{=} EX\phi(x) : \ X \in H(\Phi, S)\}.$$

with the inner product

$$\langle u_1, u_2 \rangle_{\mathcal{H}} = \langle u_1, u_2 \rangle = EX_1X_2,$$

where $u_1(x) = EX_1\phi(x)$ and $u_2(x) = EX_2\phi(x)$.

It is clear that each function $\rho(x, .)$, $x \in S$, determined by the correspondence $y \mapsto \rho(x, y)$, belongs to $\mathcal{H}(\Phi, S)$, and that (i) The map $J : X \mapsto EX\phi(x)$ determines an isometry between $H(\Phi, S)$ and $\mathcal{H}(\Phi, S)$;

(ii) $\mathcal{H}(\Phi, S)$ is spanned by the functions $\{\rho(x, .), x \in S\}$;

(iii) For each $u \in \mathcal{H}(\Phi, S)$ the reproducing property $u(x) = \langle u, \rho(x, .)\rangle$ holds. $\mathcal{H}(\Phi) = \mathcal{H}(\Phi, R^n)$ is the *reproducing kernel Hilbert space* of Φ, and ρ is the *reproducing kernel* of $\mathcal{H}(\Phi)$.

Setting

$$\overline{\mathcal{H}}(\Phi, S) = \bigcap_{S_\epsilon \supset \bar{S}} \mathcal{H}(\Phi, S_\epsilon),$$

the isometry (i), maps the spaces of random variables $H(\Phi, S)$ and $\overline{H}(\Phi, S)$ onto the function subspaces $\mathcal{H}(\Phi, S)$ and $\overline{\mathcal{H}}(\Phi, S)$ of $\mathcal{H}(\Phi)$, and from 2 it follows that

$$\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S) \quad \text{if and only if} \quad \mathcal{H}(\Phi, S) = \overline{\mathcal{H}}(\Phi, S). \tag{3}$$

The following elementary result identifies the orthogonal complement of $\mathcal{H}(\Phi, S)$ in $\mathcal{H}(\Phi)$ and is fundamental for our discussion.

Proposition 2. *For $u \in \mathcal{H}(\Phi)$, $u \in \mathcal{H}(\Phi, S)^\perp$ iff $u(x) = 0$ holds for all $x \in S$.*

P r o o f. Observe that $u \perp \mathcal{H}(\Phi, S)$ iff $\rangle u, \rho(x, .)\rangle = 0$ holds for all $x \in S$. But the reproducing property (iii) of ρ gives, $u(x) = \langle u, \rho(x, .)\rangle$ and the result follows.

This result prompts the following notation. Let

$$\mathcal{H}_0(\Phi, S) \stackrel{\text{def}}{=} \{u \in \mathcal{H}(\Phi) : u(x) = 0 \quad \text{for } x \in S\} \tag{4}$$

and

$$\mathcal{H}_{00}(\Phi, S) \stackrel{\text{def}}{=} \overline{\bigcup \mathcal{H}_0(\Phi, S_\epsilon)}, \tag{5}$$

where the union is over all ϵ neighborhoods S_ϵ of $\bar{S}$ and the closure is taken in the norm of $\mathcal{H}(\Phi)$.

Our principle result in this section is the following criterion.

Theorem 1. *For a continuous Gaussian random field Φ and a set S, $\mathcal{F}(\Phi, S) = \mathcal{F}(\Phi, S)$ iff $\mathcal{H}_0(\Phi, S) = \mathcal{H}_{00}(\Phi, S)$; that is iff each function in $\mathcal{H}_0(\Phi, S)$ is a limit in the norm of $\mathcal{H}(\Phi)$ of a sequence of functions that vanish on neighborhoods of $\bar{S}$.*

P r o o f. By (3) $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ holds iff $\mathcal{H}(\Phi, S) = \overline{\mathcal{H}}(\Phi, S)$ and thus the spaces $\mathcal{H}(\Phi, S)$ and $\overline{\mathcal{H}}(\Phi, S)$ are equal if and only if their orthogonal complements are equal. It only remains to identify the orthogonal complement of $\overline{\mathcal{H}}(\Phi, S)$ as $\mathcal{H}_{00}(\Phi, S)$, which follows directly from the definition of $\overline{\mathcal{H}}(\Phi, S)$ and Proposition 2.

Remark. In analogy with the classical results of Beurling (1948), approximation results such as Theorem 1 are called *spectral synthesis* theorems. If $\mathcal{H}_0(\Phi, S) = \mathcal{H}_{00}(\Phi, S)$ the set S is said to *admit spectral synthesis* in the function space $\mathcal{H}(\Phi)$.

3. The Bessel Random Fields

Although, as we already mentioned, the approach described in Section 2 is very general, the geometric conditions on S under which the approximation in Theorem 1 is possible depend on the structure of the reproducing kernel Hilbert space. In particular, pointwise properties such as continuity and differentiability (either everywhere or off of certain small exceptional sets) prove to be important. In this section we identify the reproducing kernel Hilbert spaces for a class of random fields $\Phi_\beta = \Phi = \{\phi(x) : x \in R^n\}$ that satisfy the stochastic pseudo-differential equations

$$(I - \Delta)^{\beta/2}\phi(x) = \dot{W}(x), \ x \in R^n, \tag{6}$$

where $\beta > n/2$ is constant and $\dot{W}(x)$ is a stationary Gaussian white noise on R^n. The case $\beta = 2$ deserves special attention. It is the simplest and most interesting of the Bessel fields and we consider it separately.

The Saces of Bessel Potentials. The spaces of Bessel potentials are defined by

$$\mathcal{L}^{\beta,2} = \mathcal{L}^{\beta,2}(R^n) = \{u : u = \mathcal{G}_\beta(g), g \in L^2(R^n)\}, \tag{7}$$

where G_β, $\beta \in R$, is the Bessel kernel of order β defined as the inverse Fourier transform of $\hat{G}_\beta(\lambda) = (1 + |\lambda|^2)^{-\frac{\beta}{2}}$. The norm in $\mathcal{L}^{\beta,2}$ is defined by $\|u\|_{\beta,2} = \|G_\beta * g\|_{\beta,2} = \|g\|_2$, where $\|.\|_2$ denotes the L^2 norm [see e.g. Adams and Hedberg (1996), p. 11, for more details].

The theory of Bessel spaces is well developed and the properties of the functions in these spaces are well known. For example, Stein (1970) and Adams and Hedberg (1996) present a good introduction to the field and contain numerous references to the original sources. We present some important facts about these spaces here, as they will be fundamental to our considerations.

In particular, it is shown in Stein (1970), p.136, that a function $u \in \mathcal{L}^{\beta,2}$ if and only if $u \in \mathcal{L}^{\beta-1,2}$ and for each j, $\partial_j u \in \mathcal{L}^{\beta-1,2}$. Moreover, the norms $\|u\|_{\beta,2}$ and $\|u\|_{\beta-1,2} + \|\nabla u\|_{\beta-1,2}$ are equivalent.

When $\beta \geq 0$ is an integer, the space $\mathcal{L}^{\beta,2}$ can be identified with the Sobolev space $W^{\beta,2}$ of weakly differentiable functions of order β:

$$W^{\beta,2}(R^n) \stackrel{\text{def}}{=} \{u \in L^2 : \int_{R^n} \sum_{0\leq|k|\leq\beta} |\nabla^k u|^2 < \infty\}, \tag{8}$$

and the norm $\|u\|_{W^{\beta,2}} = \left(\int_{R^n} \sum_{0\leq|k|\leq\beta} |\nabla^k u|^2\right)^{1/2}$ is equivalent to the norm of $\mathcal{L}^{\beta,2}$ [Stein (1970), p. 135].

Further, when $\beta > n/2$, the functions u in $\mathcal{L}^{\beta,2}(R^n)$ are continuous but not for $\beta \leq n/2$, [Adams and Hedberg (1996), Ch.6]. Thus, since in our case $\beta > n/2$, the functions in the reproducing kernel space $\mathcal{H}(\Phi_\beta)$ will be continuous but their derivaties may not be. A useful way of measuring the deviation from continuity is given by $(\beta, 2)$-*capacity* [see Adams and Hedberg (1996), Ch.2]. This capacity of a set $S \subseteq R^n$ can be defined as

$$C_{\beta,2}(S) = \inf\{\|g\|^2_{L^2} : g \geq 0,\ G_\beta * g \geq 1 \quad \text{on} \quad S\}. \tag{9}$$

In the more general context of L^p integrability, the spectral synthesis problem in the spaces $\mathcal{L}^{\beta,2}(R^n)$ has been studied extensively (see e.g. Hedberg (1980, 1981) for integral β and Adams and Hedberg (1996), Chapter 10, for the general case) and the theorem of Netrusov below (Theorem 3) gives the complete answer. Before we can present this result, it will be necessay to provide some background.

A property of points is be said to hold $(\beta, 2)$-*quasieverywhere* ($(\beta, 2)$-q.e.) if it holds for all points except those belonging to a set of $(\beta, 2)$-capacity zero. A function u, defined $(\beta, 2)$-q.e. on R^n, is called $(\beta, 2)$-*quasicontinuous*, if for each $\varepsilon > 0$ there is an open set D with $C_{\alpha,2}(D) < \varepsilon$ such that f is continuous on $R^n \setminus D$.

Theorem 2 (See Adams and Hedberg (1996), Ch.6) *Let $u \in \mathcal{L}^{\beta,2}(R^n)$ and $\beta \le n/2$. After possible redefinition on a set of measure zero, u is $(\beta, 2)$-quasicontinuous. Moreover, if u and v are two $(\beta, 2)$-quasicontinuous functions such that $u(x) = v(x)$ a.e., then $u(x) = v(x)$ $(\beta, 2)$-q.e.*

If $u \in \mathcal{L}^{\beta,2}$, $\beta < n/2$, and $S \subset R^n$ are arbitrary, then the *trace of u on S*, denoted $u\mid_S$, is defined as the restriction to S of any $(\beta, 2)$- quasicontinuous representative of u. In particular, $u\mid_S= 0$ or $\nabla u\mid_S= 0$ means these statements hold q.e.

Theorem 3 (Stoke (1984)) *Let $u \in \mathcal{L}^{\beta,2}$ where $\beta \ge 1$ is such that $\beta > n/2$ but $(\beta - 1) \le n/2$. Then u is differentiable $(\beta - 1, 2)$-q.e.*

For a set $S \subset R^n$ denote by $\mathcal{L}^{\beta,2}_{00}(S)$ the closure in $\mathcal{L}^{\beta,2}(R^n)$ of the functions $u \in \mathcal{L}^{\beta,2}$ with compact support contained in S. If S is open, then $\mathcal{L}^{\beta,2}_{00}(S)$ is the closure in $\mathcal{L}^{\beta,2}(R^n)$ of $C_0^\infty(S)$. The following is the L^2 version of a result due to Hedberg (1981) for integer order Bessel spaces (Sobolev spaces) and to Netrusov [Adams and Hedberg (1996), p. 281] for $\beta > 0$.

Theorem 4. *Let $\beta > 0$, $1 < p < \infty$, $u \in \mathcal{L}^{\beta,p}(R^n)$, and $S \subset R^n$ be arbitrary. Then the following statements are equivalent:*
(i) $D^\kappa u\mid_S=0$ for all multiindices κ, $0 \le |\kappa| < \beta$;
(ii) $u \in \mathcal{L}^{\beta,p}_{00}(S^c)$;

Remark. 1 For $\beta < n/2$, the capacity $C_{\beta,2}$ defined by (3.4) is equivalent to the classical Riesz capacity defined through the power kernel $k(x) = 1/|x|^{n-2\beta}$ from classical potential theory, [Adams and Hedberg (1996), Ch.5]. More precisely, if $k(x) \ge 0$ is a decreasing continuous extended real valued function on $[0, \infty]$, the k-energy of a positive measure μ is $\mathcal{E}(k, \mu) = \int_{R^n}\int_{R^n} k(|x - y|)d\mu(x)d\mu(y)$. Then, for compact sets $A \subseteq R^n$, define the k-capacity of A as

$$k - C(S) = \sup_\mu\{\frac{1}{\mathcal{E}(k, \mu)} : \quad \text{supp } \mu \subseteq A, \quad \mu(A) = 1\},$$

and for an arbitrary $S \subseteq R^n$,

$$k - C(S) = \sup_A \{k - C(A) : \quad A \text{ is compact}, \quad A \subseteq S\}. \tag{10}$$

When $k(x)$ is the power kernel $k(x) = 1/|x|^t$, the capacity $k - C(.)$ is the t-Riesz capacity and is also denoted by $C_t(.)$. When $\beta = n/2$, the capacity (1) is equivalent to the classical logarithmic capacity $C_{log}(S)$.

Remark. 2 If Λ_t is the t-dimensional Hausdorff measure, $\Lambda_t(S) < \infty$ implies $k - C(S) = 0$ for $k(x) = 1/|x|^t$, [Falconer (1985), Theorem 6.4], and therefore $\Lambda_{n-2\beta}(S) < \infty$ implies $C_{\beta,2}(S) = 0$.

Remark. 3 When $\beta \leq 1/2$, and S is a line segment, the energy integral for the power kernel $k(x) = 1/|x|^{n-2\beta}$ diverges and therefeore $C_{\beta,2}(S) = 0$. For $\beta > 1/2$, line segments have positive $C_{\beta,2}$ capacity.

The Whittle Field. The mean zero, stationary Gaussian field that arises as a solution of the stochastic equation

$$(I - \Delta)\phi(x) = \dot{W}(x), \ x \in R^n, \ n = 1, 2, \text{ or } 3,$$

where $\dot{W}$ is Gaussian white noise with $E\dot{W}(A)\dot{W}(B) = |A \cap B|$, was studied in Whittle (1954) and is called the Whittle field.

To identify the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$ for this field, we only consider the case of dimension $n = 2$ and recall that the operator $I - \Delta$ has an inverse given by convolution with the Bessel kernel $G_2(x)$, and we can write

$$\phi(x) = \int_{R^2} G_2(x - y)\dot{W}(y)dy,$$

and so $E\phi(x)\phi(y) = G_2 * G_2(x - y)$. Taking Fourier transforms on both sides, we obtain the spectral representation of the covariance function

$$\rho(x, y) = \frac{1}{(2\pi)^2} \int_{R^2} e^{i(x-y)\cdot\lambda}(1 + |\lambda|^2)^{-2} d\lambda = \int_{R^2} e^{i(x-y)\cdot\lambda} \Delta(\lambda) d\lambda,$$

where $\Delta(\lambda) = (2\pi(1 + |\lambda|^2))^{-2}$. The family $\{e^{ix\cdot\lambda} : x \in R^2\}$ spans $L^2(R^2, \Delta)$ and the functions $u \in \mathcal{H}(\Phi)$ are given by

$$u(x) = \int_{R^2} e^{ix\cdot\lambda} f(\lambda)\Delta(\lambda) d\lambda = G_2 * g(x), \tag{11}$$

where f satisfies $\int_{R^2} |f(\lambda)|^2(1+|\lambda|^2)^{-2} d\lambda < \infty$ and $\hat{g}(\lambda) = f(\lambda)(1+|\lambda|^2)^{-1} \in L^2$. Therefore we can identify the space $\mathcal{H}(\Phi)$ is $\mathcal{L}^{2,2}$ and $\|u\|_{2,2} = \|g\|_2$.

The following arguments identifying conditions under which a closed set S satisfyes the sharp Markov property for the Whittle field appeared in the dissertation Robeva (1997).

It follows from Theorem 4 that the Whittle field satisfies $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ for a closed set $S \subseteq R^n$, $n = 2, 3$ if and only if for each $f \in W^{2,2}$, $f\ |_S = 0$ implies $\nabla f\ |_S = 0$ where $\nabla f = \{D^\alpha f, |\alpha| = 1\}$. Note that for the dimensions under consideration, the functions in $W^{2,2}(R^n)$ are continuous and are therefore defined pointwise everywhere. Thus the trace $f\ |_S$ is defined for all $x \in S$. Since the gradient ∇f for $f \in W^{2,2}(R^n)$, $n = 2, 3$, is only quasicontinuous, it follows from Theorems 2 that ∇f is defined pointwise only $(1, 2)$-q.e. We thus have

Theorem 5. *For the Whittle field and a closed set $S \subseteq R^n$ the equality $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ holds if and only if for each $f \in W^2(R^n)$ with $f\ |_S = 0$ it follows that $C_{1,2}(S \cap \{x : \nabla f(x) \neq 0\}) = 0$.*

The following corollary is immediate.

Corollary 1. *Let Φ be the Whittle field and $S \subset R^n$, $n = 2, 3$, be such that $C_{1,2}(S) = 0$. Then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.*

As we see next, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ may also hold for large but sufficiently irregular sets S.

Following Saks (1937), p. 262, we define contingents of a set $S \subseteq R^2$. For points $x \neq y \in R^2$, we let $l(x, y)$ denote the line in R^2 that contains x and y. If x is an accumulation point of S, and if l is a line through x, we will say that l is a *contingent of* S *at* x provided there is a sequence of points $y_n \neq x$ in S that converges to x with $\lim_{n\to\infty} l(x, y_n) = l$. We let $\mathrm{Contg}(S, x)$ denote the set of all contingents to S at x. For a subset $S \subseteq R^2$, we say that S has a tangent at $x \in S$ provided that $\mathrm{Contg}(S, x)$ contains a unique line. We write $\mathcal{T}(S)$ for the set of points in S at which S has a tangent.

In the same way, if A is a subset of R^3, we say that A has a *tangent plane* h *at* x if all contingents of A at x lie in the plane h.

Let $\mathcal{T}(S)$ be the set of points in S at which S has a tangent line (if $S \subseteq R^2$) or a tangent plane (if $S \subseteq R^3$). Our next theorem shows that whether $f\ |_S = 0$ implies $\nabla f\ |_S = 0$ for a W^2 function f, depends on how big the set $\mathcal{T}(S)$ is in terms of appropriate capacities.

Theorem 6. *Let Φ be the Whittle field and $S \subseteq R^n$, $n = 2, 3$, be closed. If $C_{1,2}(\mathcal{T}(S)) = 0$, then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.*

P r o o f. To proof of Theorem 6, we use the following elementary result.

Lemma 1. *Let $S \subseteq R^n$, $n = 2, 3$, and $x_0 \in S \setminus \mathcal{T}(S)$. Let the function u, defined on S, vanish on S and be differentiable at x_0. Then $\nabla u(x_0) = 0$.*

P r o o f. Set $v = \nabla u(x_0)$. We will show that $v = 0$. Since u is differentiable at x_0,

$$u(y) = u(x_0) + < y - x_0, v > + o(|y - x_0|), \quad \text{as} \quad |y - x_0| \to 0.$$

Hence, if y approaches x_0, the unit vectors $(y - x_0)/|y - x_0|$ are asymptotically orthogonal to v. Since $x_0 \notin T(A)$, the set of cluster points of the vectors $(y - x_0)/|y - x_0|$ must contain a basis for R^n, $n = 2, 3$, all elements of which are orthogonal to v. Thus $v = \nabla u(x_0) = 0$.

As the functions in $W^2(R^n)$, $n = 2, 3$, are continuous but not necessarily differentiable, it follows from Theorem 3 that if $f \in W^2(R^n)$, $n = 2, 3$, there exists a set E such that $C_{1,2}(E) = 0$ and such that f is differentiable at all $x \in E^c$. Therefore, without loss of generality, we may assume that f is differentiable at all points $x \in S$. Further, if $f(x) = 0$ for all $x \in S$, we have by Lemma 3.15 that

$$C_{1,2}(S \setminus \{x \in S : \nabla f(x) \neq 0\}) \leq C_{1,2}(\mathcal{T}(S)) = 0.$$

We now apply Theorem 5 to complete the proof.

The Bessel Fields. The family of continuous, mean-zero, stationary Gaussian fields $\Phi_\beta = \Phi = \{\phi(x) : x \in R^n\}$ which satisfy the stochastic pseudo-differential equations (6) defines the Bessel fields. Using the same approach as for the Whittle field, the reproducing kernel Hilber spaces for these fields can be identified with the Bessel potential spaces $\mathcal{L}^{\beta,2}(R^n)$ from 7.

The following resuts concern the equality $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$ for close sets $S \subset R^2$ and will be used in Section 4. These results appeared in Pitt and Robeva (2003) and the proofs can be found there.

Combining Theorems 1 and 4, gives the following result.

Theorem 7. *Let Φ_β with $k < \beta \leq k+1$, for some integer $k \geq 1$, be a Bessel fields of order β. Let $S \subseteq R^2$. Then $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$ iff for any $u \in \mathcal{L}^{\beta,2}$, $u\,|_S = 0$ implies $\nabla^m u\,|_S = 0$ for all integers m with $0 < m \leq k$.*

In the range $1 < \beta \leq 2$ therefore, $C_{\beta-1,2}(S) = 0$ implies that $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$. For $\beta > 2$, however, $C_{\beta-1,2}(S) > 0$ for any non-empty set S (see Remark 1 following Theorem 4 above), and the condition $C_{\beta-1,2}(S) = 0$ is thus never applicable. Observe also that if $C_{\beta-m,2}(S) = 0$ for some integer m, $0 < m < k$, the conditions $\nabla^m u\,|_S = 0, \ldots, \nabla^k u\,|_S = 0$ in Theorem 9 are vacuous. In addition, when β moves from one of the ranges $k < \beta \leq k+1$, k- positive integer, to the next, the order of the derivatives involved in Theorem 9 increases by one.

As we just noted, this result immediately implies that $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$ for any sufficiently small set S satisfying, for $k < \beta \leq k+1$, $C_{\beta-k,2}(S) = 0$. As for the Whittle field, large irregular sets S may also satisfy $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.

We have

Theorem 8. *Let S be a closed set and $\mathcal{T}(S)$ be the set of all $x \in S$ for which S has a tangent line. Let $k < \beta \leq k+1$, where $k \geq 1$ is an integer. Then $C_{\beta-k,2}(\mathcal{T}(S)) = 0$ implies $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$.*

4. Other Random Processes and Fields

In this section, we identify the reproducing kernel Hilber spaces for the Ornstein-Uhlenbeck process and the Brownian motion on R^1, the Lévy Brownian motion, the fractional Brownian motion on R^n, and a solution of the stochastic heat equation. For each of these random processes and fields, we give conditions for spectral synthesis.

The Ornstein-Uhlenbeck Process on R^1.

Let Φ be the Orstein-Uhlenbeck process on the line, i.e. Φ is the continuous stationary Gaussian random process on R^1 with mean zero and covariance

$$\rho(x, y) = \rho(x-y) = e^{-|x-y|}.$$

Φ is easily seen to have a spectral density of the form $\Delta(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2}$ and therefore the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$ consists of all functions of the form

$$u(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\lambda x} f(\lambda)\frac{d\lambda}{1+\lambda^2},$$

where $\int_{-\infty}^{\infty} |f(\lambda)|^2(1+\lambda^2)^{-1}d\lambda < \infty$. Thus $\mathcal{H}(\Phi)$ is exactly the Bessel potential space $\mathcal{L}^{1,2} = W^{1,2}$.

By combining Theorems 1 and 4, the following result becomes immediate.

Theorem 9. *The Ornstein-Uhlenbeck process Φ on the line satisfies $\mathcal{F}(\Phi, S)$ $= \mathcal{F}(\Phi, S)$ for all closed sets $S \subset R^1$.*

The Brownian Motion on R^1. The Brownian motion on R^1 is defined as the real valued continuous process $\Phi = \{\phi(x), x \in R^1\}$ with independent Gaussian increments satisfying the conditions $\phi(0) = 0$, $E\phi(x) = 0$, $E|\phi(x)-\phi(y)|^2 = |x-y|$

for any $x, y \in R^1$. The distribution of Brownian motion, as with all Gaussian fields, is completely determined by its covariance function

$$\rho(x,y) = \frac{1}{2}\{x + y - |x-y|\}.$$

It is known (see for example Kailath (1971), p. 242) that the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$ for the Brownian motion consists of all absolutely continuous functions that have square integrable derivatives and that vanish at $x = 0$ i.e. $\mathcal{H}(\Phi) = \{u : u \quad \text{is absolutely continuous}, u(0) = 0, \|u'\|_{L^2} < \infty\}$. This can be easily verified by observing that

$$\rho(x,y) = \frac{1}{2}\{x + y - |x-y|\} = \min(x,y).$$

Then for any finite collection of points $0 < x_1 < x_2 < \ldots < x_k$, the linear combinations of the functions $\rho(x_1, y), \rho(x_2, y), \ldots, \rho(x_k, y)$ are piecewise linear functions equal to x_k for $y > x_k$. Thus their derivatives are step functions and the result follows by density arguments.

We next show that Brownian motion does have the sharp Markov property for each compact set Γ separating R^1 into two open sets D_1 and D_2.

We will need the following definition. For a compact set $S \subset R^n$ we say that two function spaces L_1 and L_2 are *locally equivalent near* A if there is an open neighborhood U of S for which $C_c(U) \cap L_1 = C_c(U) \cap L_2$ in the sense that they contain exactly the same functions, and that as normed function spaces the two norms are equivalent. Here we have denoted with $C_c(U)$ the space of all continuous complex valued functions with compact support contained in U.

Theorem 10. *Let Φ be Brownian motion on R^1 and $S \subset R^1$ be compact. Then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$*

Proof. It follows from Theorem 1 that $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ if and only if any function $u \in \mathcal{H}(\Phi)$ with $u\,|_S = 0$ can be approximated in $\mathcal{H}(\Phi)$ by functions that vanish on neighborhoods of S. If $0 \in S$, because for the Brownian motion $\mathcal{H}(\Phi)$ is locally equivalent to W^1 away from the origin, we get that the collection of all functions with compact support $u \in \mathcal{H}(\Phi)$ with $u\,|_S = 0$ coincides with the collection $\{u \in W^1(R) : u\,|_S = 0\}$. Thus, by Theorem 4, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.

If $0 \notin S$, then $\mathcal{H}(\Phi)$ and $W^1(R)$ are equivalent near S and, again, the result follows.

As a special case we obtain the Blumental's $0 - 1$ law for Brownian motion:

Corollary 2. *For the Brownian motion the germ σ-field for the origin is trivial, i.e.* $\overline{\mathcal{F}}(\Phi, \{0\}) = \{\Omega, \emptyset\}$.

Proof. We apply Theorem 10 for $S = \{0\}$ to get $\mathcal{F}(\Phi, \{0\}) = \overline{\mathcal{F}}(\Phi, \{0\})$. Since the normalizing condition $\phi(0) = 0$ holds, the sharp σ-field $\mathcal{F}(\Phi, \{0\})$ is trivial. Therefore the germ σ-field $\overline{\mathcal{F}}(\Phi, \{0\})$ is also trivial.

The Lévy Brownian Motion and the Fractional Brownian Motion. This is the Gaussian random field Φ on R^n of mean zero and covariance function

$$\rho(x, y) = \frac{1}{2}(|x| + |y| - |x - y|), \quad x, y \in R^n. \tag{12}$$

normalized with the condition $\phi(0) = 0$. The following property is characteristic of it (McKean, 1963): $E|\phi(x) - \phi(y)|^2 = |x - y|$. It is clear from (12) that the Lévy Brownian motion is not stationary but its increments $\phi(x) - \phi(y)$ are stationary with Gaussian distribution $N(0, |x - y|)$.

It is known that the Lévy Brownian motion has the following spectral representation (Molchan, 1966):

$$\phi(x) = \frac{1}{\sqrt{K}} \int_{R^n} \frac{e^{ix\cdot\lambda} - 1}{|\lambda|^{\frac{n+1}{2}}} \dot{W}(\lambda) d\lambda, \quad x \in R^n, \tag{13}$$

where $\dot{W}$ is Gaussian white noise.

To identify the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$, notice that, using (13), the covariance function of Φ can be written as

$$\rho(x, y) = \frac{1}{K} \int_{R^n} [e^{ix\cdot\lambda} - 1][e^{-iy\cdot\lambda} - 1] \frac{d\lambda}{|\lambda|^{n+1}}. \tag{14}$$

Further, notice that the functions $\{\frac{e^{ix\cdot\lambda}-1}{|\lambda|^{(n+1)/2}}, x \in R^n\}$ form a complete system in $L^2(R^n)$. To see this, let

$$\int \frac{e^{ix\cdot\lambda} - 1}{|\lambda|^{(n+1)/2}} f(\lambda) d\lambda = 0, \quad x \in R^n.$$

Consider the difference of the above integrals for $x = y$ and $x = y + z$. We get

$$\int e^{iy\cdot\lambda} \frac{e^{iz\cdot\lambda} - 1}{|\lambda|^{(n+1)/2}} f(\lambda) d\lambda = 0, \quad y, z \in R^n,$$

from where

$$\frac{e^{iz\cdot\lambda} - 1}{|\lambda|^{(n+1)/2}} f(\lambda) = 0 \quad \text{a.e.}, \quad z \in R^n.$$

and thus $f(\lambda) = 0$ a.e.

It follows now from (14) that an arbitrary element of the space $\mathcal{H}(\Phi)$ has the representation

$$u(x) = K \int_{R^n} \frac{[e^{ix\cdot\lambda} - 1]}{|\lambda|^{\frac{n+1}{2}}} f(\lambda) d\lambda. \tag{15}$$

where $f \in L^2(R^n)$ and K is a constant depending on the dimension n.

The Lévy Brownian motion corresponds to $\alpha = 1$ in the family of real valued Gaussian mean zero random fields Φ_α, $0 < \alpha < 2$, known as the fractional Brownian motions. The covariance function of Φ_α has the form

$$\rho(x, y) = \frac{1}{2}(|x|^\alpha + |y|^\alpha - |x - y|^\alpha), \quad x, y \in R^n.$$

It is known (Gangolli, 1967) that for this range of α, Φ_α is a random field of stationary increments and can be written as

$$\phi(x) = K \int_{R^n} \frac{e^{ix\cdot\lambda} - 1}{|\lambda|^{\frac{n+\alpha}{2}}} \dot{W}(\lambda) d\lambda, \tag{16}$$

where now $K = K_{n,\alpha}$ is a constant depending on the dimension n and α. Thus, for the covariance function, we have

$$\rho(x, y) = \frac{1}{K} \int_{R^n} [e^{ix\cdot\lambda} - 1][e^{-iy\cdot\lambda} - 1] \frac{d\lambda}{|\lambda|^{n+\alpha}}.$$

and applying the same arguments as for the Lévy Brownian motions, we arrive at the general form of the functions in the reproducing kernel space $\mathcal{H}(\Phi_\alpha)$ for the fractional Brownian motion:

$$u(x) = \int_{R^n} \frac{[e^{ix\cdot\lambda} - 1]}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda \tag{17}$$

where $f(\lambda) \in L^2(R^n)$.

The following lemma gives the connection between the reproducing kernel Hilbert spaces $\mathcal{H}(\Phi_\alpha)$ and the Bessel potential spaces.

Lemma 2. *All functions u in $\mathcal{H}(\Phi_\alpha)$ are Hölder continuous of order $\alpha/2$, satisfy $u(0) = 0$, and are locally in the Bessel potential space $\mathcal{L}^{\frac{n+\alpha}{2},2}$.*

P r o o f. It is obvious from (17) that $u(0) = 0$ for all $u \in \mathcal{H}(\Phi_\alpha)$. Next, if $u \in \mathcal{H}(\Phi_\alpha)$, it follows from our discussion in Section 2 that that u has the representation $u(x) = E\eta\phi(x)$ where $\eta \in H(\Phi_\alpha)$. Therefore

$$|u(x) - u(y)| = E\eta(\phi(x) - \phi(y)) \leq \left(E|\eta|^2\right)^{\frac{1}{2}} \left(E|\xi(x) - \xi(y)|^2\right)^{\frac{1}{2}} = K|x-y|^{\frac{\alpha}{2}}.$$

Finally, to prove the second part of the statement, let $u \in \mathcal{H}(\Phi_\alpha)$ and thus from (17) u can be written as

$$u(x) = \int_{R^n} \frac{[e^{ix\cdot\lambda} - 1]}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda,$$

for some function $f \in L^2(R^n)$. We split the above integral as $I_1(x) + I_2(x)$ where

$$\begin{aligned} I_1(x) &= K \int_{|\lambda|\leq 1} \frac{[e^{ix\cdot\lambda} - 1]}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda - \int_{|\lambda|>1} \frac{1}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda = \\ &= K \int_{|\lambda|\leq 1} \frac{[e^{ix\cdot\lambda} - 1]}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda + C, \end{aligned}$$

where C is some constant. For $I_2(x)$ we have

$$I_2(x) = K \int_{|\lambda|>1} \frac{e^{ix\cdot\lambda}}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) d\lambda = K \int_{|\lambda|>1} \frac{e^{ix\cdot\lambda}}{(1+|\lambda|^2)^{\frac{n+\alpha}{4}}} g(\lambda) d\lambda,$$

where

$$g(x) = \begin{cases} \frac{(1+|\lambda|^2)^{\frac{n+\alpha}{4}}}{|\lambda|^{\frac{n+\alpha}{2}}} f(\lambda) & \text{for } |\lambda| > 1 \\ 0 & \text{for } |\lambda| < 1 \end{cases}.$$

Therefore, since $g \in L^2(R^n)$, $I_2(x) \in \mathcal{L}^{\frac{n+\alpha}{2},2}$. Since $I_1(x)$ is real analytic, it is locally in $\mathcal{L}^{\frac{n+\alpha}{2},2}$ and the claim follows.

Next, we use Theorem 4 to determine whether the sharp and the germ σ-field for compact sets $S \subset R^n$ are equal. The next lemma is almost obvious.

Lemma 3. *The space $\mathcal{H}(\Phi_\alpha)$ is locally equivalent to the Bessel potential space $\mathcal{L}^{\frac{n+\alpha}{2},2}$ at points away from the origin.*

P r o o f. From Lemma 2 every $u \in \mathcal{H}(\Phi_\alpha)$ is locally in $\mathcal{L}^{\frac{n+\alpha}{2},2}$. Now let $v \in \mathcal{L}^{\frac{n+\alpha}{2},2}$. Then v can be written as

$$v(x) = \int \frac{e^{i\lambda\cdot x} f(\lambda)}{(1+|\lambda|^2)^{\frac{n+\alpha}{4}}} d\lambda, \quad \text{for some} \quad f \in L^2(R^n).$$

Therefore $v(x) - v(0) \in \mathcal{H}(\Phi_\alpha)$ and thus $v \in \mathcal{H}(\Phi_\alpha)$ if and only if $v(0) = 0$.

The next result follows from combinig Lemma 2 and Theorem 4.

Theorem 11. *Let Φ_α, $0 < \alpha < 2$ be a fractional Brownian motion and $S \subset R^n$ be compact. Then if for any $f \in \mathcal{L}^{\frac{n+\alpha}{2},2}$ with compact support, $f\mid_S = 0$ implies $D^\beta f\mid_S = 0$ for all multiindices β, $0 < |\beta| < (n+\alpha)/2$, the equality $\mathcal{F}(\Phi_\alpha, S) = \overline{\mathcal{F}}(\Phi_\alpha, S)$ holds.*

The following corollary generalizes Theorem 10.

Corollary 3. *If Φ_α, $0 < \alpha \le 1$ is a fractional Brownian motion over R^1 then $\mathcal{F}(\Phi_\alpha, S) = \overline{\mathcal{F}}(\Phi_\alpha, S)$ holds for any compact $S \subset R^1$.*

More detailed results for the fractional Brownian motion can be obtained in the cases when the dimension n and the parameter α are in a certain relationship.

Theorem 12. *Let Φ_α, $0 < \alpha < 2$ be a fractional Brownian motion over R^n, $S \subset R^n$ be compact and $2 < n+\alpha \le 4$. Let $\mathcal{T}(S)$ be the subset of points in S at which S has a tangent hyperplane.*

(i) If $C_{\frac{n+\alpha}{2}-1,2}(S) = 0$, then $\mathcal{F}(\Phi_\alpha, S) = \overline{\mathcal{F}}(\Phi_\alpha, S)$;

(ii) If $C_{\frac{n+\alpha}{2}-1,2}(S) > 0$ but $C_{\frac{n+\alpha}{2}-1,2}(\mathcal{T}(S)) = 0$, $\mathcal{F}(\Phi_\alpha, S) = \overline{\mathcal{F}}(\Phi_\alpha, S)$ holds.

P r o o f. The result follows from Lemma 3 and from applying Theorems 7 and 8 for $\beta = (n+\alpha)/2$.

As a corollary, we obtain that the Whittle field and the Lévy Brownian motion on R^3 are equivalent with respect to the sharp Markov property for compact sets.

Corollary 4. *Let Φ be the Lévy Brownian motion on R^3 and let Ψ be the Whittle field on R^3. Then for compact sets S, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ if and only if $\mathcal{F}(\Psi, S) = \overline{\mathcal{F}}(\Psi, S)$.*

A Solution of the Stochastic Heat Equation. Consider a continuous solution Φ with stationary increments of the stochastic differential equation

$$L(\phi) = (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})\phi(t,x) = \dot{W}(t,x), \quad (t,x) \in R^2, \tag{18}$$

normalized with the condition $\phi(0,0) = 0$. As always, $\dot{W}(t,x)$, $(t,x) \in R^2$ is a Gaussian white noise.

The following steps help to identify the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$.

Proposition 3. *The field* $\Phi = \{\phi(t,x), (t,x) \in R^2\}$ *with the representation*

$$\phi(t,x) = \frac{1}{2\pi} \int_{R^2} \frac{[e^{i(\lambda t + \mu x)} - 1]}{i\lambda + \mu^2} \dot{W}(\lambda, \mu)\, d\lambda d\mu \tag{19}$$

is a continuous solution of (19) with stationary increments, normalized with the condition $\phi(0,0) = 0$.

Proof. From (19), the covariance function $\rho[(t_1,x_1),(t_2,x_2)]$ has the form

$$\rho[(t_1,x_1),(t_2,x_2)] = \int [e^{i(\lambda t_1 + \mu x_1)} - 1][e^{i(\lambda t_2 + \mu x_2)} - 1] \frac{d\lambda d\mu}{\lambda^2 + \mu^4}. \tag{20}$$

Since

$$\int_{R^2} \frac{\lambda^2 + \mu^2}{1 + \lambda^2 + \mu^2} \cdot \frac{d\lambda d\mu}{\lambda^2 + \mu^4} < \infty,$$

a theorem of Yaglom (Yaglom, 1957) implies that ρ is a covariance of a continuous random field with stationary increments. Verifying that the random field (19) satisfies the equation (18) and that $\phi(0,0) = 0$ is straightforward.

To obtain the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$, using arguments similar to those used for the Lévy Brownian motion, one can verify that the family of functions

$$\left\{ \frac{[e^{i(\lambda t + \mu x)} - 1]}{(\lambda^2 + \mu^4)^{\frac{1}{2}}}, \quad (t,x) \in R^2 \right\}$$

is complete in $L^2(R^2)$. Therefore a function u belongs to the reproducing kernel Hilbert space $\mathcal{H}(\Phi)$ for the random field (19) if and only if u can be written as

$$u(t,x) = \int_{R^2} \frac{[e^{i(\lambda t + \mu x)} - 1]}{(\lambda^2 + \mu^4)^{\frac{1}{2}}} f(\lambda, \mu)\, d\lambda d\mu, \tag{21}$$

with $f \in L^2(R^2)$.

Notice the lack of isotropy in the space $\mathcal{H}(\Phi)$ for this field. This example presents a clear directional dependence suggesting the need for a non-isotropics capacity as a measure for the negligible sets. Classes of such non-isotropic spaces are studied in Nikolskii (1975) Chap.9, and Nikolskii (1991), Chap.5, in the context of establishing imbedding and trace-type theorems. Schmeisser and Triebel (1987), Chapter 4, study equivalent norms for anisotropic Bessel spaces and interpolation results. More recently, such anisotropic spaces are examined by Secchi (2000), and Farkash (2002) presents atomic decomposition theorems. However, to study intrinsically anisotropic fields such as (18) above, a satisfactory potential theory for the corresponding anisotropic function spaces is neeed. It appears that, to date, this general work has not been done.

Below, we definine a non-isotropic capacity that reflects the non-isotropy of the space $\mathcal{H}(\Phi)$ and we use this capacity as a measure of "smallness" for the exceptional sets. For our definition we use the kernel $L(t,x) = (t^2 + x^4)^{-\frac{1}{4}}$ to difine the non-isotropic capacity in a way analogues to the definition of the classical Riesz capacities (see Remarks 1 and 2 following Theorem 4). The general spectral synthesis problem for closed sets S remains open here, but we obtain solutions in some special cases.

For a finite positive Borel measure $dm(t,x)$ on R^2, define the *energy integral associated with the kernel* L by

$$\mathcal{E}_L(m) = \int_{R^2} \frac{dm(t,x)dm(s,y)}{[(t-s)^2 + (x-y)^4]^{\frac{1}{4}}}.$$

For a compact set $F \subset R^2$ we define the *capacity of* F, associated with K as

$$C_L(F) = \sup_m \{\frac{1}{\mathcal{E}_L(m)} : \text{ supp } m \subseteq F, \; m(F) = 1\}, \tag{22}$$

with the usual convention $1/\infty = 0$. For an arbitrary $E \subseteq R^2$ we define

$$C_L(E) = \sup\{C_L(F) : F \text{ is compact}, \; F \subset E\}.$$

Our next lemma characterizes the subsets S of the coordinate axes with $C_L(S) = 0$.

Lemma 4. *(i) If* $S \subseteq R_t = \{(t,x) : x = 0\}$*, then* $C_L(S) = 0$ *if and only if* S *has zero Riesz* $1/2$*-capacity;*

(ii) If $S \subseteq R_x = \{(t,x) : t = 0\}$*, then* $C_L(S) = 0$.

P r o o f. (i) When restricted to the t-axis the kernel $L(t,x)$ reduces to $L(t,0) = |t|^{-\frac{1}{2}}$. For $A \subseteq R_t$ we will thus have $C_L(S) > 0$ if and only if there exists a measure m supported on S for which

$$\mathcal{E}_L(m) = \int_{S\times S} \frac{dm(t)dm(s)}{|t-s|^{\frac{1}{2}}} < \infty.$$

Therefore, for such sets S, $C_L(S) > 0$ if and only if S has zero Riesz 1/2-capacity.

(ii) When restricted to the x-axis the kernel L becomes $L(0,x) = |x|^{-1}$. Thus a set $S \subseteq R_x$ will have positive capacity with respect to L if and only if it has positive Riesz 1-capacity. We now show that if $S \subseteq R_x$, S has zero Riesz 1-capacity. By definition of capacity, it is enough to prove this for compact sets S. But if $S \subset R_x$ is compact, its Hausdorff measure $\Lambda^1(S) < \infty$ and therefore implies that S has zero Riesz 1-capacity (see Remarks 1 and 2 following Theorem 4). This completes the proof of part (ii).

Our next Theorem gives a complete answer to the question of when are the sharp and the germ field equal for the random field Φ from (20) for sets S which are compact subsets of the coordinate axes.

Theorem 13. *Let S be a compact set and Φ be the solution (13) of the stochastic heat equation.*

(i) If $S \subset R_t$ then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ if and only if $C_L(S) = 0$;

(ii) If $S \subset R_x$ then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.

Proof. Since S is compact, in the same way as for the fractional Brownin motion above, it will be enough to show that for the stationary Gaussian random field

$$\psi(t,x) = \int_{R^2} \frac{e^{i(\lambda t + \mu x)}}{(1+\lambda^2+\mu^4)^{\frac{1}{2}}} \dot{W}(\lambda,\mu) d\lambda d\mu.$$

$\mathcal{F}(\Psi, S) = \overline{\mathcal{F}}(\Psi, S)$ if and only if $C_L(S) = 0$. Denote the spectral density of Ψ by $\Delta(\lambda,\mu) = (1+\lambda^2+\mu^4)^{-1}$ and the space of all square integrable with respect to $\Delta(\lambda,\mu)$ functions by $L^2(\Delta)$. Recall that for any closed set $E \subset R^2$, we defined (Section 2)

$$H(\Psi, E) = \bar{sp}\{e^{i(\lambda t + \mu s)} : (t,x) \in E\}_{L^2(\Delta)}$$

and $\overline{H}(\Psi, E) = \bigcap H(E_\epsilon)$ where the intersection is taken over all ϵ neigborhoods E_ϵ of E. Since $\mathcal{F}(\Psi, E) = \overline{\mathcal{F}}(\Psi, E)$ if and only if $H(\Psi, E) = \overline{H}(\Psi, E)$. We will prove that if S is a subset of one of the coordinate axes then $H(\Psi, S) = \overline{H}(\Psi, S)$ if and only if $C_L(S) = 0$.

(i) Let $S \subset R_t$ and $C_S(S) > 0$. We will show that there exists a function $f(\lambda,\mu) \in \overline{H}(\Psi, S)$ such that $f(\lambda,\mu) \notin H(\Psi, S)$, i.e. $H(\Psi, S) \neq \overline{H}(\Psi, S)$.

Let $f(\lambda,\mu) \in \bar{H}(\Phi, S)$. Then since $S \subset R_t$ is bounded, it follows from a version of Paley-Wiener's theorem ([Pit 73, Lemma 3.1]) that $f(\lambda,\mu)$ is an entire function of exponential type which, as a function of μ, is of minimal exponential type. But a function $f(\lambda,\mu)$ that is in $L^2(\Delta)$ must, for almost all fixed λ, be in $L^2\left(\frac{1}{1+\mu^4}\right)$. Thus, since $\mu \mapsto f(\lambda,\mu)$ is of minimal exponential type, $f(\lambda,\mu)$

must be linear as a function of μ. Therefore $f(\lambda,\mu) = g(\lambda) + h(\lambda)\mu$, where, since $f \in L^2(\Delta)$, we have

$$\int_{R^2} \frac{|h(\lambda)|^2\mu^2}{1+\lambda^2+\mu^4} d\lambda d\mu < \infty.$$

After integrating out the variable λ, the above condition gives $h \in L^2((1+\lambda^2)^{-\frac{1}{4}})$. We will show now that because $C_L(S) > 0$, $h(\lambda)$ can be taken to be the Fourier transform of a measure m, supported on S.

Since $C_L(S) > 0$, it follows from Lemma 4 (i) that there exists a measure m, supported on S, with finite 1/2-energy, i.e.

$$\mathcal{E}_{\frac{1}{2}}(S) = \int_S \int_S \frac{dm(t)dm(s)}{|t-s|^{\frac{1}{2}}} < \infty.$$

This last statement is (in Fourier variables) equivalent to

$$\mathcal{E}_{\frac{1}{2}}(S) = \int_{R^2} \frac{|\hat{m}(\lambda)|^2}{|\lambda|^{\frac{1}{2}}} d\lambda < \infty,$$

and since S is bounded, equivalent to $\hat{m}(\lambda) \in L^2((1+\lambda^2)^{-\frac{1}{4}})$. Therefore, if we take $h(\lambda) = \hat{m}(\lambda)$, the function $f(\lambda,\mu) = h(\lambda)\mu$ belongs to $\bar{H}(\Psi,S)$ but is obviously not in $H(\Psi,S)$. Thus $H(\Psi,S) \neq \bar{H}(\Psi,S)$.

Next, let $C_L(S) = 0$. We will show that $H(\Psi,S) = \bar{H}(\Psi,S)$. As in the previous case, if $f \in \bar{H}(\Psi,S)$, then $f(\lambda,\mu) = g(\lambda) + h(\lambda)\mu$ where $h \in L^2((1+\lambda^2)^{-\frac{1}{4}})$. We will show that $C_L(S)$=0 implies $h(\lambda) = 0$. Since $h \in L^2((1+\lambda^2)^{-\frac{1}{4}})$, h is the Fourier transform of a $W^{-\frac{1}{4}}(R^1)$ distribution T supported on S, i.e. $T(\phi) = 0$ if supp $\phi \in R^2 \setminus S$. Therefore $T(u) = 0$ for all $u \in W^{\frac{1}{4}}$ that can be approximated by functions in $W^{\frac{1}{4}}$ that vanish on the set S. By the classical spectral synthesis result of Beurling and Deny, this will be the whole space $W^{\frac{1}{4}}$ if $C_{\frac{1}{4},2}(S) = 0$. Since $C_{\frac{1}{4},2}(S) = 0$ if and only if the 1/2-Riesz capacity of S is zero, we get that $T(u) = 0$ for all $u \in W^{\frac{1}{4}}$, which shows that T and thus h vanish identically. We therefore have $f(\lambda,\mu) = g(\lambda)$. Since $f \in L^2(\Delta)$,

$$\int_{R^2} \frac{|g(\lambda)|^2}{1+\lambda^2+\mu^4} d\lambda d\mu = C \int_{R^2} \frac{|g(\lambda)|^2}{(1+\lambda^2)^{\frac{3}{4}}} d\lambda < \infty,$$

which shows that $g(\lambda)$ is a Fourier transform of a $W^{-\frac{3}{4}}(R^1)$ distribution supported on S. Thus $g(\lambda)$ can be approximated in $L^2(\frac{1}{1+\lambda^{\frac{3}{2}}})$ by Fourier transforms of measures supported on S, which shows that $f(\lambda,\mu)=g(\lambda)\in H(\Psi,S)$. Since f was arbitrary, we get $H(\Psi,S)=\bar{H}(\Psi,S)$. This completes the proof of part (i).

(ii) Let $S\subset R_x$. We will show that $H(\Psi,S)=\bar{H}(\Psi,S)$.

Let $f\in\bar{H}(\Psi,S)$. Since $S\subset R_x$ and S is compact, it follows from the Paley-Wiener theorem that $f(\lambda,\mu)$ is an entire function of exponential type, which is of minimal type as a function of λ. Since $f\in L^2(\Delta)$, $\Delta=(1+\lambda^2+\mu^4)^{-1}$, for almost all fixed μ we have $f\in L^2(\frac{1}{1+\lambda^2})$. Because the function $\lambda\mapsto f(\lambda,\mu)$ is of minimal exponential type, we must have $f(\lambda,\mu)=g(\mu)$ and therefore

$$\int_{R^2}\frac{g(\mu)}{1+\lambda^2+\mu^4}d\lambda d\mu<\infty.$$

Integrating out the variable λ, this gives that $g\in L^2(\frac{1}{1+\mu^2})$. Thus $g(\mu)$ is the Fourier transform of a W^{-1} distribution supported on S and is therefore a limit in $L^2(\frac{1}{1+\mu^2})$ of Fourier transforms of measures supported on S. This shows that $f(\lambda,\mu)=g(\mu)\in H(\Psi,S)$. Since f was arbitrary, we obtain $\bar{H}(\Psi,S)=H(\Psi,S)$.

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