PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **16** (2004), 147-157

PLISKA
STUDIA MATHEMATICA
BULGARICA

DISCRETE TIME BISEXUAL BRANCHING PROCESSES IN VARYING ENVIRONMENTS

Manuel Molina, Manuel Mota, Alfonso Ramos[1]

This paper concerns with the bisexual branching process in varying environments introduced in [16]. For such a model a survey of results is provided. Previously, brief descriptions about the bisexual branching process and some bisexual models derived from it are given.

1. Introduction.

Branching processes theory provides appropriate mathematical models for description of the probabilistic evolution of systems whose components, after certain life period, reproduce and die. It is well-known that, from the classical Bienaymé-Galton-Watson process, several asexual models have been investigated. In the Symposium held in 1966 at the Wistar Institute of Anatomy and Biology, S.M. Ulam pointed up the necessity of developing a corresponding sexual branching processes theory. In 1968, D.J. Daley introduced the bisexual branching process as a two-type discrete time stochastic model $\{(F_n, M_n)\}_{n\geq 1}$ defined in the form:

(1)

$$Z_0 = N, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}), \quad Z_{n+1} = L(F_{n+1}, M_{n+1}),\ n = 0, 1, \ldots$$

[1]Research supported by the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica, grant BFM2003-06074.

2000 *Mathematics Subject Classification:* 60J80

Key words: Discrete time branching processes. Bisexual processes. Branching processes in varying environments

where the empty sum is considered to be $(0,0)$, N is a positive integer, $\{(f_{ni}, m_{ni}),$ $n = 0, 1, \dots;\ i = 1, 2, \dots\}$ is a sequence of i.i.d. nonnegative, integer-valued random variables, their common probability law is called offspring probability distribution, and the mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is assumed to be monotonic nondecreasing in each coordinate, integer-valued on integer arguments and such that $L(x, y) \leq xy$. Intuitively, (f_{ni}, m_{ni}) represents the number of females and males produced by the i-th mating unit in the n-th generation and consequently, from (1), (F_{n+1}, M_{n+1}) will be the total number of females and males in the $(n + 1)$-th generation. These females and males form $Z_{n+1} = L(F_{n+1}, M_{n+1})$ mating units, which reproduce independently of all other mating units with the same offspring probability distribution for each generation. It can be easily proved that $\{(F_n, M_n)\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 0}$ are homogeneous Markov chains. The bisexual branching process is a reasonable model to describe the probabilistic behaviour of populations with sexual reproduction, it has received some attention in the literature (see e.g. [1]-[8],[13] and [14]) and moreover, it served as a base for a variety of other bisexual branching models. In particular, a bisexual process which allows the offspring probability distribution to vary from generation to generation has been introduced in [16]. In section 3, a survey of results about this bisexual process is given. Previously, in section 2, some bisexual processes derived from Daley's model general context are briefly described.

2. Some bisexual models investigated.

Recently, in order to describe the probabilistic evolution of more complicated bisexual populations, from (1), some bisexual branching processes have been introduced. Next, we provide the formal definition and some information about them:

A) Bisexual process with immigration of females and males.

Introduced in [9], the bisexual process with immigration of females and males is defined in the form:

$$Z_0 = N, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}) \ + \ (F^I_{n+1}, M^I_{n+1}),$$

$$Z_{n+1} = L(F_{n+1}, M_{n+1}), \quad n = 0, 1, \dots$$

where $\{(F^I_n, M^I_n)\}_{n\geq 1}$ is a sequence of i.i.d. nonnegative, integer-valued random variables independent of $\{(f_{ni}, m_{ni}),\ n = 0, 1, \dots;\ i = 1, 2, \dots\}$. Intuitively, (F^I_n, M^I_n) represents the number of immigrant females and males in the n-th generation. The classification of its states, relations among its probability generating functions and some inferential results have been established in [9] and some limiting theorems have been proved in [10] and [12].

B) Bisexual process with immigration of mating units.

This bisexual branching model was also introduced in [9] in the following manner:

$$Z_0 = N, \ (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}),$$
$$Z_{n+1} = L(F_{n+1}, M_{n+1}) + I_{n+1} \ , \ n = 0, 1, \dots$$

where $\{I_n\}_{n\geq 1}$ is a sequence of i.i.d. nonnegative, integer-valued random variables independent of $\{(f_{ni}, m_{ni}), \ n = 0, 1, \dots ; \ i = 1, 2, \dots\}$. Intuitively, I_n represents the number of immigrant mating units in the n-th generation. Results concerning with the classification of its states and some relations about its probability generating functions have been provided in [9] and its limiting behaviour has been investigated in [11].

C) Bisexual process with population-size dependent mating.

This branching model has been formally defined in [15] as follows:

$$Z_0 = N, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}),$$
$$Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1}), \ n = 0, 1, \dots$$

where $\{L_k\}_{k\geq 0}$ is a sequence of mating functions. Assuming some conditions about the sequence of mating functions, results concerning with its extinction probability and limiting behaviour have been obtained in [15] and [19].

D) Bisexual process in varying environments.

Introduced in [16], this branching process is defined in the form:

$$Z_0 = N, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}) \ , \tag{2}$$
$$Z_{n+1} = L(F_{n+1}, M_{n+1}) \ , \quad n = 0, 1, \dots$$

where, for every $n = 0, 1, \dots$, $\{(f_{ni}, m_{ni})\}_{i\geq 1}$, is a sequence of i.i.d. nonnegative, integer-valued random variables. Thus, for this process, unlike Daley's bisexual model defined in (1), the offspring probability distribution varies from generation to generation and therefore, it is deduced that $\{(F_n, M_n)\}_{n\geq 1}$ and its associated sequence of mating units $\{Z_n\}_{n\geq 0}$ are Markov chains not necessarily homogeneous. This lack of homogeneity establishes an important difference with the previous bisexual processes.

3. Results concerning bisexual processes in varying environments.

In this section, we shall present a survey of results concerning with the bisexual process in varying environments (BPVE) defined in (2). For details about the proofs of such results we refer the reader to the papers [16]-[18].

A) Probability generating functions and moments.

Firstly, we shall provide some relations among the probability generating functions of the random variables involved in a BPVE. As consequence, some relations among their main moments will be obtained.

Proposition 1. *Given a BPVE it is verified that:*

(i)

$$E[s_1^{f_{n1}} s_2^{m_{n1}}] = \pi_n(\alpha_n s_1 + (1-\alpha_n)s_2), \quad s_1, s_2 \in [0,1], \quad n = 1, 2, \ldots$$

where, for $s \in [0,1]$, $\pi_n(s) := E[s^{t_{n1}}]$ *with* $t_{n1} := f_{n1} + m_{n1}$, *and* $\alpha_n \in (0,1)$ *denotes the probability to obtain a descendant female in generation* n,

(ii)

$$E[s_1^{F_{n+1}} s_2^{M_{n+1}}] = h_n(E[s_1^{f_{n1}} s_2^{m_{n1}}]), \quad s_1, s_2 \in [0,1], \quad n = 0, 1, \ldots$$

where $h_n(s) := E[s^{Z_n}]$, $s \in [0,1]$.

Let us denote by μ_n, Σ_n and Γ_n, Δ_n the mean vector and the covariance matrix of (f_{n1}, m_{n1}) and (F_n, M_n) respectively. From Proposition 1, it is matter of straightforward computation to deduce, for $n = 0, 1, \ldots$, the relations:

(i) $\mu_n = E[t_{n1}]\alpha_n$, $\Sigma_n = \mathrm{Var}[t_{n1}]\alpha_n^t \alpha_n + E[t_{n1}]\alpha_n(1-\alpha_n)\beta^t\beta$,

(ii) $\Gamma_{n+1} = E[Z_n]\mu_n$, $\Delta_{n+1} = E[Z_n]\Sigma_n + \mathrm{Var}[Z_n]\mu_n^t\mu_n$,

where $\alpha_n := (\alpha_n, 1-\alpha_n)$, $\beta := (1, -1)$ and α_n^t, β^t and μ_n^t denote the transpose vectors of α_n, β and μ_n respectively.

Definition 1. *A BPVE is said to be superadditive when its mating function* L *is superadditive, namely, for* $k = 2, 3, \ldots$, *it verifies:*

$$L\left(\sum_{i=1}^k x_i, \sum_{i=1}^k y_i\right) \geq \sum_{i=1}^k L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, \quad i = 1, \ldots, k.$$

Proposition 2. *For a superadditive BPVE it follows, for $s \in [0,1]$ and $n = 0,1,\dots,$ that:*

(i) $h_{n+1}(s) \leq h_n(g_n(s))$,

(ii) $h_{n+1}(s) \leq ((g_0 \circ \cdots \circ g_n)(s))^N$,

where $g_n(s) := E[s^{L(f_{n1},m_{n1})}]$, i.e. the probability generating function associated to the number of mating units originated by the offspring of a mating units in generation n.

B) Extinction probability.

We will now present some necessary and sufficient conditions for the almost sure extinction of a superadditive BPVE.

Definition 2. *For a BPVE we define the mean growth rates per mating unit as:*

$$r_{nj} := j^{-1} E[Z_{n+1} | Z_n = j], \quad n = 0,1,\dots\ ;\ j = 1,2,\dots$$

Proposition 3. *For a superadditive BPVE it is verified, for $n = 0,1,\dots,$ that:*

(i) $r_{n1} = \inf_{j>0} r_{nj}$,

(ii) $r_n := \lim_{j\to\infty} r_{nj}$ *exists and* $r_n = \sup_{j>0} r_{nj}$,

(iii) $r_n < \infty$ *implies that* $E[Z_n] < \infty$.

Sufficient conditions for the almost sure extinction are established in the following result.

Theorem 1. *Consider a superadditive BPVE such that:*

(i) $\limsup_{n\to\infty} r_n < 1$,

(ii) $P(Z_n \to 0 \mid Z_0 = N) + P(Z_n \to \infty \mid Z_0 = N) = 1$,

then $P(Z_n \to 0 \mid Z_0 = N) = 1, \quad N = 1,2,\dots$

Before providing sufficient conditions for the survival of a superadditive BPVA with positive probability, we need to introduce the following conditioned variances per mating unit:

$$\sigma_{nj} = j^{-1} E[\ (Z_{n+1} - j r_{nj})^2 \mid Z_n = j\ ]\ ,\ n = 0,1,\dots\ ;\ j = 1,2,\dots$$

Theorem 2. *Consider a superadditive BPVE such that:*

(i) $\lim_{j\to\infty} r_{kj} = r_k$ *uniformly in* $k = 1, 2, \dots,$

(ii) $\sigma_k := \sup_{j>0} \sigma_{kj} < \infty, \quad k = 0, 1, \dots,$

(iii) $\underline{r} := \liminf_{n\to\infty} r_n > 1$ *and* $\sum_{k=0}^{\infty} (\underline{r} - \delta)^{-k} \sigma_k < \infty$*, for some* $\delta > 0$*,*

then, for some positive integer j_0*, it is verified that*

$$P(Z_n \to 0 \mid Z_0 = N) < 1, \quad N > j_0.$$

Given a BPVE we define its associated asexual process in varying environments, denoted as $\{\widetilde{Z}_n\}_{n\geq 0}$, in the form:

$$\widetilde{Z}_0 = Z_0 = N \ , \quad \widetilde{Z}_{n+1} = \sum_{i=1}^{\widetilde{Z}_n} L(f_{ni}, m_{ni}) \ , \quad n = 0, 1, \dots$$

Using the fact that $P(Z_n \to 0 \mid Z_0 = N) \leq P(\widetilde{Z}_n \to 0 \mid \widetilde{Z}_0 = N)$, $N = 1, 2, \dots$ and applying some results from asexual branching processes in varying environment theory, sufficient conditions for the existence of a positive probability of non-extinction have been obtained in [17]. On the other hand, sufficient conditions for the almost sure extinction of a superadditive BPVE have been derived in [16] by considering the probability generating functions $E[s^{Z_{n+1}} \mid Z_n = j]$, $s \in [0, 1]$, $n = 0, 1, \dots; j = 1, 2, \dots$ and making use of results about fractional linear functions theory.

C) Limiting behaviour.

Finally, we shall provide some results on the limiting behaviour of a superadditive BPVE suitably normed. We will assume that the process starts with a number of mating units N large enough such that $P(Z_n \to \infty \mid Z_0 = N) > 0$. Sufficient conditions for this assumption to hold have been established in [16].

Definition 3. *A sequence of positive constants* $\{B_n\}_{n\geq 0}$ *is said to be a rate of growth for the BPVE if the sequence* $\{B_n^{-1} Z_n\}_{n\geq 0}$ *converges to a finite and nondegenerate at 0 random variable.*

In order to obtain rates of growth for the BPVE, let us consider the mean growth rates introduced in Definition 2. Taking into account that

$$N \prod_{k=0}^{n-1} r_{k1} \leq E[Z_n \mid Z_0 = N] \leq N \prod_{k=0}^{n-1} r_k, \quad n = 0, 1, \dots$$

if we consider the sequences $\{m_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ where $m_0 = c_0 := 1$ and $m_n := \prod_{k=0}^{n-1} r_{k1}$, $c_n := \prod_{k=0}^{n-1} r_k$, $n = 1, 2, \ldots$, it can be expected that, under some conditions, they will be rates of growth. Next, we will study each one of them.

Limiting behaviour of the sequence $\{m_n^{-1}Z_n\}_{n\geq 0}$.

Let us consider the sequence $\{\overline{W}_n\}_{n\geq 0}$ where $\overline{W}_n := m_n^{-1}Z_n$, $n = 0, 1, \ldots$

Theorem 3. *If* $\sum_{k=0}^{\infty}(1 - r_k^{-1}r_{k1}) < \infty$ *then,* $\{\overline{W}_n\}_{n=0}^{\infty}$ *converges almost surely, as* $n \to \infty$*, to a finite and nonnegative random variable* $\overline{W}$*. Moreover it is verified that* $E[\overline{W} \mid Z_0 = N] < \infty$.

This result is derived from martingale convergence theorem, using the fact that $\{\overline{W}_n\}_{n\geq 0}$ is a submartingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$, where $\mathcal{F}_n$ denotes the σ-algebra generated by $Z_0, \ldots, Z_n$ and taking into account that, under the required assumption, $\sup_{n\geq 0} E[\overline{W}_n \mid Z_0 = N] < \infty$.

In order to assure that $\{m_n\}_{n\geq 0}$ is a rate of growth for $\{Z_n\}_{n\geq 0}$ it rests to prove that $\overline{W}$ is non degenerate at 0. For this, additional assumptions will be necessary.

Definition 4. *The associated asexual process* $\{\widetilde{Z}_n\}_{n\geq 0}$ *is said to be uniformly supercritical if there exist constants* $A > 0$ *and* $c > 1$ *such that* $m_j^{-1}m_{n+j} \geq Ac^n$, $j = 1, 2, \ldots; n = 0, 1, \ldots$

Theorem 4. *If* $\{\widetilde{Z}_n\}_{n\geq 0}$ *is uniformly supercritical and each random variable of the sequence* $\{r_{n1}^{-1}L(f_{n1}, m_{n1})\}_{n\geq 0}$ *is stochastically smaller than an integer-valued random variable* X *verifying that* $E[X \log^+ X] < \infty$ *then,* $P(\overline{W} > 0) > 0$.

It can be proved that if, for each $n = 0, 1, \ldots$, the sequence $\{r_{nj}\}_{j\geq 1}$ is nondecreasing then, there exists a nondecreasing function λ_n on $\mathbb{R}^+$ such that $\lambda_n(j) \leq r_{nj}$, $j = 1, 2, \ldots$ A necessary condition for the L^1-convergence of $\{\overline{W}_n\}_{n\geq 0}$ is derived in the following result.

Theorem 5. *Suppose that, for* $n = 0, 1, \ldots$, $\{r_{nj}\}_{j=1}^{\infty}$ *is a nondecreasing sequence. If* $\{\overline{W}_n\}_{n=0}^{\infty}$ *converges in* L^1*, as* $n \to \infty$*, to a random variable* $\overline{W}$ *with* $E[\overline{W}] < \infty$*, then:*

$$\sum_{n=0}^{\infty} \left(1 - \frac{r_{n1}}{\lambda_n(E[Z_n])}\right) < \infty.$$

being λ_n *the function above-mentioned.*

In order to obtain sufficient conditions for the L^1-convergence of $\{\overline{W}_n\}_{n\geq 0}$ we shall introduce, for each $n = 0, 1, \ldots$, the sequence $\{R_{nj}\}_{j\geq 1}$ where

$$R_{nj} := j^{-1} E\left[\, \mid Z_{n+1} - r_{n1} Z_n \mid \mid Z_n = j \,\right], \quad j = 1, 2, \ldots$$

If $\{R_{nj}\}_{j\geq 1}$ is nonincreasing for any $n = 0, 1, \ldots$ then, it can be derived the existence of a nonincreasing function ξ_n on $\mathbb{R}^+$ such that $\xi_n(j) \geq R_{nj}$, $j = 1, 2, \ldots$

Theorem 6. *If $\{R_{nj}\}_{j\geq 1}$ is nonincreasing for any $n = 0, 1, \ldots$, $\sum_{k=0}^{\infty}(1 - r_k^{-1} r_{k1}) < \infty$ and $\sum_{n=0}^{\infty} r_{n1}^{-1} \xi_n(N m_n) < \infty$ then, it is verified that $\{\overline{W}_n\}_{n=0}^{\infty}$ converges in L^1, as $n \to \infty$, to a nondegenerate at 0 random variable $\overline{W}$ such that $E[\overline{W} \mid Z_0 = N] < \infty$.*

Limiting behaviour of the sequence $\{c_n^{-1} Z_n\}_{n\geq 0}$.

Now, let us consider the sequence $\{\widehat{W}_n\}_{n\geq 0}$ with $\widehat{W}_n := c_n^{-1} Z_n$, where recall that $c_0 = 1$, $c_n = \prod_{k=0}^{n-1} r_k$, $n = 1, 2, \ldots$ and $r_k = \sup_{j>0} r_{kj}$, $k = 0, 1, \ldots$. It will be assumed throughout, that for $n = 0, 1, \ldots$, $\{r_{nj}\}_{j\geq 1}$ is a nondecreasing sequence and therefore, it is guaranteed the existence of the function λ_n. It is not difficult to verify that $\{\widehat{W}_n\}_{n\geq 0}$ is a supermartingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$ and therefore, by martingale convergence theorem, it is derived the following result.

Theorem 7. *$\{\widehat{W}_n\}_{n\geq 0}$ converges almost surely, as $n \to \infty$, to a finite and nonnegative random variable $\widehat{W}$.*

In order to establish that $\{c_n\}_{n\geq 0}$ is a rate of growth for the BPVE it rests to determine some conditions guaranteeing $P(\widehat{W} > 0) > 0$. For this, we introduce the sequences $\{\sigma_{nj}\}_{j\geq 1}$, $n = 0, 1, \ldots$ where:

$$\sigma_{nj} := j^{-2}\, \mathrm{Var}[Z_{n+1} \mid Z_n = j], \quad j = 1, 2, \ldots$$

and we will suppose that, for each $n = 0, 1, \ldots$, there exists σ_n such that $\sigma_{nj} \leq \sigma_n$, $j = 1, 2, \ldots$. It is clear that

$$\sigma_{nj} = d_{nj} - r_{nj}^2 \quad \text{where} \quad d_{nj} := j^{-2} E[Z_{n+1}^2 \mid Z_n = j]$$

Theorem 8. *If $\sum_{n=0}^{\infty} r_n^{-2} \sigma_n < \infty$ and $\sum_{n=0}^{\infty} \left(1 - r_n^{-1} \lambda_n(E[Z_n])\right) < \infty$ then, $\{\widehat{W}_n\}_{n\geq 0}$ converges in L^1, as $n \to \infty$, to the random variable $\widehat{W}$. Moreover $P(\widehat{W} > 0) > 0$.*

Let us denote by:

$$\varepsilon_{nj} := r_n - r_{nj} \ , \quad j = 1, 2, \ldots \ ; \quad n = 0, 1, \ldots$$

From Proposition 3(ii), $\lim_{j\to\infty} \varepsilon_{nj} = 0$ and $\{\varepsilon_{nj}\}_{j\geq 1}$ is a nonincreasing sequence, so it can be deduced the existence of a nonincreasing function τ_n on $\mathbb{R}^+$ verifying that $\tau_n(j) \geq \varepsilon_{nj}$. Next theorem establishes sufficient conditions for the L^2-convergence of $\{\widehat{W}_n\}_{n\geq 0}$.

Theorem 9. *If* $\sum_{n=0}^{\infty} r_n^{-2} \sigma_n < \infty$ *and* $\sum_{n=0}^{\infty} r_n^{-1} \tau_n(E[Z_n]) < \infty$ *then,* $\{\widehat{W}_n\}_{n=0}^{\infty}$ *converges in* L^2*, as* $n \to \infty$*, to a non degenerate at 0 random variable.*

Acknowledgements.
The authors would like to thank to the referee for a thorough reading of the paper and the constructive suggestions which have improved it.

REFERENCES

[1] G. Alsmeyer, U. Rösler, The bisexual Galton–Watson process with promiscuous mating: extinction probabilities in the supercritical case. *Ann. Appl. Probab.* **6** (1996), 922–939.

[2] J.H. Bagley, On the asymptotic properties of a supercritical bisexual branching process. *J. Appl. Probab.* **23** (1986), 820–826.

[3] F.T. Bruss, A note on extinction criteria for bisexual Galton–Watson processes. *J. Appl. Probab.* **21** (1984), 915–919.

[4] D.J. Daley, Extinction conditions for certain bisexual Galton–Watson branching processes. *Z. Wahrsch. Verw. Gebiete.* **9** (1968), 315–322.

[5] D.J. Daley, D.M. Hull, J.M. Taylor, Bisexual Galton–Watson branching processes with superadditive mating functions. *J. Appl. Probab.* **23** (1986), 585–600.

[6] A. González-Fragoso, Ratio estimation for the offspring means of a bisexual Galton–Watson branching process. *J. Inter-American Stat. Inst.* **47** (1995), 17–36.

[7] M. González, M. Molina, On the limit behaviour of a superadditive bisexual Galton–Watson branching process. *J. Appl. Probab.* **33** (1996), 960–967.

[8] M. González, M. Molina, On the L^2–convergence of a superadditive bisexual Galton–Watson branching process. *J. Appl. Probab.* **34** (1997), 575–582.

[9] M. González, M. Molina, M. Mota, A note on bisexual branching models with immigration. *J. Inter-American Stat. Inst.* **49** (1999), 81–107.

[10] M. González, M. Molina, M. Mota, Limit behavior for a subcritical bisexual Galton–Watson branching process with immigration. *Statist. Prob. Lett.* **49** (2000), 19–24.

[11] M. González, M. Molina, M. Mota, On the limit behaviour of a supercritical bisexual Galton–Watson branching process with immigration of mating units. *Stochastic Anal. Appl.,* **19** (2001b), 933–943.

[12] M. González, M. Molina, M. Mota, Bisexual Galton–Watson process with immigration of females and males. Asymptotic behaviour. *Markov Process. Related Fields,* **8** (2002), 651–663.

[13] D.M. Hull, A necessary condition for extinction in those bisexual Galton–Watson branching processes governed by superadditive mating functions. *J. Appl. Probab.* **19** (1982), 847–850.

[14] M. Molina, M. González, M. Mota, Bayesian inference for bisexual Galton–Watson processes. *Comm. Statist. Theory Methods,* **27** (1998), 1055–1070.

[15] M. Molina, M. Mota, A. Ramos, Bisexual Galton–Watson branching process with population-size-dependent mating. *J. Appl. Probab.,* **39** (2002), 479–490.

[16] M. Molina, M. Mota, A. Ramos, Bisexual Galton–Watson branching process in varying environments. *Stochastic Anal. Appl.,* **21** (2003), 1353–1367.

[17] M. Molina, M. Mota, A. Ramos, On the extinction probability for bisexual branching processes in varying environments. *Serdica Math. J.,* **29** (2003), 187–194.

[18] M. Molina, M. Mota, A. Ramos, Limiting behaviour for superadditive bisexual Galton–Watson processes in varying environments. *Test.* **13** (2004). (to appear)

[19] M. Molina, M. Mota and A. Ramos, Limit behaviour for a supercritical bisexual Galton–Watson branching process with population-size-dependent mating. *Stochastic Process. Appl.* (to appear)

Manuel Molina
Faculty of Sciences
University of Extremadura
06071 Badajoz. Spain.
e-mail: mmolina@unex.es

Manuel Mota
Faculty of Sciences
University of Extremadura
06071 Badajoz. Spain.
e-mail: mota@unex.es

Alfonso Ramos
Faculty of Sciences
University of Extremadura
06071 Badajoz. Spain.
e-mail: aramos@unex.es