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Pliska Stud. Math. Bulgar. **15** (2003), 67–84

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HYPOELLIPTICITY OF ANISOTROPIC PARTIAL DIFFERENTIAL EQUATIONS

Giuseppe De Donno

Abstract. We propose an approach based on methods from microlocal analysis, for characterizing the hypoellipticity in C^∞ and Gevrey G^λ classes of semilinear anisotropic partial differential operators with multiple characteristics, in dimension $n \geq 3$. Conditions are imposed on the lower order terms of the linear part of the operator; we also consider C^∞ nonlinear perturbations, see Theorem 1.1 and Theorem 1.4 below.

1. Introduction

We consider a class of semilinear anisotropic equations with multiple characteristics in n variables $z = (x, y) = (x_1, \dots, x_{n'}, y_1, ..., y_{n''})$, $n \geq 3$ (for related results in the case $n = 2$, see De Donno-Oliaro [3]), belonging to the set Ω, neighborhood of a point $z_0 \in \mathbb{R}^n$, in the case when no assumptions of Levi-type are imposed on the lower order terms; then, as well known, the main properties of the operators depend heavily on the lower order terms of their symbol. We consider operators of the form:

$$P(x, y, D_x, D_y)u + G(x, y; \partial_x^\gamma \partial_y^j u)|_{\left|\frac{\gamma}{\rho'}\right|+|j|<k^*} = 0, \tag{1.1}$$

where the linear part is given by:
(1.2)

$$P(x, y, D_x, D_y) = \sum_{|\alpha|=m} a_\alpha(z) D_y^\alpha - \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) D_x^\beta + \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z) D_x^\gamma D_y^j$$

2002 *Mathematics Subject Classification:* 35S05
Key words: Partial differential equations, hypoellipticity, Gevrey spaces

with $m \in \mathbb{Z}_+$, $m \geq 4$, and the anisotropic weight $\rho = (\rho', 1) = (\rho_1, ..., \rho_{n'}, \underbrace{1, ..., 1}_{n''})$, $0 < \rho_i \leq 1$, $i = 1, ..., n'$; $\alpha = (\alpha_1, ..., \alpha_{n''}), j = (j_1, ..., j_{n''}) \in \mathbb{Z}_+^{n''}$, $\beta = (\beta_1, ..., \beta_{n'}), \gamma = (\gamma_1, ..., \gamma_{n'}) \in \mathbb{Z}_+^{n'}$, $0 < k^* < m$, $\left|\frac{\beta}{\rho'}\right| := \sum_{i=1}^{n'} \beta_i \frac{1}{\rho_i}$; we shall also say that $\left|\frac{\gamma}{\rho'}\right| + |j|$ is the anisotropic order of $D_x^\gamma D_y^j$, so the nonlinearity involves derivatives of anisotropic order less than k^*. We give for (1.1) and (1.2) results of hypoellipticity and for (1.2) of Gevrey hypoellipticity too; the arguments in our proofs are based mainly on microlocal tools, allowing relevant simplifications in the study: pseudo-differential operators, wave front sets and $S^m_{\rho,\delta}$ techniques. We consider C^∞ nonlinearity G, C^∞ coefficients in (1.2) and in the following we also suppose that the principal symbol of (1.2) is real and elliptic (with respect to the η variables), i.e.:

$$c_1|\eta|^m \leq |\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha| \leq C_1|\eta|^m, \tag{1.3}$$

for positive constants c_1, C_1, and

$$\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \neq 0 \text{ for } \xi \neq 0, z \in \Omega, \tag{1.4}$$

$$G(z;t) = \sum_{r \in \mathbb{Z}_+^M} C_r(z)t^r, \ C_r \in C^\infty(\Omega), \ t \in \mathbb{Z}^M, \tag{1.5}$$

where, for every compact $K \subset \Omega$, $\sup_{z\in K} |D^\alpha C_r(z)| \leq C_{\alpha,K}\lambda_r$ and moreover $\tilde{F}(t) = \sum_r \lambda_r t^r$ is entire analytic.

We recall that the nonzero hypothesis on $\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta$ is a nondegeneracy condition with invariant meaning, usually required in the study of hypoellipticity (and local solvability) of the linear operator (1.2) in C^∞ and G^λ Gevrey spaces, see for example Liess-Rodino [10], De Donno-Rodino [4], concerning Gevrey hypoellipticity for 2 variables PDE's with higher multiplicity. As standard, the Gevrey anisotropic space $G^\lambda(\Omega)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, is defined by the estimates:

$$\sup_K |\partial_z^\alpha f(z)| \leq C_K^{|\alpha|+1}(\alpha_1!)^{\lambda_1} \cdots (\alpha^n!)^{\lambda_n}, \text{ for every } K \subset\subset \Omega, \tag{1.6}$$

where $\lambda_i \geq 1$ for $i = 1, \ldots, n$. Let us also observe that if $\Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \neq 0$ then the operator is quasi-elliptic; the results of hypoellipticity (and local solvability) are well known in this case.

Moreover we define the anisotropic characteristic manifold

$$\Sigma := \{(z,\zeta) \in \Omega \times (\mathbb{R}^n \setminus 0) : \sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta = 0\}. \tag{1.7}$$

We may regard the next results as an extension of De Donno-Oliaro [3] in which hypoellipticity (and local solvability) are proved in the case of 2-variables equations. Let us state the main results.

Theorem 1.1. *Let us fix k^* in* (1.1) $m - \frac{1}{2} < k^* < m$ *in such a way that there exists at least one n-uple $(\gamma^*, j^*) \in \mathbb{Z}_+^{n'} \times \mathbb{Z}^{n''}$ such that $\left|\frac{\gamma^*}{\rho'}\right| + |j|^* = k^*$. We suppose $a_\alpha(z)$, $b_\beta(z)$, $c_{\gamma j}(z) \in C^\infty(\Omega)$, and assume that for $(z,\zeta) \in \Sigma$ the following conditions hold:*

i) $\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j|^*=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \neq 0$ *for* $\xi \neq 0$, $\eta \neq 0$;

ii) $\Im \sum_{k^*<\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j \cdot \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j|^*=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \geq 0$;

iii) $\Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \cdot \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j|^*=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \leq 0$.

Assume moreover that (1.3) *and* (1.4) *hold. Then* (1.2) *is C^∞-hypoelliptic.*

Remark 1.2. *The operator $P(x, y, D_x, D_y)$ is hypoelliptic with loss of regularity of $\iota = m - k^*$ derivatives.*

Remark 1.3. *Taking analytic coefficients in Theorem 1.1 we obtain G^λ-hypoellipticity of the operator P in* (1.2) *for $\lambda_i \geq \frac{1}{k^*-(m-1)}$, cf. De Donno-Rodino [4] regarding the isotropic case.*

In the following it will be convenient to use the Sobolev anisotropic space H^s_ρ, where as before $\rho = (\rho', 1)$, defined by

$$\|f\|_{H^s_\rho} := \Big(\int (1 + \sum_{i=1}^{n'} |\xi_i|^{2\rho_i} + |\eta|^2)^s |(\mathcal{F}_{z\to\zeta} f)(\zeta)|^2 \, d\zeta\Big)^{\frac{1}{2}} < \infty$$

$\mathcal{F}_{z\to\zeta} f$ being the Fourier transform of $f(z)$. For $s > \frac{1+\sum_{i=1}^{n'} \frac{1}{\rho_i}}{2}$, the space H^s_ρ is an algebra, cf. the inhomogeneous Schauder estimates in Garello [6].

Theorem 1.4. *Under the above assumptions on $P(x,y,D_x,D_y)$ and G, let u be a solution of (1.1) which belongs to $H^s_{\rho,\mathrm{loc}}(\Omega)$, for $s \geq s_0$, where s_0 is a sufficiently large fixed real number. Then $u \in C^\infty(\Omega)$.*

As examples of operators satisfying Theorem 1.1 we consider in $\mathbb{R}^4$:

(1.8)
$$\left(D^2_{y_1}+D^2_{y_2}\right)^{bp} - (1-i|z|^2)\left(D^{2a}_{x_1}+D^{2b}_{x_2}\right)^{p-1} + i\left(D^{2a}_{x_1}+D^{2b}_{x_2}\right)^{p-2}\left(D^2_{y_1}+D^2_{y_2}\right)^{b},$$

where $p,a,b \in \mathbb{N}$, $p \geq 4b+2$, $1 \leq a \leq b$; we have $\rho_1 = \frac{a(p-1)}{bp}$, $\rho_2 = \frac{p-1}{p}$, $k^* = 2bp - \frac{2b}{p-1}$, $(\gamma^*, j^*) = (2a(p-2-k), 2bk, 2(b-h), 2h)$ for $k = 0,\dots,p-2$, $h = 0,...,b$. In $\mathbb{R}^3$, of even order operator we take

$$D^{2bp}_{y_1} - (1-i|z|^{2l})(D^{2a}_{x_1}+D^{2b}_{x_2})^{p-1} + i(D^{2a}_{x_1}+D^{2b}_{x_2})^{p-2}D^{2b}_{y_1}, \tag{1.9}$$

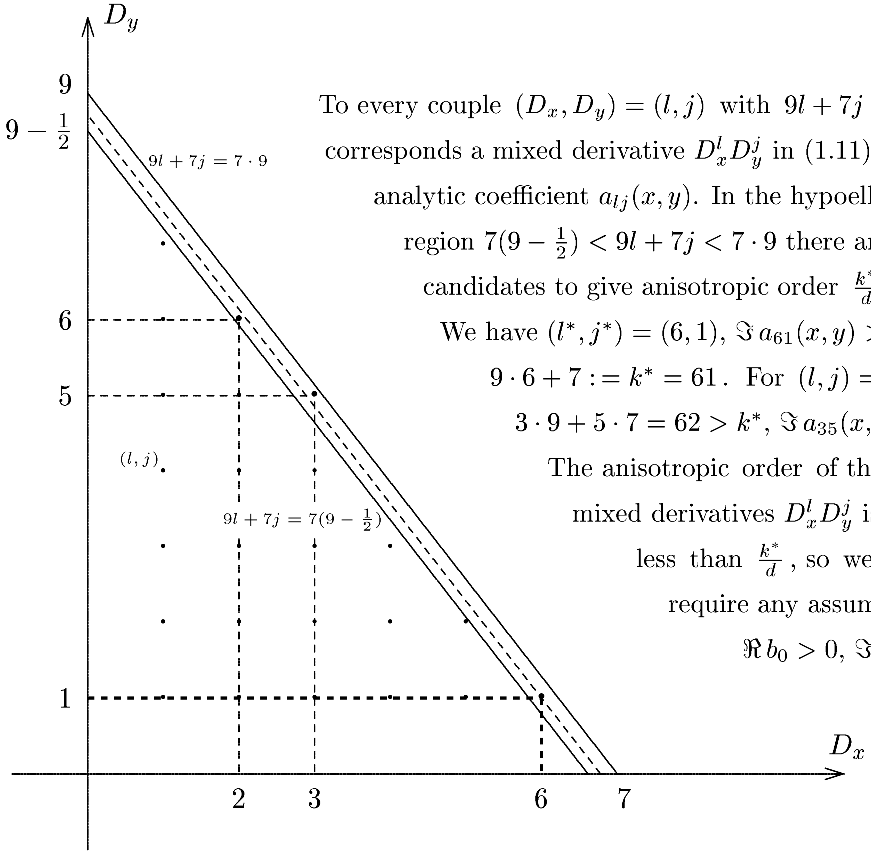
where p,a,b,ρ_1,ρ_2,k^* are the same of the previous example, but $(\gamma^*, j^*) = (2a(p-2-k), 2bk, 2b)$ for $k = 0,\dots,p-2$. About of odd order operator we consider

$$D^{2bp+1}_{y_1} - (1-i|z|^{2l})(D^{2a}_{x_1}+D^{2b}_{x_2})^{p} + i(D^{2a}_{x_1}+D^{2b}_{x_2})^{p-1}D^{2b}_{y_1}, \tag{1.10}$$

where $p,a,b \in \mathbb{N}$, $p \geq 3$, $1 \leq a \leq b$; we have $\rho_1 = \frac{2ap}{2bp+1}$, $\rho_2 = \frac{2bp}{2bp+1}$, $k^* = 2bp+1-\frac{1}{p}$, $(\gamma^*, j^*) = (2a(p-1-k), 2bk, 2b)$ for $k = 0,\dots,p-1$. We may add in (1.8)-(1.10) arbitrary nonlinear C^∞ perturbation of lower anisotropic order satisfying the hypotheses (1.5), and we obtain that (1.8)-(1.10) are C^∞ hypoelliptic. In the following picture, which resembles the Newton polygon pictures, we show in the case of two variables (x,y) the geometrical meaning of the hypothesis *ii)* in Theorem 1.1. We consider the operator of order $m = 9$ with $d = 7$:

$$D^9_y - (1-iy^{2k})D^7_x + y^h D^3_x D^5_y + iD^6_x D_y + \sum_{\frac{9}{7}l+j<\frac{61}{7}} a_{lj}(x,y)D^l_x D^j_y, \tag{1.11}$$

that is C^∞ and Gevrey hypoelliptic.



To every couple $(D_x, D_y) = (l, j)$ with $9l + 7j < 7 \cdot 9$, corresponds a mixed derivative $D_x^l D_y^j$ in (1.11) having analytic coefficient $a_{lj}(x, y)$. In the hypoellipticity region $7(9-\frac{1}{2}) < 9l + 7j < 7 \cdot 9$ there are three candidates to give anisotropic order $\frac{k^*}{d} = \frac{k^*}{7}$. We have $(l^*, j^*) = (6, 1)$, $\Im a_{61}(x, y) > 0$ and $9 \cdot 6 + 7 := k^* = 61$. For $(l, j) = (3, 5)$, $3 \cdot 9 + 5 \cdot 7 = 62 > k^*$, $\Im a_{35}(x, y) \equiv 0$. The anisotropic order of the other mixed derivatives $D_x^l D_y^j$ is $\frac{9l+7j}{d}$, less than $\frac{k^*}{d}$, so we do not require any assumptions. $\Re b_0 > 0$, $\Im b_0 \leq 0$.

In the next section 2. we prove Theorem 1.1 using $S^m_{\rho,\delta}$ estimates; Theorem 1.4 is proved in Section 3..

2. Hypoellipticity for a class of differential polynomials.

In this Section we begin to prove $S^m_{\rho,\delta}$ estimates for a pseudo-differential model in n variables, $n = n' + n''$, $n \geq 3$ (for related results in the case $n = 2$ see De Donno-Oliaro [3]). We recall that an operator P is said to be hypoelliptic at (a neighborhood $\Omega \subseteq \mathbb{R}^n$ of) a point z_0 when *sing supp* $Pu =$ *sing supp* u for all $u \in \mathcal{E}'(\Omega)$. We take $m \in \mathbb{Z}_+$, $m \geq 4$ and the anisotropic weight $\rho = (\rho', 1) = (\rho_1, ..., \rho_{n'}, \underbrace{1, ..., 1}_{n''})$, $0 < \rho_i \leq 1$, $i = 1, ..., n'$.

Let the function in $\Omega \times \mathbb{R}^n$

$$
\begin{aligned}
p(z,\zeta) \;=\; & \sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha \;-\; \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \,+ \\
& \qquad + \sum_{k^*\le\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j \;+\; \sigma(z,\zeta)\ ,
\end{aligned}
\tag{2.1}
$$

be the symbol of the pseudo-differential operator

$$
\begin{aligned}
P(z,D) \;=\; & \sum_{|\alpha|=m} a_\alpha(z)D_y^\alpha \;-\; \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)D_x^\beta \,+ \\
& \qquad + \sum_{k^*\le\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)D_x^\gamma D_y^j \;+\; \Xi(z,D)\ ,
\end{aligned}
$$

where $z=(x,y)\in\mathbb{R}^{n'}\times\mathbb{R}^{n''}$, $\zeta=(\xi,\eta)=(\xi_1,...,\xi_{n'},\eta_1,...,\eta_{n''})\in\mathbb{R}^{n'}\times\mathbb{R}^{n''}$ is the dual variable of z; $\left|\frac{\beta}{\rho'}\right| := \sum_{i=1}^{n'}\beta_i\frac{1}{\rho_i}$; $a_\alpha:\Omega\to\mathbb{R}$, $b_\beta, c_{\gamma j}:\Omega\to\mathbb{C}$, are in $C^\infty(\Omega)$, $\beta=(\beta_1,...,\beta_{n'}),\gamma=(\gamma_1,...,\gamma_{n'})\in\mathbb{Z}_+^{n'}$, $\alpha=(\alpha_1,...,\alpha_{n''}), j=(j_1,...,j_{n''})\in\mathbb{Z}_+^{n''}$. We define the following sets for $k\in\mathbb{Q}_+$, $0<k<m$:

$$
I_k \;=\; \left\{(\gamma,j)\;\in\;\mathbb{Z}_+^{n'}\times\mathbb{Z}_+^{n''},: \left|\frac{\gamma}{\rho'}\right| + |j| = k\right\}
$$

and let k^* be such that $(m-\frac{1}{2})<k^*<m$. We use the notation k^- for all $k<k^*$ and k^+ for all $k>k^*$. The symbol $\sigma(z,\zeta)$ in (2.1) belonging in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is such that

$$
|D_z^{\mathrm{p}}D_\zeta^{\mathrm{q}}\sigma(z,\zeta)| \le C_{\mathrm{p},\mathrm{q}}\langle\zeta\rangle_\rho^{\overline{k}-\left|\frac{\mathrm{q}}{\rho}\right|}
$$

where we understand $\mathrm{p}=(\mathrm{p}',\mathrm{p}'')$, $\mathrm{q}=(\mathrm{q}',\mathrm{q}'')\in\mathbb{Z}_+^{n'}\times\mathbb{Z}_+^{n''}$, $\left|\frac{\mathrm{q}}{\rho}\right| := \sum_{i=1}^{n'}\mathrm{q}_i'\frac{1}{\rho_i}+\sum_{i=1}^{n''}\mathrm{q}_i''$; with $\overline{k}<k^*$; $\langle\zeta\rangle_\rho=\langle\xi\rangle_{\rho'}+|\eta|=\sum_{i=1}^{n'}(1+|\xi_i|^{\rho_i})+|\eta|$ is the anisotropic norm. Let Λ be a neighborhood of the anisotropic characteristic manifold Σ, (see (1.7)), and let Γ the set $\Omega\times\Lambda$. Then we state the following:

Theorem 2.1. *Assume I_{k^*} is not empty, and moreover for $(z,\zeta)\in\Gamma$:*

$$
(2.2)\quad
\begin{aligned}
&i)\ \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \neq 0\ ,\ \xi\neq 0,\eta\neq 0\\
&ii)\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\Im \sum_{\left|\frac{\gamma}{\rho'}\right|+|j|=k^+} a_{\gamma j}(z)\xi^{\gamma}\eta^{j}\geq 0\\
&\text{for every } k_+\\
&iii)\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_{\beta}(z)\xi^{\beta}\leq 0\,,\\
&iv)\ \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_{\beta}(z)\xi^{\beta}\neq 0\,,\ \xi\neq 0
\end{aligned}
$$

Then for all $\mathrm{p},\mathrm{q}\in\mathbb{Z}^n_+$*, for all* $K\subset\subset\Omega$ *there exist positive constants* $L_{\mathrm{p,q}}$ *and* B *such that:*

$$
(2.3)\qquad \frac{|D_z^{\mathrm{p}} D_\zeta^{\mathrm{q}} p(z,\zeta)|\,\langle\zeta\rangle_\rho^{\mu\left|\frac{\mathrm{q}}{\rho}\right|-\delta\left|\frac{\mathrm{p}}{\rho}\right|}}{|p(z,\zeta)|}\leq L_{\mathrm{p,q}}\ ,z\in K,|\zeta|>B,\zeta\in\mathbb{R}^n,
$$

with $\mu=k^*-(m-1)$, $\delta=m-k^*$. *Observe that* $\delta<\mu$ *since we have assumed* $k^*>(m-\frac{1}{2})$

Remark 2.2. *Hypothesis ii) implies that* $\Im\sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}$ *and* $\Im\sum_{\left|\frac{\gamma}{\rho'}\right|+|j|=k^+} c_{\gamma j}(z)\xi^{\gamma}\eta^{j}$ *are both positive or both negative (we observe that the sum* $\Im\sum_{\left|\frac{\gamma}{\rho'}\right|+|j|=k^+} c_{\gamma j}(z)\xi^{\gamma}\eta^{j}$ *may vanish, too).*

Remark 2.3. *We may obtain the estimates* (2.3) *for* $p(z,\zeta)$ *also in the case when the set* I_{k^*} *is empty, by requiring* $|\Im\sigma(z,\zeta)|\geq\langle\zeta\rangle_\rho^{\overline{k}}$, $\overline{k}>m-\frac{1}{2}$ *and by replacing* $\Im\sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j*}$ *with* $\Im\sigma(z,\zeta)$ *in the hypotheses* $ii),iii)$. *We will give more details at the end of this section.*

Remark 2.4. *By formula* (2.3) *and by well known result, see for example Rodino-Mascarello ([12],Theorem 3.3.6), we have that the operator* $P(z,D)$*, associated to the symbol* $p(z,\zeta)$ *in* (2.1)*, is* C^∞*-hypoelliptic and for analytic coefficients Gevrey-hypoelliptic in* $G^\lambda(\Omega)$, $\lambda=(\lambda_1,...,\lambda_n)$, $\lambda_i\geq\frac{1}{k^*-m+1}$. *The estimates* (2.3) *guarantee also the existence of a parametrix of* $P(z,D)$.

Remark 2.5. *We confine ourselves for simplicity to prove the estimates (2.3) for* $|\mathrm{p}|+|\mathrm{q}| = |\mathrm{p}'|+|\mathrm{p}''|+|\mathrm{q}'|+|\mathrm{q}''| = 1$. *The case* $|\mathrm{p}|+|\mathrm{q}| > 1$ *does not involve actual complications; cf. Wakabayashi([16], Theorem 2.6), Kajitani-Wakabayashi([11], Theorem 1.9) for the analytic frame.*

Proof of theorem 2.1. We estimate first the numerator of (2.3) and then we give some lemmas to estimate the denominator.

If $|\mathrm{p}'| = 1$, we get

$$|D_{x_l}p(z,\zeta)|\,\langle\zeta\rangle_\rho^{-\delta\frac{1}{\rho_l}} \leq L_1\left(\left(|\eta|^m + \langle\xi\rangle_{\rho'}^m + \langle\xi\rangle_{\rho'}^{\overline{m}}|\eta|^{|j|}\right)\langle\zeta\rangle_\rho^{-\delta\frac{1}{\rho_l}} + \langle\zeta\rangle_\rho^{\overline{k}-\delta\frac{1}{\rho_l}}\right)$$

where $l = 1,...,n'$, $\overline{m} < m-|j|$ and $\overline{k} < k^*$; moreover, for $|\mathrm{p}''| = 1$

$$|D_{y_h}p(z,\zeta)|\,\langle\zeta\rangle_\rho^{-\delta} \leq L_2\left(\left(|\eta|^m + \langle\xi\rangle_{\rho'}^m + \langle\xi\rangle_{\rho'}^{\overline{m}}|\eta|^{|j|}\right)\langle\zeta\rangle_\rho^{-\delta} + \langle\zeta\rangle_\rho^{\overline{k}-\delta}\right)$$

with $h = 1,...,n''$, for suitable constants $L_1,\ L_2$.

If $|\mathrm{q}'| = 1$,

$$|D_{\xi_l}p(z,\zeta)|\langle\zeta\rangle_\rho^{\mu\frac{l}{\rho_l}} \leq L_3\left(\left(\langle\xi\rangle_{\rho'}^{m-\frac{1}{\rho_l}} + \langle\xi\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}}|\eta|^{|j|}\right)\langle\zeta\rangle_\rho^{\mu\frac{1}{\rho_l}} + \langle\zeta\rangle_\rho^{\overline{k}-\frac{1}{\rho_l}(1-\mu)}\right),$$

$l = 1,...,n'$; and for $|\mathrm{q}''| = 1$

$$|D_{\eta_h}p(z,\zeta)|\,\langle\zeta\rangle_\rho^{\mu} \leq L_4\left(\left(|\eta|^{m-1} + \langle\xi\rangle_{\rho'}^{\overline{m}}|\eta|^{|j|-1}\right)\langle\zeta\rangle_\rho^{\mu} + \langle\zeta\rangle_\rho^{\overline{k}-1+\mu}\right)$$

where $h = 1,..,n''$, with suitable constants $L_3,\ L_4$.

Therefore, we note that $\overline{k}-(1-\mu) \geq \overline{k}-\frac{1}{\rho_l}(1-\mu)$, $l = 1,...,n'$ and $\overline{k}-(1-\mu) = \overline{k}-\delta > \overline{k}-\frac{1}{\rho_l}\delta$ since $\rho+\delta = 1$. To prove (2.3), it will be then sufficient to show the boundedness in $\mathbb{R}^n$, for $|\zeta| > B$, of the functions

$$Q_1(\zeta) = \frac{\left(|\eta|^m + \langle\xi\rangle_{\rho'}^m + \langle\xi\rangle_{\rho'}^{\overline{m}}|\eta|^{|j|}\right)\langle\zeta\rangle_\rho^{-\delta}}{|p(z,\zeta)|},$$

$$Q_2(\zeta) = \frac{\left(\langle\xi\rangle_{\rho'}^{\overline{m}}|\eta|^{|j|-1} + |\eta|^{m-1}\right)\langle\zeta\rangle_\rho^{\mu}}{|p(z,\zeta)|},$$

$$Q_3(\zeta) = \frac{\left(\langle\xi\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}}|\eta|^{|j|} + \langle\xi\rangle_{\rho'}^{m-\frac{1}{\rho_l}}\right)\langle\zeta\rangle_\rho^{\mu\frac{1}{\rho_l}}}{|p(z,\zeta)|},\ l = 1,...,n'$$

$$Q_4(\zeta) = \frac{\langle\zeta\rangle_\rho^{\bar{k}-1+\mu}}{|p(z,\zeta)|}.$$

First we introduce three regions:

$$(2.4) \qquad \begin{array}{ll} R_1: & c\,\langle\xi\rangle_{\rho'} \le |\eta| \le C\,\langle\xi\rangle_{\rho'} \\ R_2: & |\eta| \ge C\,\langle\xi\rangle_{\rho'} \\ R_3: & |\eta| \le c\,\langle\xi\rangle_{\rho'} \end{array},$$

for suitable constants c, C to be determined precisely later on, satisfying $0 < c << min_{z\in K}\,G(z)$, $G(z) = min\{|b_\beta(z)|\}_{\left|\frac{\beta}{\rho'}\right|=m}$, and $C >> max_{z\in K}\,F(z)$, $F(z) = max\{|b_\beta(z)|\}_{\left|\frac{\beta}{\rho'}\right|=m}$. We understand the neighborhood $\Lambda \subset R_1$.

The following inequalities then hold:

$$(2.5) \qquad \langle\zeta\rangle_\rho^{-\delta} \le \begin{cases} C^\delta\,|\eta|^{-\delta} &, \quad \xi \in R_1 \qquad (I) \\ |\eta|^{-\delta} &, \quad \xi \in R_2 \qquad (II) \\ \langle\xi\rangle_{\rho'}^{-\delta} &, \quad \xi \in R_3\,; \qquad (III) \end{cases}$$

and,

$$\langle\zeta\rangle_\rho^{\mu} \le \begin{cases} C_1\,|\eta|^{\mu} &, \quad \xi \in R_1 \\ C_2\,|\eta|^{\mu} &, \quad \xi \in R_2 \\ C_3\,\langle\xi\rangle_{\rho'}^{\mu} &, \quad \xi \in R_3\,; \end{cases}$$

note that (II) and (III) in (2.5) hold for all $\zeta \in \mathbb{R}^n$, but for our aim we may limit ourselves to consider them respectively in R_2 and in R_3. By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega\times R_1, \Omega\times R_2, \Omega\times R_3$; recall that $\Gamma = \Omega\times\Lambda$.

Lemma 2.6. *Let $p(z,\zeta)$ be the function* (2.1) *and assume that i), ii), iii) in* (2.2) *hold. Then there are positive constants $K_1 < 1$, B, such that:*

$$(2.6) \quad |p(z,\zeta)| \ge K_1 \left| \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \right| \qquad (z,\zeta) \in \Gamma \bigcap R_1,\ |\zeta| > B.$$

P r o o f. We have that

(2.7)

$$|p(z,\zeta)|^2 = \left(\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \Re \sum_{k^*\le\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j + \Re\sigma(z,\zeta)\right)^2 +$$

$$+\left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} + \right.$$

$$\left. +\Im \sum_{k^*<\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j - \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \Im\sigma(z,\zeta)\right)^2 .$$

By removing the terms rising from the real part of $p(z,\zeta)$, we can write

$$|p(z,\zeta)|^2 \ge \left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\right)^2 + \sum_{i=1}^{4} W_i(z,\zeta)$$

where

$$W_1(z,\zeta) = \left(\Im \sum_{k^*<\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j - \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \Im\sigma(z,\zeta)\right)^2, \tag{2.8}$$

$$W_2(z,\zeta) = 2\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \Im \sum_{k^*<\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j , \tag{2.9}$$

$$W_3(z,\zeta) = -2\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*} \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta , \tag{2.10}$$

$$W_4(z,\zeta) = 2\Im\sigma(z,\zeta)\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}. \tag{2.11}$$

The function (2.8) is non-negative, (2.9) and (2.10) are also non-negative by hypotheses *(ii)*, *(iii)* for all $(z,\zeta) \in R_1$.

Concerning (2.11), it holds

$$\left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\right)^2 + W_4(z,\zeta) \geq$$

$$\geq (1-\epsilon)\left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\right)^2.$$

In fact, for $|\zeta|$ sufficiently large

$$\frac{|W_4(z,\zeta)|}{\left|\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\right|^2} \leq const\, \frac{\langle\xi\rangle_{\rho'}^{k^*-|j^*|}|\eta|^{|j^*|}\langle\xi\rangle_{\rho'}^{\overline{k}}}{|\eta|^{2|j^*|}\langle\xi\rangle_{\rho'}^{2k^*-2|j^*|}} \leq$$

$$\leq const\, \frac{|\eta|^{k^*-\overline{k}}}{\eta^{2k^*}} < \epsilon, \quad |\zeta| > B;$$

since $\overline{k} < k^*$.

Then

$$|p(z,\zeta)| \geq K_1 \left|\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^* j^*}(z)\xi^{\gamma^*}\eta^{j^*}\right|, \quad (z,\zeta) \in R_1,\ |\zeta| > B,$$

for a suitable positive constant K_1. □

Lemma 2.7. *Let $p(z,\zeta)$ be the function* (2.1)*. Then there are positive constants $K_2 < 1$, B, such that:*

$$|p(z,\zeta)| \geq K_2\, |\eta|^m, \quad (z,\zeta) \in R_2,\ |\zeta| > B. \tag{2.12}$$

P r o o f. We write $|p(z,\zeta)|^2$ as in (2.7); by removing the terms arising from the imaginary part of $p(z,\zeta)$, we get

(2.13)
$$|p(z,\zeta)|^2 \geq \left(\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta\right)^2 + W_1(z,\zeta) + W_2(z,\zeta) + W_3(z,\zeta)$$

where

$$W_1(z,\zeta) = \left(\Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z)\xi^\gamma\eta^j + \Re\sigma(z,\zeta)\right)^2, \tag{2.14}$$

(2.15)
$$W_2 = 2\Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z)\xi^\gamma\eta^j \sum_{|\alpha|=m} a_\alpha\eta^\alpha - \\ - 2\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} a_{\gamma j}(z)\xi^\gamma\eta^j$$

$$W_3 = 2\sum_{|\alpha|=m} a_\alpha\eta^\alpha \Re\sigma(z,\zeta) - 2\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \Re\sigma(z,\zeta). \tag{2.16}$$

Observe first that for $\lambda > 0$ sufficiently small

$$\left(\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta\right)^2 > \lambda\,\eta^{2m};$$

(2.14) is non-negative. We denote (2.15) by $\Upsilon_1(z,\zeta) - \Upsilon_2(z,\zeta)$ and (2.16) by $\Upsilon_3(z,\zeta) - \Upsilon_4(z,\zeta)$. Then

$$|p(z,\zeta)|^2 \geq \lambda \left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|^2 + \Upsilon_1(z,\zeta) - \Upsilon_2(z,\zeta) + \Upsilon_3(z,\zeta) - \Upsilon_4(z,\zeta).$$

Arguing on $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ in the same way as we have done in Lemma 2.6, it is possible to show that for all $\epsilon > 0$

$$\frac{\lambda}{2}\left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|^2 + \Upsilon_1(z,\zeta) - \Upsilon_2(z,\zeta) \geq \frac{(\lambda-\epsilon)}{2}\left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|^2, \ (z,\zeta) \in R_2,$$

and

$$\frac{\lambda}{2}\left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|^2 + \Upsilon_3(z,\zeta) - \Upsilon_4(z,\zeta) \geq \frac{(\lambda-\epsilon)}{2}\left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|^2, (z,\zeta) \in R_2.$$

Thus

$$|p(z,\zeta)| \geq K_2\,\eta^m, \quad (z,\zeta) \in R_2, \ |\zeta| > B.$$

□

Lemma 2.8. *Let $p(z,\zeta)$ be the function (2.1), such that iv) in (2.2) holds. Then there are positive constants $K_3 < 1$, B, such that:*

(2.17) $$|p(z,\zeta)| \geq K_3\,\langle\xi\rangle_{\rho'}^m, \quad (z,\zeta) \in R_3, \ |\zeta| > B.$$

Proof. We apply again (2.13), (2.14), (2.15), (2.16) to $|p(z,\zeta)|^2$. Observe that in R_3, arguing as above, since $c << min_{z\in K} G(z)$, we obtain for a suitable constant $\mu > 0$

$$\left(\sum_{|\alpha|=m} a_\alpha\eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta\right)^2 > \mu\,\langle\xi\rangle_{\rho'}^{2m}.$$

About the terms in (2.14), (2.15) and (2.16), the remarks we have done in Lemma 2.7 hold by replacing $\lambda\left|\sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha\right|$ with $\mu\,\langle\xi\rangle_{\rho'}^{2m}$. Then we have

$$|p(z,\zeta)| \geq K_3\,\langle\xi\rangle_{\rho'}^m, \quad (z,\zeta) \in R_3, \ |\zeta| > B.$$

□

Remark 2.9. *In the previous lemmas we have estimated the symbol function $|p(z,\zeta)|$ in (2.1), separately in the three regions (2.4). It is possible to obtain the following global result on $|p(z,\zeta)|$: there exists $m' \in \mathbb{R}$, $d > 0$, $B > 0$, such that*

(2.18) $$|p(z,\zeta)| \geq d\langle\zeta\rangle_\rho^{m'}, \text{ in } \Gamma \text{ for } |\zeta| > B.$$

In fact, by remembering that under assumptions $i), ii), iii), iv)$ *in theorem 2.1 the estimates* (2.6), (2.12), (2.17) *hold, we obtain that* $|p(z,\zeta)| \geq const\,|\eta|^{k^*}$ *in* R_1 *and* R_2. *Since in these regions* $|\eta|^{k^*} = \frac{1}{2}|\eta|^{k^*} + \frac{1}{2}|\eta|^{k^*} \geq c\langle\xi\rangle_{\rho'}^{k^*} + \frac{1}{2}|\eta|^{k^*} \sim const\,(\langle\xi\rangle_{\rho'} + |\eta|)^{k^*} = const\,\langle\zeta\rangle_{\rho}^{k^*}$, *we have that*

$$|p(z,\zeta)| \geq d\,\langle\zeta\rangle_{\rho}^{k^*}.$$

In the same way we get

$$|p(z,\zeta)| \geq d\,\langle\zeta\rangle_{\rho}^{m}$$

in R_3. *Because* $k^* < m$, *we have* $m' = k^*$ *in* (2.18).

We first consider $Q_1(\zeta)$ separately in the regions R_1, R_2, R_3, to prove boundedness.

In R_1 by (2.5),(2.6) we get easily:

$$Q_1(\zeta) \leq const\,\frac{|\eta|^{\overline{m}+j-\delta} + |\eta|^{m-\delta}}{|\eta|^{k^*}} \leq L\ , |\zeta| > B$$

since $\delta \geq m - k^*$. We recall that $\overline{m} + j < m$. In the region R_2 we have that $|p(z,\zeta)| \geq |\eta|^m > |\eta|^{k^*}$. In R_3, by using (2.5),(2.17), we have for a constant $\epsilon > 0$ which we may take as small as we want by fixing B sufficiently large:

$$Q_1(\zeta) \leq const\,\frac{\langle\xi\rangle_{\rho'}^{\overline{m}+j-\delta} + \langle\xi\rangle_{\rho'}^{m-\delta}}{\langle\xi\rangle_{\rho'}^{m}} < \epsilon\ , |\zeta| > B$$

We have therefore proved that $Q_1(\zeta)$ is bounded. Let us estimate $Q_2(\zeta)$ and $Q_3(\zeta)$. In the region R_2 we argue as before; in the regions R_1, R_3 we obtain respectively

$$Q_2(\zeta) \leq const\,\frac{|\eta|^{\overline{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + |\eta|^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{|\eta|^{k^*}} < \epsilon\ ,$$

in R_1 for $|\zeta| > B$, since $\mu \leq k^* - (m-1)$, $\overline{m} < m - j$,

$$Q_2(\zeta) \leq const\,\frac{\langle\xi\rangle_{\rho'}^{\overline{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + \langle\xi\rangle_{\rho'}^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{\langle\xi\rangle_{\rho'}^{k^*}} < \epsilon\ ,$$

in R_3 for $|\zeta| > B$.

For Q_3 we obtain that

(2.19) $$Q_3(\zeta) \le const \frac{|\eta|^{\overline{m}+j-1+\mu} + |\eta|^{m-1+\mu}}{|\eta|^{k^*}} \le L \,, |\zeta| > B$$

since $\mu \le k^* - m + 1$ in R_1 and

(2.20) $$Q_3(\zeta) \le const \frac{\langle\xi\rangle_{\rho'}^{\overline{m}+j-1+\mu} + \langle\xi\rangle_{\rho'}^{m-1+\mu}}{\langle\xi\rangle_{\rho'}^{m}} < \epsilon \,,$$

in R_3 for $|\zeta| > B$.

For Q_4, we get $\frac{|\eta|^{\overline{k}-(1-\mu)}}{|\eta|^{k^*}} < \epsilon$ in R_1 since $\overline{k} < k^*$, $\mu < 1$. In R_3: $\frac{\langle\xi\rangle_{\rho'}^{\overline{k}-(1-\mu)}}{\langle\xi\rangle_{\rho'}^{k^*}} < \epsilon$, $|\zeta| > B$

Now Lemma 2.6, Lemma 2.7, Lemma 2.8 and the estimate (2.18) complete the proof.

We shall use also the following variant of Theorem 1.1, where the role of I_{k^*}-terms is played by the pseudo-differential term $\sigma(z,\zeta)$ (see Remark 2.3). Namely, we fix now t with $0 < t < \frac{1}{2}$ and assume that $|\Im\sigma(z,\zeta)| \ge \langle\zeta\rangle_\rho^{\overline{k}}$ where $m - \frac{1}{2} < \overline{k} < m - t$, considering a symbol of the form:

(2.21)
$$p(z,\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \\ + \sum_{m-t\le\left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j + \sigma(z,\zeta) \,,$$

Theorem 2.10. *Let $p(z,\zeta)$ be the function* (2.21) *such that for* $(z,\zeta) \in \Gamma$

(2.22)
i) $|\Im\sigma(z,\zeta)| \ge \langle\zeta\rangle_\rho^{\overline{k}}$,

ii) $\Im\sigma(z,\zeta)\Im \sum_{\left|\frac{\gamma}{\rho'}\right|+|j|=k^+} c_{\gamma j}(z)\xi^\gamma\eta^j \ge 0$, for every $k_+ \ge m - t$,

iii) $\Im\sigma(z,\zeta)\Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \le 0$,

iv) $\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta \ne 0$, $\xi \ne 0$

Then for all $\mathrm{p},\mathrm{q} \in \mathbb{Z}_+^n$, *for all* $K \subset\subset \Omega$ *there exist such positive constants* $L_{\mathrm{p},\mathrm{q}}$ *and* B *that:*

$$\frac{|D_z^{\mathrm{p}} D_\zeta^{\mathrm{q}} p(z,\zeta)| \, \langle\zeta\rangle_\rho^{\mu\left|\frac{\mathrm{q}}{\rho}\right| - \delta\left|\frac{\mathrm{p}}{\rho}\right|}}{|p(z,\zeta)|} \leq L_{\mathrm{p},\mathrm{q}} \; , z \in K, |\zeta| > B, \zeta \in \mathbb{R}^n, \tag{2.23}$$

with $\mu = \overline{k} - (m-1)$, $\delta = m - \overline{k}$. *Observe that* $\delta < \mu$ *since we have assumed* $\overline{k} > (m - \frac{1}{2})$.

P r o o f. We have $\overline{k}$ in the role of k^* in the proof of Theorem 2.1, by observing that in R_1

$$\begin{aligned} &|p(z,\zeta)|^2 \geq \\ &\left(\Im \sum_{m-t<\left|\frac{\gamma}{\rho'}\right|+|j|<m} a_{\gamma j}(z)\xi^\gamma \eta^j - \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \Im\sigma(z,\zeta) \right)^2 \\ &\geq (\Im\sigma(z,\zeta))^2 \geq \langle\zeta\rangle_\rho^{2\overline{k}} \geq |\eta|^{2\overline{k}} \end{aligned} \tag{2.24}$$

since $ii), iii)$ and $i)$ hold; then arguing as in the proof of theorem2.1 we obtain our result. Of course, the power m' in Remark 2.9 is given by $\overline{k}$:

$$|p(z,\zeta)| \geq d\langle\xi\rangle_\rho^{\overline{k}}, \;\; |\zeta| > B. \tag{2.25}$$

□

3. The semilinear version

Let us consider now the semilinear equation

$$P(x,y,D_x,D_y)u + G(x,y;\partial_x^\gamma \partial_y^j u)|_{\left|\frac{\gamma}{\rho'}\right|+|j|<k^*} = 0, \tag{3.1}$$

where $P(x,y,D_x,D_y)$ is the model operator considered in (1.2) having the symbol $p(z,\zeta)$ in $\Omega \times \mathbb{R}^n$:

$$p(z,\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\eta^\alpha - \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z)\xi^\beta + \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right|+|j|<m} c_{\gamma j}(z)\xi^\gamma\eta^j \tag{3.2}$$

and such that the hypotheses of the Theorem 1.1 hold. Moreover G is of the type:

$$G(z;t) = \sum_{r\in\mathbb{Z}_+^M} C_r(z)t^r, \;\; C_r \in C^\infty(\Omega), \;\; t \in \mathbb{Z}^M,$$

where, for every compact $K \subset \Omega$, $\sup_{z\in K} |D^\alpha C_r(z)| \leq C_{\alpha,K}\lambda_r$ and moreover $\tilde{F}(t) = \sum_r \lambda_r t^r$ is entire analytic.

Theorem 3.1. *Under the above assumptions on $P(x, y, D_x, D_y)$ and G, let u be a solution of* (3.1) *which belongs to $H^s_{\rho,loc}(\Omega)$, for $s \geq s_0$, where s_0 is a sufficiently large fixed real number. Then $u \in C^\infty(\Omega)$.*

P r o o f. Observe that $D^\gamma_x D^j_y u \in H^{s-(\left|\frac{\gamma}{\rho'}\right|+|j|)}_{\rho,loc}(\Omega)$. Note that $\left|\frac{\gamma}{\rho'}\right| + |j| < k^*$ actually implies $\left|\frac{\gamma}{\rho'}\right| + |j| \leq k^* - \varepsilon$, with $\varepsilon > 0$. Then

$$P(x, y, D_x, D_y)u = -G(x, y; \partial^\gamma_x \partial^j_y u)|_{\left|\frac{\gamma}{\rho'}\right|+|j|<k^*} \in H^{s-k^*+\varepsilon}_{\rho,loc}(\Omega)$$

using here the assumption $s \geq s_0$, see Garello ([6], remark 2.4). By remark 1.2 we have that $u \in H^{s+\gamma}_{\rho,loc}(\Omega)$. Using again Garello ([6], remark 2.4) we get that $P(x, y, D_x, D_y)u \in H^{s-k^*+2\varepsilon}_{\rho,loc}(\Omega)$ and in its own turn $u \in H^{s+2\gamma}_{\rho,loc}(\Omega)$. Repeating now the preceding argument we obtain $u \in \cap_{t\in\mathbb{R}^+} H^t_{\rho,loc}(\Omega)$, that is $u \in C^\infty(\Omega)$. $\square$

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