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ON PRINCIPLE EIGENVALUE FOR LINEAR SECOND ORDER ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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Abstract. The principle eigenvalue and the maximum principle for second-order elliptic equations is studied. New necessary and sufficient conditions for symmetric and nonsymmetric operators are obtained. Applications for the estimation of the first eigenvalue are given.

1. Introduction

The aim of this paper is to investigate the principal eigenvalue and the related maximum principle for linear second order uniformly elliptic equations in divergence form

$$Lu = -\left(a_j^k(x)u_{x_k} + a_j^0(x)u\right)_{x_j} + b^j(x)u_{x_j} + b^0(x)u \;\text{ in } \Omega, \tag{1}$$

$$a_j^k(x)\xi^j\xi^k \geq \mu\,|\xi|^2 \;\text{ for every } x \in \bar{\Omega},\;\; \xi \in R^n,\;\; \mu = \text{ const} > 0. \tag{2}$$

Here Ω is a bounded domain in R^n,

$$a_j^k,\; a_j^0 \in C^1(\bar{\Omega}),\; b^j \in C(\bar{\Omega}),\; b^0 \in L^\infty(\Omega),\; \partial\Omega \in C^1,\; \left\{a_j^k\right\} = \left\{a_k^j\right\} \tag{3}$$

and under the repeating indices the summation convention is understood.

The case of L^∞ coefficients or domains with weaker regularity assumptions can be considered in a similar way as in [2] but for simplicity we omit it.

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Let us recall that the maximum principle for the operator L holds if every weak subsolution $u \in H_0^1(\Omega)$ of (1) is nonpositive, $u \leq 0$ in Ω. The function $u \in H_0^1(\Omega)$ is a weak subsolution of (1) if the integral inequality

$$\int_\Omega \left(a_j^k u_{x_k} w_{x_j} + a_j^0 u w_{x_j} + b^j w u_{x_j} + b^0 u w \right) dx \leq 0$$

is satisfied for every nonnegative function $w \in C_0^1(\Omega)$.

The motivation for investigation of this problem is the comparison principle for quasilinear second-order uniformly elliptic equations in divergence form

$$Q(u) = -\frac{\partial}{\partial x_j} a_j(x, u, Du) + b(x, u, Du) \quad \text{in } \Omega. \tag{4}$$

In fact the maximum principle in the linear case is the base for the validity of the comparison principle for weak $C^1(\bar{\Omega})$ smooth sub- and supersolutions of (4). More precisely, if

$$\begin{aligned} a_j^k(x) &= \int_0^1 \frac{\partial a_j}{\partial p_k}(x, S_t, P_t)\, dt, \ a_j^0(x) = \int_0^1 \frac{\partial a_j}{\partial u}(x, S_t, P_t)\, dt, \\ b^j(x) &= \int_0^1 \frac{\partial b}{\partial p_j}(x, S_t, P_t)\, dt, \ b^0(x) = \int_0^1 \frac{\partial b}{\partial u}(x, S_t, P_t)\, dt \end{aligned} \tag{5}$$

where $S_t = v(x) + t(u(x) - v(x)$, $P_t = Dv(x) + t(Du(x) - Dv(x)$ and $u, v \in C^1(\bar{\Omega})$ are weak sub-and supersolutions of (4), then the validity of the comparison principle for (4) is reduced to the validity of the maximum principle for linear equation (1) with the above coefficients (5). As a consequence of the maximum principle we get immediately the uniqueness and the continuous dependence on the data of the weak solutions of (1) and (4). Moreover, using suitable chosen barrier functions one can estimate the amplitude of the weak solutions of (1) or (4) which is an important step in the proof of the existence of a classical solution by means of the Leray-Schauder fixed point theorem. The maximum principle is important also in the investigations of the asymptotic behaviour of the solutions of linear and quasilinear parabolic equations in divergence form which appear in the population dynamics modeling a population which will persist or will go extinct.

There are two type of conditions guaranteeing the validity of the maximum principle. The first of them are necessary and sufficient and are given in [1] for linear equations in divergence form and in [2] for general nondivergence form equations. One of the main results in [1] and [2] is that the maximum principle for the operator L holds if and only if the first eigenvalue λ_L of L with zero Dirichlet data is positive. It is clear that the positiveness of the first eigenvalue λ_L is not easy checkable condition so that this result is more useful for some theoretical investigations. However, there are some qualitative properties of λ_L which can be used one to find out lower and upper bounds for the first eigenvalue. For example, λ_L is an increasing function with respect to the coefficient b^0 and a decreasing one with respect to the domain inclusions, i.e. $\lambda_L(b^0) \geq \lambda_L(\bar{b}^0)$ if $b^0 \geq \bar{b}^0$ and $\lambda_L(\Omega) \leq \lambda_L(\bar{\Omega})$ if $\Omega \supset \bar{\Omega}$. Moreover, λ_L is Lipschitz continuous with respect to the coefficients a^0_j, b^j, b^0 (using the L^∞ norm) and concave function of b^0 (see [2]).

There are also second type results which are only sufficient but easy checkable conditions for wide class of equations. They are given, for example, in [4], [6], [8] (see also the references there) and guarantee the maximum principle for (1) if one of the following assumptions is satisfied:

$$
\begin{array}{lll}
 & \text{(i)} & b^0 - \operatorname{div} a^0 \geq 0 \;\text{ in } \Omega,\; a^0 = (a^0_1, \cdots, a^0_n); \\
 & \text{(ii)} & b^0 - \operatorname{div} b \geq 0 \;\text{ in } \Omega,\; b = (b^1, \cdots, b^n); \\
(6) & \text{(iii)} & \text{The matrix } \; A + A^* \; \text{ is a nonnegative one, where} \\
 & & A = \begin{pmatrix} a^k_j & b^j \\ a^0_k & b^0 \end{pmatrix} \;\text{ and } A^* \;\text{ is the conjugate matrix of } A.
\end{array}
$$

Unfortunately, conditions $(6)_{\mathrm{i}}$, $(6)_{\mathrm{ii}}$ are not useful for quasilinear equations (4) because the derivatives of the coefficients a^0_j, b^j given by (5) are not under control. That is why $(6)_{\mathrm{i}}$, $(6)_{\mathrm{ii}}$ are replaced in the nonlinear case with some additional structure assumptions guaranteeing that a^0_j or b^j are identically equal to zero (see theorem 9.5 in [6]). By the way, $(6)_{\mathrm{i}}$, $(6)_{\mathrm{ii}}$ are not sharp even in the linear case because they guarantee that the discrete spectrum of the operator L (or of the formal self-adjoint operator L^* of L) is on the right hand side of the origin. However, it is possible the first eigenvalue of L (or L^*) to be far from the origin.

As for $(6)_{\mathrm{iii}}$, it seems to be the most general sufficient condition but it is not sharp, too. Following the idea in [7] we obtain that $(6)_{\mathrm{iii}}$ is not invariant if equation (1) is rewritten in an equivalent way, for example with

$$
Lu = -\left(a^k_j u_{x_k} + (a^0_j + f^j)\, u\right)_{x_j} + (b^j + f^j)\, u_{x_j} + (b^0 + \operatorname{div} f)\, u \tag{7}
$$

for arbitrary vector $f(x)$, $f^j \in C^{0,1}(\bar{\Omega})$. Now $(6)_{\mathrm{iii}}$ for equation (1) in the new form (7) is

$$\text{(8)}\qquad \begin{gathered}\text{The matrix } A_f + A_f^* \text{ is a nonnegative one, where}\\ A_f = \begin{pmatrix} a_j^k & b^j + f^j \\ a_k^0 + f^k & b^0 + \operatorname{div} f \end{pmatrix}.\end{gathered}$$

Condition (8) can be better than $(6)_{\text{iii}}$ for some special choice of f.

Starting from the idea of Protter in [7] we consider the whole class of equations (7) instead of (1) and sufficient conditions (8) instead of $(6)_{\text{iii}}$. Moreover, we prove in section 2 that (8) is also a necessary condition for the validity of the maximum principle for symmetric operators if (8) is taken over the set of all vectors $f(x)$, $f^j \in C^{0,1}(\bar{\Omega})$. Unfortunately, the same result is not true for nonsymmetric operators. The reason is that the matrix $\frac{1}{2}(A_f + A_f^*)$ in (8) corresponds exactly to the symmetric part $L_0 = \frac{1}{2}(L + L^*)$ of the operator L and the first eigenvalue of Lcan be far from the first eigenvalue of L_0, see theorem 3 in section 3.

However, over the set of nondegenerate transformations of the operator L preserving the first eigenvalue of L, for example, $\tilde{L}u = e^{-z}L(ue^z)$ for $z \in C^1(\bar{\Omega})$, we get as in the previous case a necessary and sufficient condition for the maximum principle for nonsymmetric operators.

In this way we prove in section 2 several equivalent formulas for the first eigenvalue λ_L which are different from the well known results and in many cases are more convenient for lower and upper estimates for λ_L.

Using the new expressions for λ_L we get in section 3 some quantitative properties of the first eigenvalue λ_L with respect to the coefficients a_j^0, b^j as well as with respect to the matrix $\left\{a_j^k\right\}$.

2. Main results and definitions

In this section we will recall some definitions for the first eigenvalue λ_L. If the operator L is a symmetric one, i.e. $a_j^0 = b^j$, then the variational formulation of λ_L is given in the following way

$$\text{(9)}\qquad \lambda_L = \inf_v \int_\Omega \left(a_j^k v_{x_j} v_{x_k} + 2b^j v v_{x_j} + b^0 v^2\right) dx$$

where the infinum is taken over all functions $v \in H_0^1(\Omega)$, $\int_\Omega v^2\, dx = 1$.

As it is well known (see [5]) the above infinum is attained for a function $\phi \in H_0^1(\Omega)$, $\phi > 0$ in Ω, which solves the equation $L\phi = \lambda_L \phi$ in Ω, $\phi = 0$ on $\partial\Omega$

in a weak sense. The function ϕ is the first eigenfunction of L and every weak solution $\psi \in H_0^1(\Omega)$ of the above equation is a multiple of ϕ.

When L is a nonsymmetric operator then the following "max-min" representation formulae for the first eigenvalue λ_L holds

$$\lambda_L = \sup_v \inf_x \, (Lv/v)\,, \; v \in W^{2,n}(\Omega), \; v > 0 \;\text{ in } \bar{\Omega}. \tag{10}$$

In order to formulate our results we will rewrite the operator L in the following equivalent way

$$Lu = -\left(a_j^k u_{x_k} + (g^j - c^j)\, u\right)_{x_j} + (g^j + c^j)\, u_{x_j} + b^0\, u \tag{11}$$

where $g^j = \dfrac{1}{2}(b^j + a_j^0)$, $c^j = \dfrac{1}{2}(b^j - a_j^0)$.

The reason is that the influence on λ_L of the coefficients g^j from the symmetric part L_0 of L is quite different in comparison with the coefficients c^j forming the nonsymmetric part $\frac{1}{2}(L - L^*)$ of L.

For every Lipschitz continuous vector $f(x)$, $f^j \in C^{0,1}(\bar{\Omega})$ let us introduce the notations

$$\begin{aligned} &\sigma_{L_0}(f) = \operatorname*{ess\,inf}_{x\in\Omega} \left(b^0 + \operatorname{div} f - \alpha_j^k (f^j + g^j)(f^k + g^k)\right) \\ &\sigma_{L_0} = \sup_{f^j \in C^{0,1}(\bar{\Omega})} \sigma_L(f) \end{aligned} \tag{12}$$

where $\left\{\alpha_j^k\right\} = \left\{a_j^k\right\}^{-1}$ and L_0 is the symmetric operator

$$L_0 u = -\left(a_j^k u_{x_k} + g^j u\right)_{x_j} + g^j u_{x_j} + b^0 u.$$

Now for symmetric operator L_0 we have the following result.

Theorem 1. *Let the operator L_0 satisfy (2) and (3). Then $\sigma_{L_0} = \lambda_{L_0}$ and hence the maximum principle for L_0 holds if and only if $\sigma_{L_0} > 0$.*

Proof. For arbitrary $f(x)$, $f^j \in C^{0,1}(\bar{\Omega})$ we get from (9) the inequalities

$$\begin{aligned} \lambda_{L_0} &= \inf_v \int_\Omega \left(a_j^k v_{x_j} v_{x_k} + 2g^j v v_{x_j} + (f^j v^2)_{x_j} + b^0 v^2\right) dx \\ &= \inf_v \int_\Omega \Big\{ a_j^k \left[v_{x_j} + \alpha_j^m (g^m + f^m) v\right] \left[v_{x_k} + \alpha_k^s (g^s + f^s) v\right] \\ &\quad + \left[b^0 + \operatorname{div} f - \alpha_j^k (f^j + g^j)(f^k + g^k)\right] v^2 \Big\}\, dx \geq \sigma_{L_0}(f) \end{aligned}$$

i.e. $\lambda_{L_0} \geq \sup_f \sigma_{L_0}(f) = \sigma_{L_0}$.

In order to prove the opposite inequality we will use a special choice of f.

For every positive constant $\delta > 0$, there exists from (1.10) in [2] a function $u^\delta \in C^\infty(\bar\Omega)$, $u^\delta > 0$ in Ω, such that $L_0 u^\delta \geq (\lambda_{L_0} - \delta)\, u^\delta$. Now for $\bar f^j = -a_j^k\, u^\delta_{x_k} \Big/ u^\delta - g^j$ we get from (12) the estimate

$$\sigma_{L_0} \geq \sigma_{L_0}(\bar f) = \operatorname*{ess\,inf}_x \left(L_0 u^\delta \Big/ u^\delta \right) \geq \lambda_{L_0} - \delta.$$

After the limit $\delta \to 0$ we have the desired inequality $\sigma_{L_0} \geq \lambda_{L_0}$ and hence

$$\sigma_{L_0} = \lambda_{L_0}. \qquad \square$$

As for general nonsymmetric operators L, an equivalent definition of λ_L by means of σ_L is a little bit more complicated. More precisely, let us introduce for every Lipschitz functions $z(x)$, $f^j(x) \in C^{0,1}(\bar\Omega)$ the notation

(13)
$$\begin{aligned} \sigma_L(f,z) &= \operatorname*{ess\,inf}_{x\in\Omega} \left(b^0 + \operatorname{div} f - a_j^k (f^j + g^j)(f^k + g^k) + c^j z_{x_j} - \tfrac{1}{4} a_j^k\, z_{x_j} z_{x_k} \right) \\ \sigma_L &= \sup_{z, f^j \in C^{0,1}(\bar\Omega)} \sigma_L(f,z) \end{aligned}$$

The following theorem gives the relation between the first eigenvalue λ_L of the nonsymmetric operator L and the first eigenvalues of the family of suitable chosen symmetric operators.

Theorem 2. *Let the nonsymmetric operator L satisfies (2) and (3). Then $\sigma_L = \lambda_L$ and hence the maximum principle for L holds if and only if $\sigma_L > 0$. Moreover, the following identity takes place*

$$\sigma_L = \sup_{z \in C^{0,1}(\Omega)} \lambda_{M_z} \tag{14}$$

where operator M_z is defined as

$$\begin{aligned} M_z u &= \frac{1}{2} \left(e^{-z/2} L(e^{z/2} u) + e^{z/2} L^*(e^{-z/2} u) \right) \\ &= \frac{1}{2} (L + L^*)\, u + \left(c^j z_{x_j} - \frac{1}{4}\, a_j^k z_{x_j} z_{x_k} \right) u. \end{aligned}$$

P r o o f. From (9) and the chain of inequalities

$$\begin{aligned} \lambda_{M_z} &= \inf_{v \in H_0^1(\Omega)} \int_\Omega v\, M_z v \, dx = \inf_{v \in H_0^1(\Omega)} \int_\Omega e^{-z/2} v\, L(e^{z/2} v)\, dx \\ &\leq \int_\Omega e^{-z/2} w L(e^{z/2} w)\, dx = \left(\int_\Omega e^{-z} \phi\, L\phi \, dx \right) \left(\int_\Omega e^{-z} \phi^2 \, dx \right)^{-1} = \lambda_L \end{aligned}$$

where $w = e^{-z/2}\phi \left(\int\limits_{\Omega} e^{-z}\phi^2\,dx\right)^{-1}$ and ϕ is the first eigenfunction of L, we get the estimate

$$\sup_z \lambda_{M_z} \le \lambda_L. \tag{15}$$

In order to prove the opposite inequality let us consider a sequence Ω_j of expanding C^∞ smooth subdomains of Ω, $\cup\Omega_j = \Omega$, $\lambda_L(\Omega) = \lim\limits_{j\to\infty} \lambda_L(\Omega_j)$, where $\lambda_L(\Omega_j)$ is the first eigenvalue of L in Ω_j. If $\phi > 0$, $\psi > 0$ are the first eigenfunctions of L and L^* respectively, we consider the truncated functions

$$z^j = \begin{cases} k_j \quad \text{for } x\in\Omega \quad \text{and} \ln(\phi/\psi) \ge k_j, \\ \ln(\phi/\psi) \quad \text{for } x\in\Omega \quad \text{and } m_j < \ln(\phi\psi) < k_j, \\ m_j \quad \text{for } x\in\Omega \quad \text{and } \ln(\phi/\psi) \le m_j, \end{cases} \tag{16}$$

where $k_j = \sup\limits_{\bar\Omega_j} \ln(\phi/\psi)$, $m_j = \inf\limits_{\bar\Omega_j} \ln(\phi/\psi)$.

Simple computations give us the identity $M_{zj}v = \lambda_L v$ in $\bar\Omega_j$, $v = (\phi\psi)^{1/2}$. Since $v > 0$ in Ω_j it follows from corollary 2.1 in [2] that $\lambda_{M_{zj}} \ge \lambda_L$. Using the monotonicity of $\lambda_{M_{zj}}$ with respect to the domain inclusions, after the limit $j\to\infty$ we get the inequality $\sup\limits_z \lambda_{M_z}(\Omega_k) \ge \lambda_L$ for every $k = 1, 2, \cdots$.

From theorem 1 we have

$$\begin{aligned}\sup_z \lambda_{M_z}(\Omega_k) &= \sup_{z,f} \operatorname*{ess\,inf}_{x\in\Omega_k} \Big[b^0 + c^j z_{x_j} - \frac14 a_j^k z_{x_j} z_{x_k} + \operatorname{div} f - \alpha_j^k (f^j + g^j)(f^k + g^k)\Big] \\ &= \sigma_L(\Omega_k)\end{aligned}$$

i.e. $\sigma_L(\Omega_k) = \sup\limits_z \lambda_{M_z}(\Omega_k) \ge \lambda_L$.

After the limit $k\to\infty$, from (15) we obtain the final result (14)

$$\sigma_L = \lambda_L = \sup_z \lambda_{M_z}. \qquad \square$$

Using theorem 2 we will give here different variants of σ_L or equivalently for λ_L which are useful for the investigations of the qualitative properties of λ_L in section 3.

Proposition 1. *Let the operator L satisfy (2), (3) and g^j, $c^j \in C^{0,1}(\bar\Omega)$. Then the identity*

$$\lambda_L = \sigma_L = \sup_{z\in C^{0,1}(\bar\Omega)} \operatorname*{ess\,inf}_{x\in\Omega} \Big[b^0 + div\, f + \alpha_j^k c^j c^k - \alpha_j^k (f^j + g^j)(f^k + g^k)\Big] \tag{17}$$

holds, where $f^j = \pm c^j - g^j + a_j^k z_{x_k}$.

Remark 1. For the special choice of f and z in (17), $f^j = c^j - g^j = -a_j^0\, z = 0$ we get immediately from (17) the condition $(6)_{\text{i}}$ and for $f^j = -c^j - g^j = -b^j$, $z = 0$ respectively, the condition $(6)_{\text{ii}}$.

3. Properties of the principal eigenvalue

In this section we will give some applications of theorems 1, 2 and propositions 1 for qualitative properties of λ_L. For this purpose let us recall the well known monotonicity and concavity properties of λ_L with respect to b^0: λ_L is an increasing and concave function with respect to b^0.

For the time being it is not known whether a similar monotonicity result for λ_L is true with respect to the matrix $\left\{a_j^k\right\}$ or coefficients a_j^0, b^j, respectively g^j, c^j. To give some particular answer of these questions we will need the following properties of λ_L.

Theorem 3. *Let the operator L satisfy (2), (3). Then the inequalities*

$$\lambda_{L_0} \leq \lambda_L \leq \lambda_{L_1} \tag{18}$$

hold, where L_0 is the symmetric part of L and $L_1 = L_0 + \alpha_j^k c^j c^k$.

Moreover, if additionally a_j^k, g^j, $c^j \in C^1(\bar\Omega)$ then

(19)

(i) $\lambda_L = \lambda_{L_0} \Leftrightarrow \phi_L = \phi_{L_0}$ *in* Ω $\Leftrightarrow div\,(c\phi_{L_0}^2) = 0$ *in* Ω, *where ϕ_L, ϕ_{L_0} are the first eigenfunctions of L and L_0 respectively*;

(ii) $\lambda_L = \lambda_{L_1} \Leftrightarrow c^j = \dfrac{1}{2}a_j^k z_{x_k}$ *for some* $z \in C^1(\bar\Omega)$ *and more precisely,* $z = \ln(\phi_L/\psi_{L^*})$

P r o o f. By integration by parts we get immediately the estimate

$$\begin{aligned}\lambda_L &= \int_\Omega \phi_L L\phi_L \, dx = \int_\Omega \left[a_j^k (\phi_L)_{x_j}(\phi_L)_{x_k} + 2g^j\phi_L(\phi_L)_{x_j} + b^0\phi_L^2\right] dx \\ &\geq \inf_{v\in H_0^1(\Omega)} \int_\Omega \left(a_j^k v_{x_j} v_{x_k} + 2g^j v v_{x_j} + b^0 v^2\right) dx = \lambda_{L_0}.\end{aligned}$$

Since $c^j z_{x_j} - \frac{1}{4}a_j^k z_{x_j} z_{x_k} \leq \alpha_j^k c^j c^k$ we get from (14) and theorem 2 the inequalities $\lambda_L = \sigma_L = \sup\limits_z \lambda_{M_z} \leq \lambda_{L_1}$, where M_z is defined in theorem 2.

Now let us suppose that $\operatorname{div}(c\phi_{L_0}^2) = 0$ in Ω and for simplicity let us denote $\phi_{L_0} = \phi_0$ and $\lambda_{L_0} = \lambda_0$. Since $Lu = L_0 u + \frac{1}{u}\operatorname{div}(cu^2)$ it follows that $L\phi_0 = \lambda_0\phi_0$, $\phi_0 = 0$ on $\partial\Omega$, $\phi_0 > 0$ in Ω, i.e. ϕ_0 is the first eigenfunction of L, $\phi_0 = \phi_L$ and $\lambda_L = \lambda_0 = \lambda_{L_0}$.

Suppose that $\phi_0 = \phi_L$. An easy calculations give us the identity

$$\lambda_L\phi_0 = \lambda_L\phi_L = L\phi_L = L\phi_0 = L_0\phi_0 + \frac{1}{\phi_0}\operatorname{div}(c\phi_0^2) = \lambda_0\phi_0 + \frac{1}{\phi_0}\operatorname{div}(c\phi_0^2)$$

i.e. $(\lambda_L - \lambda_0)\,\phi_0^2 = \operatorname{div}(c\phi_0^2)$ in Ω. Integrating the above expression in Ω we get immediately that $\lambda_L = \lambda_0 = \lambda_{L_0}$ and $\operatorname{div}(c\phi_0^2) = \operatorname{div}(c\phi_{L_0}^2) = 0$ in Ω.

Finally, let us suppose that $\lambda_L = \lambda_0$. By integration by parts we have

$$\int_\Omega \phi_0 L_0 \phi_0 \, dx = \lambda_0 = \lambda_L = \int_\Omega \phi_L L \phi_L \, dx = \int_\Omega \phi_L L_0 \phi_L \, dx$$

and from theorem 2 in section 6.5 in [5], it follows that $\phi_L = \phi_0$.

To prove (20) $_{\text{ii}}$, let us suppose that $c^j = a_j^k z_{x_k}$ for some $z \in C^1(\bar{\Omega})$. Since the operator $e^z L(u\, e^{-z}) = L_1 u$ has the same first eigenvalue as the operator L we have $\lambda_L = \lambda_{L_1}$. Moreover, if ϕ_{L_1} is the first eigenfunction of L_1 then $\phi_L = e^z \phi_{L_1}$, $\phi_{L^*} = e^{-z} \phi_{L_1}$ and $z = \ln(\phi_L / \phi_{L^*})$.

The rest of the proof of $(20)_{ii}$ follows by means of (14) and the special choice (16) of z^j in the proof of theorem 2. $\square$

Using theorem 3 we will give some partial results about the monotonicity of λ_L with respect to the matrix $\left\{a_j^k\right\}$. For this purpose we introduce the operator

$$Mu = -\left(m_j^k u_{x_k} + (g^j - c^j)\, u\right)_{x_j} + (g^j + c^j)\, u_{x_j} + b^0 u.$$

Proposition 2. *Let the operators L and M satisfy (2) and (3) and $\left\{a_j^k\right\} \geq \left\{m_j^k\right\}$. Suppose that one of the following assumptions is satisfied:*

i) L and M are symmetric operators;

ii) $\lambda_M = \lambda_{M_0}$, $M_0 = \frac{1}{2}(M + M^*)$;

iii) $c^j = a_j^k z_{x_k}$ for some $z \in C^{0,1}(\bar{\Omega})$ and $\mu_j^k c^j c^k = a_j^k c^j c^k$ for a.e. $x \in \bar{\Omega}$ where $\left\{\mu_j^k\right\} = \left\{m_j^k\right\}^{-1}$;

iv) $\left\{a_j^k\right\} \geq \left\{m_j^k\right\} + kI$, $k = const > 0, I$ is the unit matrix and $\mu_j^k c^j c^k \leq k\, (\omega_n / |\Omega|)^{n/2}$ for a.e. $x \in \bar{\Omega}$, where ω_n is the volume of the unit ball in R^n and $|\Omega| = mess\, \Omega$.

Then the inequality $\lambda_L \geq \lambda_M$ holds.

As for the monotonicity of λ_L with respect to g^j and c^j, it is trivially to prove that λ_L increases when $div\, g$ decreases. However, the monotonicity of λ_L with respect to c^j is not clear. For convenience we will denote the operator L with L_c and with λ_c and ϕ_c the first eigenvalue and the first eigenfunction of L_c, respectively, when the coefficients a_j^k, g^j, b^0 are fixed and c^j vary.

Proposition 3. *Let the operator L_c satisfy (2) and (3). Then the following inequality holds:*

(20) $\lambda_{tc} \geq \lambda_c$ *for every* $|t| > 1$ *where* $\lambda_{tc} = \lambda_c \Leftrightarrow \lambda_{sc} = \lambda_0$ *for every* $s \in R$.

As for the concavity of λ_c with respect to the coefficients c we have such result only in fixed directions tc, $t \in R$. In different directions c, $\bar{c}$ a similar result is true with a correction term. More precisely, we get the following result.

Proposition 4. *Let the operators L_c and $L_{\bar{c}}$ satisfy (2) and (3). Then λ_{ct} is a concave function of t^2. If $c \neq \bar{c}$ then for every $0 < t < 1$ the inequality*

$\lambda_S \geq (1-t)\lambda_{L_c} + t\lambda_{L_{\bar{c}}}$ *holds, where* $Su = L_{(1-t)c+t\bar{c}}u + t(1-t)\alpha_j^k(c^j - \bar{c}^j)(c^k - \bar{c}^k)u.$

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