PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

For further information on
Pliska Studia Mathematica Bulgarica
visit the website of the journal http://www.math.bas.bg/~pliska/
or contact: Editorial Office
Pliska Studia Mathematica Bulgarica
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: pliska@math.bas.bg

Pliska Stud. Math. Bulgar. **16** (2004), 43-54

PLISKA
STUDIA MATHEMATICA
BULGARICA

RECENT RESULTS FOR SUPERCRITICAL CONTROLLED BRANCHING PROCESSES WITH CONTROL RANDOM FUNCTIONS

Miguel González, Manuel Molina, Inés del Puerto[1]

In this paper we are concerned with the controlled branching processes with random control function. Recently, we have considered them under the condition of asymptotically linear growth of the mathematical expectations associated to the random control variables. We present a review of the main results obtained until now, mainly, in the supercritical case.

1. Introduction

Sevast'yanov and Zubkov (1974) introduced the controlled branching process (CBP) as a model defined by the iterative relation:

$$Z_0 = N, \quad Z_{n+1} = \sum_{j=1}^{\phi(Z_n)} X_{nj} \qquad n = 0, 1, \ldots \tag{1}$$

where the empty sum is defined to be 0, N is a positive integer, $\{X_{nj} : n = 0, 1, \ldots, j = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.), integer valued random variables and the control function ϕ, with range

[1]Research supported by the Ministerio de Ciencia y Tecnología and the FEDER through the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica, grant BFM2003-06074.

2000 *Mathematics Subject Classification:* 60J80, 60F05

Key words: Controlled branching process. Extinction problem. Limiting behaviour.

and domain $\mathbb{R}^+$, is assumed to be integer valued for integer valued arguments. Intuitively, X_{nj} is interpreted as the number of individuals originated by the j-th individual in the n-th generation and Z_n represents the population size in the n-th generation. Each individual generates, independently of all others and with identical probability distribution, new individuals but, in contrast with the standard Galton-Watson process, the population size in the $(n+1)$-th generation is controlled through the function ϕ. For the CBP, some questions about the almost sure extinction, the limit behaviour and the estimation of its main parameters have been studied (Bagley (1986), Molina et al. (1998), Zubkov (1974) and González et al. (2003)) and various extensions from it have been developed: controlled processes with multiple control function (Sevast'yanov and Zubkov (1974), Zubkov (1974)), multitype controlled processes (Sevast'yanov and Zubkov (1974)), controlled processes in a random environment (Holzheimer (1984)), controlled processes with random control function (Bruss (1980), Dion and Essebbar (1995), Nakagawa (1994), Yanev (1975)), controlled processes with multiple random control function (Yanev and Yanev (1989)) or controlled processes with random control function in a random environment (Yanev and Yanev (1990)).

We study a class of controlled branching processes introduced by Yanev (1975) in which the control functions are random variables. Recently, González et al. (2002, 2003a) have considered them under the condition of asymptotically linear growth of the mathematical expectations of the control variables. In this paper we review recent results for both limiting behaviour and extinction in the supercritical case. In Section 2, we give the description of the CBP model. Section 3 is devoted to the extinction problem and finally, in Section 4, the limiting behaviour of the process, suitably normed, is studied.

2. The probability model

Yanev (1975) defined the CBP with random control function as follows: On the same space, consider two independent sets of non negative, integer valued random variables $\{X_{nj} : n = 0, 1, \dots ; j = 1, 2, \dots\}$ and $\{\phi_n(k) : n, k = 0, 1, \dots\}$. The variables X_{nj} are i.i.d. with probability law $\{p_k\}_{k\geq 0}$, $p_k := P(X_{01} = k)$, $k = 0, 1, \dots$, known as the offspring probability distribution.

For $n = 0, 1, \dots$, $\{\phi_n(k)\}_{k\geq 0}$ are independent stochastic processes with identical one dimensional probability distributions, i.e. $P(\phi_n(k) = j) = P_k(j)$, $n, j, k = 0, 1, \dots$ The CBP with random control function is then defined by:

$$Z_0 = N, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj} \quad n = 0, 1, \dots$$

with the empty sum defined to be 0. The intuitively interpretation of the random variables X_{nj} and Z_n is the same as for the CBP (1) introduced by Sevast'yanov and Zubkov (1974), but now, if in a certain generation n there are k individuals, i.e. $Z_n = k$, then $\phi_n(k)$, identically distributed for each n, controls in the process as follows. When $\phi_n(k) = j$, an event that happens with probability $P_k(j)$, j individuals will take part in the reproduction process that will determinate Z_{n+1}, i.e. the total number of individuals making up the $n+1$-th generation. Hence, this is a branching model that, in our opinion, can describe reasonably the probabilistic evolution of populations in which, for various reasons of an environmental, social or other nature, there exits a random mechanism for the number of progenitors in each generation.

Taking into account the properties of this model, it is easy to verify that the CBP with random control function is a homogeneous Markov chain. Moreover, from now on, we will consider CBPs with random control function such that $P_0(0) = 1$, i.e. 0 is an absorbent state, and at least one of the following conditions

$$(i) \quad p_0 > 0 \quad \text{or} \quad (ii) \quad P_k(0) > 0, \quad k = 1, 2, \ldots$$

holds.

Under these assumptions, Yanev (1975) proved that the positive states are transient and, as a consequence, the classical extinction-explosion duality in branching processes theory, namely $P(Z_n \to 0) + P(Z_n \to \infty) = 1$, is verified.

Let us denote $m := E[X_{01}]$, $\sigma^2 := Var[X_{01}]$, $\varepsilon(k) := E[\phi_0(k)]$ and $\nu^2(k) := Var[\phi_0(k)]$, $k = 0, 1, \ldots$, the mean and the variance of the offspring probability distribution and the control random variables, respectively. Assume all of them to be finite.

3. Extinction problem

Introduce the notation: $q_N := P(Z_n \to 0 | Z_0 = N)$, $N = 1, 2, \ldots$, i.e. the extinction probability when initially there are N individuals in the population. All the proofs of the results, presented in this section, can be found in González et al. (2002).

In order to study the extinction problem for $\{Z_n\}_{n \geq 0}$ we introduce the following quantity, for $k = 1, 2, \ldots$,

$$\mu(k) := k^{-1} E[Z_{n+1} | Z_n = k] = k^{-1} \varepsilon(k) m.$$

For each k, $\mu(k)$ represents the mean growth rate. Intuitively, it can be interpreted as the average offspring per individual for generation of size k.

First we state a sufficient condition for the almost sure extinction of $\{Z_n\}_{n\geq 0}$. In complete analogy with the standard Galton–Watson process (GWP) situation, we shall obtain that the almost sure extinction will occur if the mean growth rates, defined above, are all less than or equal to 1.

Theorem 1. *If $\mu(k) \leq 1$, for all $k = 1, 2, \ldots$, then $q_N = 1$ for all $N \geq 1$.*

The following result shows that, in certain situations, it is possible to weaken the condition of Theorem 1.

Theorem 2. *If*

$$\limsup_{k\to\infty} \mu(k) < 1$$

holds then $q_N = 1$ for all $N \geq 1$.

Based on Theorem 2, one can expect that if $\liminf_{k\to\infty} \mu(k) > 1$, then there exists a positive probability of non-extinction. The following example shows that, in general, this statement is not true.

Example 1. Let $0 < \epsilon < 1$. We consider a CBP with random control function such that $\phi_0(0) = 0$ a.s. and for each $k = 1, 2, \ldots$,

$$\begin{aligned} P_k(0) &= \epsilon^{1/k} \\ P_k(k[(1-\epsilon^{1/k})^{-1}]) &= 1 - \epsilon^{1/k} \end{aligned}$$

where $[x]$ denotes the greatest integer less than or equal to x.
Let us study the behaviour of the sequence $\{k^{-1}\varepsilon(k)\}_{k\geq 1}$,

$$\varepsilon(k) = E[\phi_0(k)] = k[(1-\epsilon^{1/k})^{-1}](1-\epsilon^{1/k})$$

Let us prove that $\lim_{k\to\infty} k^{-1}\varepsilon(k) = 1$. Indeed, it is deduced using the inequalities

$$(1-\epsilon^{1/k})\left(\frac{1}{1-\epsilon^{1/k}} - 1\right) \leq (1-\epsilon^{1/k})\left[\frac{1}{1-\epsilon^{1/k}}\right] < (1-\epsilon^{1/k})\left(\frac{1}{1-\epsilon^{1/k}} + 1\right)$$

and letting $k \to \infty$.
We now prove by induction that

$$P[Z_n > 0 \mid Z_0 = N] < (1-\epsilon)^n, \quad n = 1, 2, \ldots \tag{2}$$

First, note that, for $n = 0, 1, \ldots$

$$P[Z_{n+1} > 0 \mid Z_n = k] = 1 - P\left[\sum_{i=1}^{\phi_n(k)} X_{ni} = 0\right] \leq 1 - P[\phi_0(k) = 0] = 1 - \epsilon^{1/k}$$

Let us prove (2) for $n = 1$,

$$P[Z_1 > 0 \mid Z_0 = N] \le 1 - \epsilon^{1/N} < 1 - \epsilon$$

Suppose that (2) is true for $n-1$. We have that

$$\begin{aligned} P[Z_n > 0 \mid Z_0 = N] &= \sum_{j=1}^{\infty} P[Z_n > 0 \mid Z_{n-1} = j] P[Z_{n-1} = j \mid Z_0 = N] \\ &\le \sum_{j=1}^{\infty} (1 - \epsilon^{1/j}) P[Z_{n-1} = j \mid Z_0 = N] < (1-\epsilon)(1-\epsilon)^{n-1} \\ &= (1-\epsilon)^n \end{aligned}$$

Therefore choosing a reproduction law such that its mean $m > 1$, one will have that $\liminf_{k\to\infty} \mu(k) > 1$ but, however, from (2) it follows that $q_N = 1$ for all $N \ge 1$.

The example above shows that $\liminf_{k\to\infty} \mu(k) > 1$ does not guarantee a positive probability of non–extinction, i.e., $q_N < 1$. In the theorem below we obtain $q_N < 1$ under some additional conditions upon the first two moments of the control functions. For simplicity, from now on, we denote, for $k \ge 1$, $\tau_k := k^{-1}\varepsilon(k)$ and $\sigma_k := k^{-1}\nu^2(k)$.

Theorem 3. *Assume that*

1. $\{\tau_k\}_{k\ge1}$ *and* $\{\sigma_k\}_{k\ge1}$ *are bounded sequences,*

2. $$\liminf_{k\to\infty} \mu(k) > 1.$$

Then there exists an $N_0 \ge 1$ *such that* $q_N < 1$, *for all* $N \ge N_0$.

Remark. Note that if all non-null states communicate and it is verified that the process starting with a large enough number of individuals does not die out, then for any initial number of progenitors there is a positive probability of non-extinction. Indeed, suppose that there exists an $N_0 \in \mathbb{N}$ such that $q_N < 1$ for all $N \ge N_0$. Let i be a non–null state such that $i < N_0$, let us prove that $q_i < 1$. Denote $p_{ij}^{(n)} := P(Z_{m+n} = j \mid Z_m = i)$. Taking into account that all non–null states are communicating, there exists some $m > 0$ such that

$$p_{iN_0}^{(m)} = P[Z_m = N_0 \mid Z_0 = i] > 0$$

For $n > m$,

$$p_{i0}^{(n)} = \sum_{k=0}^{\infty} p_{k0}^{(n-m)} p_{ik}^{(m)} = p_{i0}^{(m)} + \sum_{k=1}^{\infty} p_{k0}^{(n-m)} p_{ik}^{(m)}$$

Taking the limit as $n \to \infty$,

$$\begin{aligned} q_i &= p_{i0}^{(m)} + \sum_{k=1}^{\infty} q_k p_{ik}^{(m)} \leq p_{i0}^{(m)} + \sum_{k=1}^{N_0-1} p_{ik}^{(m)} + \sum_{k=N_0}^{\infty} q_k p_{ik}^{(m)} \\ &< p_{i0}^{(m)} + \sum_{k=1}^{N_0-1} p_{ik}^{(m)} + \sum_{k=N_0}^{\infty} p_{ik}^{(m)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} = 1 \end{aligned}$$

The last inequality is true without more ado than using the hypothesis $q_N < 1$ for all $N \geq N_0$ and $p_{iN_0}^{(m)} > 0$.

Remark. Let us show that the control functions in the above example does not verify conditions imposed in Theorem 1. Indeed,

$$\begin{aligned} \nu^2(k) &= E[\phi_0(k)^2] - (\varepsilon(k))^2 \\ &= k^2[(1-\epsilon^{1/k})^{-1}]^2(1-\epsilon^{1/k}) - k^2[(1-\epsilon^{1/k})^{-1}]^2(1-\epsilon^{1/k})^2 \\ &= k^2[(1-\epsilon^{1/k})^{-1}]^2(1-\epsilon^{1/k})\epsilon^{1/k} \end{aligned}$$

Taking into account that $\lim_{k\to\infty} \epsilon^{1/k} = 1$, $\lim_{k\to\infty}(1-\epsilon^{1/k})[(1-\epsilon^{1/k})^{-1}] = 1$ and $\lim_{k\to\infty} k[(1-\epsilon^{1/k})^{-1}] = \infty$, it follows that

$$\lim_{k\to\infty} \sigma_k = \infty.$$

Remark. It can be deduced from, the proof of Theorem 3, that (i) in such a theorem can be replaced by the condition

$$E[|Z_{n+1} - m\varepsilon(Z_n)|^{1+\delta} \mid Z_n = k] = O(k^{\delta}), \text{ for some } \delta \geq 1. \tag{3}$$

Obviously, (3) may be difficult to check in a practical situation. Hence, it is interesting to look for sufficient conditions, easy to verify, which guarantee that the assumption about the $(1+\delta)$-th conditional absolute moment of $Z_{n+1} - m\varepsilon(Z_n) =: \xi_{n+1}$ holds. In this sense, for instance, using the fact that $|a+b|^r \leq C_r(|a|^r + |b|^r)$, $r > 0$, for some positive constant C_r (called C_r-inequality) and Marcinkiewicz-Zygmund's inequality, we obtain, as $k \to \infty$, that:

$$\begin{aligned} &E[|\xi_{n+1}|^{1+\delta} \mid Z_n = k] \leq \\ &\qquad \leq C_{1+\delta}\left(E[|\sum_{i=1}^{\phi_n(k)} (X_{ni} - m)|^{1+\delta}] + m^{1+\delta} E[|\phi_n(k) - \varepsilon(k)|^{1+\delta}] \right) \end{aligned}$$

and

$$E[|\sum_{i=1}^{\phi_n(k)}(X_{ni}-m)|^{1+\delta}] = O(E[(\phi_n(k))^{(1+\delta)/2}]).$$

Hence, the conditions

$$E[(\phi_0(k))^{(1+\delta)/2}] = O(k^\delta) \quad \text{and} \quad E[|\phi_0(k)-\varepsilon(k)|^{1+\delta}] = O(k^\delta)$$

imply (3). Note that for $\delta = 1$, this gives the condition (i) in Theorem 3

From the results described up till now, we can deduce that in relation to the possible process' extinction, the behaviour is well or "nearly well" defined in the cases in which $\liminf_{k\to\infty}\mu(k) > 1$ and $\limsup_{k\to\infty}\mu(k) < 1$. In analogy with the classification of the standard Galton-Watson process, we propose the following classification for a CBP with random control function. We say that the process:

1. is *subcritical* if $\limsup_{k\to\infty}\mu(k) < 1$.
2. is *critical* if $\liminf_{k\to\infty}\mu(k) \leq 1 \leq \limsup_{k\to\infty}\mu(k)$.
3. is *supercritical* if $\liminf_{k\to\infty}\mu(k) > 1$.

Theorem 2 implies that every subcritical CBP with random control function dies out with probability one. In general, and unlike what happens in supercritical standard GWPs, the non–extinction with positive probability is not guaranteed for supercritical CBPs with random control function. In Example 1, we showed a supercritical control process which dies out with probability one. However if the control variances verify certain properties of regularity, according to Theorem 3, for those processes there is a positive probability of non–extinction. The research concerning with the critical case has been considered recently in González et al. (2003b). It is convenient to establish such a classification because it allows a reasonable and coherent description of the class of CBPs, and it is useful to show in an organized way the results relating to the limit behaviour, question that is investigated in the next section for the supercritical case.

4. Limiting behaviour of supercritical processes

In this section we assume that $\{\tau_k\}_{k\geq 1}$ is a convergent sequence and we denote $\tau := \lim_{k\to\infty}\tau_k < \infty$. Under these conditions we investigate the limit behaviour of the sequence $\{W_n\}_{n\geq 0}$, where

$$W_n := \frac{Z_n}{(\tau m)^n} \qquad n = 0, 1, \ldots$$

Theorem 4. *Assumed that one of the following conditions holds:*

1. $\{\tau_k\}_{k\geq 1}$ *is nondecreasing and convergent,*

2. $\{\tau_k\}_{k\geq 1}$ *is nonincreasing and convergent and* $\{E[W_n]\}_{n\geq 0}$ *is a bounded sequence.*

Then, there exists a non-negative and finite random variable, W*, such that* $\{W_n\}_{n\geq 1}$ *converges to* W *almost surely as* $n \to \infty$*.*

Note that if $\tau m \leq 1$, Theorem 1 implies that $P(W = 0) = 1$. On the other hand, for $\tau m > 1$, assuming that the sequence $\{\sigma_k\}_{k\geq 1}$ is bounded, Theorem 3 states that, for Z_0 large enough, there exists a positive probability of survival.

We study the supercritical case, i.e. $\tau m > 1$. We present conditions for the random variable W be non-degenerate and, as a consequence we obtain a geometric rate of growth in the population.

First, we establish that $\delta = \tau m$ is the only constant for which $\{\delta^{-n} Z_n\}_{n\geq 0}$ converges a.s., as $n \to \infty$, to a finite and non-degenerate at zero random variable.

Theorem 5. *Suppose that* $\{\sigma_k\}_{k\geq 1}$ *is a bounded sequence and* $P(Z_n \to \infty) > 0$*, then*

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \tau m \quad a.s. \quad on \quad [Z_n \to \infty]$$

Next result establishes a necessary condition.

Theorem 6. *Under the assumptions in Theorem 4, supposed that* $P(W > 0) > 0$*, then*

$$\sum_{k=0}^{\infty} \left| \tau - \frac{\varepsilon(Z_k)}{Z_k} \right| < \infty \quad a.s. \quad on \quad [W > 0]$$

From the above theorem it follows that the condition

$$P\left(\omega \in [Z_n \to \infty] \colon \sum_{k=0}^{\infty} \left| \tau - \frac{\varepsilon(Z_k(\omega))}{Z_k(\omega)} \right| < \infty \right) > 0$$

is necessary for W be non-degenerate.
A necessary and sufficient condition, based on the $X \log^+ X$ criterion, is now established.

Theorem 7. *Under the assumptions in Theorem 4, suppose that* $\{\sigma_k\}_{k\geq 1}$ *is a bounded sequence. Then*

$P(W > 0) > 0$ *if and only if* $E[X_{01}\log^+ X_{01}] < \infty$ *and*

$$P\left(\omega \in [Z_n \to \infty]\colon \sum_{k=0}^{\infty}\left|\tau - \frac{\varepsilon(Z_k(\omega))}{Z_k(\omega)}\right| < \infty\right) > 0$$

The proofs of Theorems 4, 5, 6 and 7 are given in González et al. (2002).

From Theorem 4, it follows that if $\{\tau_k\}_{k\geq 0}$ converges in a monotonic way to a finite value, the L^1-convergence of $\{W_n\}_{n\geq 0}$ implies the almost sure convergence, and both limits are almost sure equal. By this, we now consider the L^1-convergence in order to obtain new conditions for W be non-degenerate.

In order to extend the class of CBP with random control function considered until now, we assume,

> *H1.* $\{\tau_k\}_{k\geq 1}$ converges to a finite value τ, with $\tau m > 1$, in such a manner that $\{|\rho(k)|\}_{k\geq 1}$ is a non-increasing sequence, where $\rho(0) := 0,\ \rho(k) := \tau - \tau_k,\ k = 1, 2, \ldots$.

Before investigating necessary or sufficient conditions for the L^1-convergence of the sequence $\{W_n\}_{n\geq 0}$, a condition for the existence of $\lim_{n\to\infty} E[W_n]$ is established. Such a limit will be positive and finite if the process starts with a large enough number of individuals to guarantee non-extinction with a positive probability.

Theorem 8. *Under H1, if* $\sum_{k=1}^{\infty} k^{-1}|\rho(k)| < \infty$ *then it is verified that*

1. $0 < \lim\limits_{n\to\infty} E[W_n \mid Z_0 = N] < \infty$ *for all* $N \geq 1$ *such that* $q_N < 1$.

2. *There exists a finite random variable* W*, such that* $\lim\limits_{n\to\infty} W_n = W$ *a.s.*

Note that the condition $\sum_{k=1}^{\infty} k^{-1}|\rho(k)| < \infty$ has allowed to prove the almost sure convergence of the sequence $\{W_n\}_{n\geq 0}$ for a new and more general situation, not investigated until now, namely when $\{\tau_k\}_{k\geq 1}$ converges to τ, with $\tau m > 1$, in such a manner that $\{|\rho(k)|\}_{k\geq 1}$ is a non-increasing sequence.

The following theorem provides a necessary condition for the L^1-convergence of $\{W_n\}_{n\geq 0}$ to a non-degenerate at zero random variable.

Theorem 9. *Assume that* $\{\rho(k)\}_{k\geq 1}$ *is a monotonic sequence and* $\tau m > 1$. *If* $\{W_n\}_{n\geq 0}$ *converges in* L^1, *as* $n \to \infty$, *to a finite and non-degenerate in 0 random variable, then it is verified that*

$$\sum_{n=1}^{\infty} n^{-1}|\rho(n)| < \infty$$

In Theorem 9 it is necessary to assume that $\{\rho(k)\}_{k\geq 1}$ is a monotonic sequence because in the proof, which can be found in Theorem 2 of González et al.(2003a), it is required that the values $\rho(k),\ k = 1, 2, \ldots$ have the same sign.

In order to determine a sufficient condition for the L^1-convergence of $\{W_n\}_{n\geq 0}$ to a non-degenerate at zero limit, it will be necessary to introduce the following expectations:

$$R(0) := 0, \quad R(k) := E[|Z_{n+1} - m\varepsilon(k)| \mid Z_n = k] \quad k = 1, 2, \ldots$$

Since $E[Z_{n+1} \mid Z_n = k] = m\varepsilon(k)$, the value $R(k)$ is interpreted as a mean deviation, when there are k individuals in the population.

Theorem 10. *Under H1, assuming that* $P(Z_n \to \infty) > 0$ *and* $\{k^{-1}R(k)\}_{k\geq 1}$ *is a nonincreasing sequence. If*

$$\sum_{n=1}^{\infty} n^{-1}|\rho(n)| < \infty, \qquad \sum_{n=1}^{\infty} n^{-2}R(n) < \infty$$

then $\{W_n\}_{n\geq 0}$ *converges in* L^1, *as* $n \to \infty$, *to a non-degenerate at zero random variable* W.

The condition $\sum_{n=1}^{\infty} n^{-2}R(n) < \infty$ can be difficult to verify. In the following result, based again on the logarithmic criterion, an alternative manner, easier to check, the L^1-convergence of $\{W_n\}_{n\geq 0}$ to a non-degenerate in 0 limit, is provided.

Theorem 11. *Under H1, assuming that* $\sum_{k=1}^{\infty} k^{-1}|\rho(k)| < \infty$ *and* $\{\sigma_k\}_{k\geq 1}$ *is bounded sequence, it is verified that* $\{W_n\}_{n\geq 0}$ *converges in* L^1, *as* $n \to \infty$, *to a non-degenerate in 0 random variable* W *if and only if* $E[X_{01} \log^+ X_{01}] < \infty$.

For details about the proofs of Theorems 8, 9, 10 and 11 we refer the reader to González et al. (2003a).

Acknowledgements

The authors would like to thank to the referee for a thorough reading of the paper and the constructive suggestions which have improved it.

REFERENCES

[1] J.H. Bagley. On the almost sure convergence of controlled branching processes. *J. Appl. Probab.* **23** (1986), 827–831.

[2] F.T. Bruss. A counterpart of the Borel-Cantelli lemma. *J. Appl. Probab.* **17** (1980), 1094–1101.

[3] J.P. Dion and B. Essebbar. On the statistics of controlled branching processes. *Lecture Notes Statist. 99, ed. C.C. Heyde, Springer, Berlin* (1995), 14–21.

[4] M. González, R. Martínez, I. del Puerto. Nonparametric estimation of the offspring distribution and the mean for a controlled branching process. *Test* (2003)(to appear).

[5] M. González, M. Molina, I. del Puerto. On the class of controlled branching processes with random control functions. *J. Appl. Probab.* **39** (2002), 804–815.

[6] M. González, M. Molina, I. del Puerto. On the geometric growth for a controlled branching process with random control function. *J. Appl. Probab.* **40** (2003a), 995-1006.

[7] M. González, M. Molina, I. del Puerto. On the critical controlled branching processees with random control function. *Preprint n^o 76 Department of Mathematics, University of Extremadura (Spain)*(2003b), 1–18.

[8] J. Holzheimer. ϕ-branching processes in a random environment. *Zastos. Mat.* **18**(1984), 351–358.

[9] M. Molina, M. González, M. Mota. Some theoretical results about superaddive controlled Galton-Watson branching processes. *In Proc. Prague Stoch.'98, v.1, ed. M. Husková, P. Lachout and J.A. Visek, Union of Czech Mathematicians and Physicist*, (1998), 413–418.

[10] T. Nakagawa. The $L^\alpha (1 < \alpha \leq 2)$ convergence of a controlled branching process in a random environment. *Bull. Gen. Ed. Dokkyo Univ. School Medicine,* **17** (1994), 17–24.

[11] B.A. Sevast'yanov, A.M. Zubkov. Controlled branching processes. *Theory Probab. Appl.* **19** (1974), 14–24.

[12] N.M. Yanev. Conditions for degeneracy of φ-branching processes with random φ. *Theory Probab. Appl.* **20** (1975), 421–428.

[13] G.P. Yanev, N.M. Yanev. Conditions for extinction of controlled branching processes. *In Mathematics and Education in Mathematics, ed. G. Gerov, Bulgarian Akad. Nauk., Sofía*, (1989), 550–555.

[14] G.P. Yanev, N.M. Yanev. Extinction of controlled branching processes in random environments. *Math. Balkanica* **4** (1990), 368–380.

[15] A.M. Zubkov. Analogies between Galton-Watson processes and φ-branching processes. *Theory Probab. Appl.* **19** (1974), 309–331.

Miguel González
Department of Mathematics
University of Extremadura
06071 Badajoz-Spain
e-mail: `mvelasco@unex.es`

Manuel Molina
Department of Mathematics
University of Extremadura
06071 Badajoz-Spain
e-mail: `mmolina@unex.es`

Inés del Puerto
Department of Mathematics
University of Extremadura
06071 Badajoz-Spain
e-mail: `idelpuerto@unex.es`