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Pliska Stud. Math. Bulgar. **16** (2004), 137-145

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LIMITING DISTRIBUTIONS FOR LIFETIMES IN ALTERNATING RENEWAL PROCESSES

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The spent life time and the residual life time are well investigated characteristics of an ordinary renewal process. In the present paper a generalization of these lifetime processes associated with an alternating renewal process is considered. Limiting distributions are presented in the case of infinite mean renewal periods.

1. Introduction and definitions

Two important characteristics of renewal processes are the so called "spent life time" and "residual life time". For the ordinary renewal processes these characteristics are widely studied in the literature (see e.g. [1], [3], [4], [9]). In the recent papers [7], [8] some processes of this kind were defined and limit theorems were proved in the more general case of alternating renewal processes. The aim of the present paper is to represent in a systematic way the main results obtained up to now in this area.

On the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, let

$$X = \{X_i : i = 1, 2, \dots\} \text{ and } T = \{T_i : i = 1, 2, \dots\}$$

[1]The paper is supported by NFSI-Bulgaria, Grant No. MM-1101/2001.
2000 *Mathematics Subject Classification:* 60K05
Key words: Alternating Renewal Processes, Spent Working Time, Spent Waiting Time, Residual Working Time, Limiting Distributions, Infinite Mean Renewal Periods

be two independent sets of independent, non-negative random variables (r.v.) with distribution functions (d.f.)

$$A(t) = \mathbf{P}\{X_i \le t\} \text{ and } F(t) = \mathbf{P}\{T_i \le t\}$$

respectively.

Denote by $Y_i = X_i + T_i$, $i = 1, 2, \dots$, the sequence of independent, identically distributed, non-negative r.v. with d.f.

$$G(t) = \mathbf{P}\{Y_i \le t\} = (A * F)(t) \equiv \int_0^t A(t-u)F(du).$$

Define the ordinary renewal process

$$S_0 = 0, \quad S_{n+1} = S_n + Y_{n+1}, \quad n = 0, 1, 2, \dots ,$$

and the corresponding counting process

$$N(t) = \max\{n : S_n \le t\}.$$

The moments S_n are usually interpreted as times when an element in a system is replaced instantly by another one of the same type. Two well known processes related to this sequence are the spent life time $(t - S_{N(t)}, t \ge 0)$ and the residual life time $(S_{N(t)+1} - t, t \ge 0)$. These processes are widely studied in the literature. (See e.g. [1], [2], [4], [5], [6], [10]).

In our setting the sequence

$$\{S_n, S'_{n+1}\}_{n=0}^{\infty}$$

where

$$S'_{n+1} = S_n + X_{n+1}, \quad n = 0, 1, 2, \dots ,$$

is called alternating renewal process.

In the terms of reliability theory one can interpret X_i as the time for the installation or the repairing time of the $i-$th element in a system and T_i as the working time of the same element. So, in the sequence $\{S_n, S'_{n+1}\}$ there are two types of renewal events: S_n the beginning of the installation and $S'_{n+1} = S_n + X_{n+1}$ the beginning of the work of the $n-$th element.

For the alternating renewal process $\{S_n, S'_{n+1}\}_{n=0}^{\infty}$ the following random variables can be defined:

i) the spent working time

$$\sigma(t) = t - S'_{N(t)+1} = t - S_{N(t)} - X_{N(t)+1}, \quad t \geq 0;$$

ii) the residual working time

$$\tau(t) = \min\{S_{N(t)+1} - t, T_{N(t)+1}\}, \quad t \geq 0;$$

iii) the spent waiting time

$$\omega(t) = \min\{t - S_{N(t)}, X_{N(t)+1}\}, \quad t \geq 0.$$

It is clear from the definitions that the following inequalities hold almost surely:

$$-\infty < \sigma(t) \leq t, \quad t \geq 0,$$

$$0 \leq \tau(t) < \infty, \quad t \geq 0,$$

$$0 \leq \omega(t) \leq t, \quad t \geq 0.$$

Obviously, one obtains a classical renewal process if all X_i are identically equal to zero. In this case $\sigma(t) = t - S_{N(t)} \geq 0$ a.s. is the spent life time, $\tau(t) = S_{N(t)+1} - t$ a.s. is the residual life time, and $\omega(t) = 0$ a.s., i.e. there is no waiting time.

For the alternating renewal processes, $\sigma(t)$ can take both negative and positive values:

$$\sigma(t) = \sigma^+(t) - \sigma^-(t),$$

where $\sigma^+(t) = \max\{\sigma(t), 0\}$ and $\sigma^+(t) = \max\{-\sigma(t), 0\}$.

Note that $\sigma^+(t)$ can be interpreted as the spent lifetime and $\sigma^-(t)$—as the residual waiting period.

The limiting behaviour of the spent and residual life times in an ordinary renewal process is well studied in both cases when the interarival times T_i have finite and infinite mathematical expectations. (See e.g. [1], [2], [4], [6], [9], [10]). In the recent papers [7] and [8] we investigated the limiting behaviour of the life time processes defined for an alternating renewal process in the case when at least one of the mathematical expectations EX_i or ET_i is infinite. In the present paper the results of those papers together with new ones are given without proof. The rest of the paper is organized as follows. Section 2 contains the basic conditions and equations for interrarival times. The results and comments are stated in Section 3.

2. Basic Conditions and Equations

Further we will suppose some of the following basic conditions:

(1) $$A(0) = F(0) = 0, \quad A(x) \text{ and } F(x) \text{ are non-lattice d.f.;}$$

(2) $$m_A = \mathbf{E}X_i < \infty;$$

or

(3) $$\mathbf{E}X_i = \infty,\ \bar{A}(t) = 1 - A(t) \sim t^{-\alpha} L_A(t),\ t \to \infty, \quad \alpha \in (0,1),$$

where $L_A(.)$ is a s.v.f., and for each $h > 0$ fixed $A(t) - A(t-h) = O(1/t), \ \ t > 0$;

(4) $$m_F = \mathbf{E}T_i < \infty;$$

or

(5) $$\mathbf{E}T_i = \infty, \quad \bar{F}(t) = 1 - F(t) \sim t^{-\beta} L_F(t), \quad t \to \infty, \quad \beta \in (0,1),$$

where $L_F(.)$ is a s.v.f.

Finally it is assumed that

(6) $$\lim_{t\to\infty} \frac{\bar{A}(t)}{\bar{F}(t)} = c, \quad 0 \le c \le \infty.$$

Note that a sufficient condition for $A(t) - A(t-h) = O(1/t), \ \ t > 0$, is that $L_A(t)$ to be non-decreasing in $t \in [0, \infty)$.

In the papers [7] and [8] the following inequalities are required $1/2 < \alpha, \beta \le 1$. It appears that these constrains are not necessary in all the cases. So they will be explicitly stated only in the theorems where they are needed.

The following notations are needed:

$$U(t) = \sum_{n=0}^{\infty} \mathbf{P}\{S_n \le t\} = \sum_{n=0}^{\infty} G^{*n}(t)$$

and

$$V(t) = \sum_{n=0}^{\infty} \mathbf{P}\{S'_{n+1} \le t\} = \sum_{n=0}^{\infty} \mathbf{P}\{S_n + X_{n+1} \le t\} = \int_0^t A(t-u)U(du).$$

The first theorem is auxiliary. It provides the functional equations for the distributions of $\sigma(t), \tau(t)$ and $\omega(t)$, which are the basic tools for the investigation.

Theorem 1. *i). The distribution function of $\sigma(t)$ is*

$$\begin{aligned} &\mathbf{P}\{\sigma(t) \le x\} \\ &= \int_0^t \bar{A}(\max(t, t-x) - u) U(du) \\ &+ \int_{\min(t-x,t)}^t \bar{F}(t-u) V(du), \quad -\infty < x \le t. \end{aligned} \tag{7}$$

ii). The survival function of $\tau(t)$ is

$$\mathbf{P}\{\tau(t) \ge x\} = \int_0^t \bar{F}(t+x-u) V(du) + \bar{F}(x)\mathbf{P}\{\sigma(t) < 0\}, \quad x \ge 0. \tag{8}$$

iii). The survival function of $\omega(t)$ is

$$\begin{aligned} &\mathbf{P}\{\omega(t) > x\} \\ &= \int_x^t \left[\int_0^{t-v} \bar{F}(t-v-u) U(du)\right] A(dv) + \int_0^{t-x} \bar{A}(t-u) U(du), \ 0 \le x \le t. \end{aligned} \tag{9}$$

iv). The joint distribution of $(\sigma(t), \tau(t))$ has the following representation:

$$\begin{aligned} &\mathbf{P}\{\sigma(t) \le x, \tau(t) > y\} \\ &= \bar{F}(y) \int_0^t \bar{A}(\max(t, t-x) - u) U(du) \\ &+ \int_{\min(t,t-x)}^t \bar{F}(t+y-u) V(du), \quad -\infty < x \le t, \quad y > 0. \end{aligned} \tag{10}$$

3. Statement of the limiting results

The limit theorems stated below depends on the limit (6). So, provided that $c = \infty$, the time for installation X_i has a heavier tail than the time for work T_i, and the condition $0 \le c < \infty$ gives the opposite.

The next theorem establishes proper limiting distributions under appropriate normalization for $\sigma(t)$ in the case $c = \infty$. The first assertion improves Theorem 2.1 in [7] which is proved under the restriction $\alpha, \beta > 1/2$. The assertion iii) is the result of Theorem 2.2(ii) in [7]. The limiting distribution (11) is new.

Theorem 2. *Assume (1), (3), either (4) or (5), and (6) with $c = \infty$.*
i) There exist the limits

$$\lim_{t\to\infty} \mathbf{P}\{\sigma(t) \ge 0\} = 0, \quad \lim_{t\to\infty} \mathbf{P}\{\sigma(t) < 0\} = 1.$$

ii) For $-\infty < x < \infty$,

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \leq x\} = \frac{\sin \pi\alpha}{\pi} \int_0^{1/\max\{1,1-x\}} (1-u)^{-\alpha} u^{\alpha-1} du. \tag{11}$$

iii) If (5) with $1/2 < \beta < 1$ *holds then*

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \leq x | \sigma(t) \geq 0\} = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^x u^{\alpha-1}(1-u)^{-\beta} du.$$

The limiting distributions given in the following two theorems are new. They describe the behavior of $\tau(t), \omega(t)$, and that of the vector $(\sigma(t), \tau(t))$.

Theorem 3. *Assume (1), (3), either (4) or (5), and (6) with* $c = \infty$.
i) For $x > 0$,

$$\lim_{t\to\infty} \mathbf{P}\{\tau(t) \leq x\} = F(x). \tag{12}$$

ii) For $-\infty < x < \infty, \quad y > 0$,

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \leq x, \tau(t) > y\} = \bar{F}(y)\frac{\sin \pi\alpha}{\pi} \int_0^{1/\max\{1,1-x\}} (1-u)^{-\alpha} u^{\alpha-1} du. \tag{13}$$

Theorem 4. *Assume (1), (3) with* $\alpha \in (1/2, 1)$, *and (6) with* $c = \infty$. *For* $0 < x < 1$,

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\omega(t)}{t} \leq x\} = \frac{\sin \pi\alpha}{\pi} \int_0^x u^{-\alpha}(1-u)^{\alpha-1} du. \tag{14}$$

It is evident that in the case when the installation time X_i is "longer" then the time for work T_i the limiting distributions depend on the tail parameter α of X_i. Only the conditional distribution in Theorem 2 depends on both tails. It turns out that the restriction $\alpha, \beta > 1/2$ is not necessary for most of the results in the above theorems. From the first part of Theorem 3 it becomes clear that $\sigma(t)$ and $\tau(t)$ are asymptotically independent. The joint limit distribution in (13) is the product of the limit distributions of $\sigma(t)$ and $\tau(t)$ given by (11) and (12) respectively.

The following theorems give the limiting distributions of $\sigma(t), \tau(t)$ and $\omega(t)$ for the case $0 \leq c < \infty$.

Theorem 5. *Assume (1), (5), and (6) with* $0 \le c < \infty$.
i) The following limits exist

$$(15)\qquad \left.\begin{aligned} \lim_{t\to\infty} P\{\sigma(t) < 0\} &= \frac{c}{1+c}\ , \\ \lim_{t\to\infty} P\{\sigma(t) \ge 0\} &= \frac{1}{1+c}\ . \end{aligned}\right\}$$

ii) If $\beta \in (1/2, 1)$ *then for* $\infty < x < \infty$

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \le x\} = \frac{c \sin \pi\beta}{\pi(1+c)} \int_0^{1/\max\{1,1-x\}} (1-u)^{-\beta} u^{\beta-1} du$$

$$(16)\qquad + \frac{\sin \pi\beta}{\pi(1+c)} \int_0^{g(x)} (1-u)^{\beta-1} u^{-\beta} du,$$

where

$$g(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

iii) If $\beta \in (1/2, 1)$ *then for* $0 \le x \le 1$

$$(17)\qquad \lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \le x\} = \frac{c}{1+c} + \frac{c \sin \pi\beta}{\pi(1+c)} \int_0^x (1-u)^{-\beta} u^{\beta-1} du$$

and

$$(18)\qquad \lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \le x | \sigma(t) \ge 0\} = \frac{\sin \pi\beta}{\pi} \int_0^x (1-u)^{-\beta} u^{\beta-1} du$$

The first assertion of Theorem 5 improves Theorem 2.3 by removing the condition $\alpha, \beta > 1/2$. The third assertion is the result of Theorem 2.3(ii) in [7]. The second one is new.

The last three theorems state the limiting behavior of $\tau(t), \omega(t)$ and the vector $(\sigma(t), \tau(t))$ in the case when the working time T_i is "longer" then the time for installation X_i. All the results are new.

Theorem 6. *Assume (1), (5), and (6) with* $0 \le c < \infty$. *Then for* $y > 0$

$$\lim_{t\to\infty} \mathbf{P}\{\tau(t) \le y\} = \frac{c}{1+c} F(y),$$

and

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\tau(t)}{t} \le y\} = 1 - \frac{\sin \pi\beta}{\pi(1+c)} \int_0^{1/(1+y)} (1-u)^{-\beta} u^{\beta-1} du.$$

Note that there are two singular limiting distributions for $\tau(t)$ under different normalization. The first one has an atom at infinity equal to $1/(1+c)$ over which is the support of the second distribution with an atom at zero equal to $c/(1+c)$.

Theorem 7. *Assume (1), (5) with* $\beta \in (1/2, 1)$, *and (6) with* $0 \le c < \infty$. *Then for* $x \in (0, 1)$

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\omega(t)}{t} \le x\} = \frac{1}{1+c} + \frac{c \sin \pi\beta}{\pi(1+c)} \int_0^{x} u^{-\beta} (1-u)^{\beta-1} du.$$

Theorem 8. *Assume (1), (5) with* $\beta \in (1/2, 1)$, *and (6) with* $0 \le c < \infty$. *Then for* $y \ge 0, x \in \mathbf{R}$

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \le x, \tau(t) > y\} = \bar{F}(y) \frac{c \sin \pi\beta}{\pi(1+c)} \int_0^{1/\max\{1,1-x\}} (1-u)^{-\beta} u^{\beta-1} du$$

$$+ \frac{\sin \pi\beta}{\pi(1+c)} \int_0^{g(x)} (1-u)^{\beta-1} u^{-\beta} du,$$

and

$$\lim_{t\to\infty} \mathbf{P}\{\frac{\sigma(t)}{t} \le x, \frac{\tau(t)}{t} > y\} = \frac{\sin \pi\beta}{\pi(1+c)} \int_{\frac{1-g(x)}{1+y}}^{\frac{1}{1+y}} (1-u)^{-\beta} u^{\beta-1} du.$$

The only case which is not considered here is the boundary case when $\alpha = 1$ or $\beta = 1$ and the corresponding mathematical expectations are infinite. This case is under consideration and the results will be published later.

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