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UPPER AND LOWER BOUNDS FOR RUIN PROBABILITY

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In this note we discuss upper and lower bound for the ruin probability in an insurance model with very heavy-tailed claims and interarrival times.

1. Backgrounds

The framework of our study is set by a given Bernoulli point process (Bpp) $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ on the time-state space $\mathcal{S} = (0, \infty) \times (0, \infty)$. By definition (cf. Balkema and Pancheva 1996) \mathcal{N} is simple in time ($T_k \neq T_j$ a.s. for $k \neq j$), its mean measure is finite on compact subsets of \mathcal{S} and all restrictions of \mathcal{N} to slices over disjoint time intervals are independent. We assume that:

- a) the sequences $\{T_k\}$ and $\{X_k\}$ are independent and defined on the same probability space;
- b) the state points $\{X_k\}$ are independent and identically distributed random variables (iid rv's) on $(0, \infty)$ with common distribution function (df) F which is asymptotically continuous at infinity;
- c) the time points $\{T_k\}$ are increasing to infinity, i.e. $0 < T_1 < T_2 < \dots$, $T_k \rightarrow \infty$ a.s.

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The main problem in the Extreme Value Theory is the asymptotic of the extremal process $\{\bigvee_k X_k : T_k \leq t\} = \bigvee_{k=1}^{N(t)} X_k$, associated with \mathcal{N} , for $t \rightarrow \infty$. Here the maximum operation between rv's is denoted by “ \vee ” and $N(t) := \max\{k : T_k \leq t\}$ is the counting process of \mathcal{N} . The method usually used is to choose proper time-space changes $\zeta_n = (\tau_n(t), u_n(x))$ of \mathcal{S} (i.e. strictly increasing and continuous in both components) such that for $n \rightarrow \infty$ and $t > 0$ the weak convergence

$$(1) \quad \tilde{Y}_n(t) := \{\bigvee_k u_n^{-1}(X_k) : \tau_n^{-1}(T_k) \leq t\} \Longrightarrow \tilde{Y}(t)$$

to a non-degenerate extremal process holds. (For weak convergence of extremal processes consult e.g. Balkema and Pancheva 1996.)

In fact, the classical Extreme Value Theory deals with Bpp's $\{(t_k, X_k) : k \geq 1\}$ with deterministic time points t_k , $0 < t_1 < t_2 < \dots$, $t_k \rightarrow \infty$. One investigates the weak convergence to a non-degenerate extremal process

$$(2) \quad Y_n(t) := \{\bigvee_k u_n^{-1}(X_k) : t_k \leq \tau_n(t)\} \Longrightarrow Y(t)$$

under the assumption that the norming sequence $\{\zeta_n\}$ is regular. The later means that for all $s > 0$ and for $n \rightarrow \infty$ there exist point-wise

$$\lim_{n \rightarrow \infty} u_n^{-1} \circ u_{[ns]}(x) = U_s(x)$$

$$\lim_{n \rightarrow \infty} \tau_n^{-1} \circ \tau_{[ns]}(t) = \sigma_s(t)$$

and $(\sigma_s(t), U_s(x))$ is a time-space change. As usual “ \circ ” means the composition and $[s]$ the integer part of s . The family $\mathcal{L} = \{(\sigma_s(t), U_s(x)) : s > 0\}$ forms a continuous one-parameter group w.r.t. composition.

Let us denote the (deterministic) counting function $k(t) = \max\{k : t_k \leq t\}$, and put $k_n(t) := k(\tau_n(t))$, $k_n := k_n(1)$. The df of the limit extremal process in (2) we denote by $g(t, x) := \mathbf{P}(Y(t) < x)$, and set $G(x) := g(1, x)$. Then necessary and sufficient conditions for convergence (2) are the following

1. $F^{k_n}(u_n(x)) \xrightarrow{w} G(x)$, $n \rightarrow \infty$
2. $\frac{k_n(t)}{k_n} \longrightarrow \lambda(t)$, $n \rightarrow \infty$, $t > 0$.

The regularity of the norming sequence $\{\zeta_n\}$ has some important consequences (cf. Pancheva 1998). First of all, the limit extremal process $Y(t)$ is self-similar w.r.t. \mathcal{L} , i.e.

$$U_s \circ Y(t) \stackrel{d}{=} Y \circ \sigma_s(t), \quad \forall s > 0.$$

Furthermore:

$$0. \quad \frac{k_{[ns]}}{k_n} \longrightarrow s^a, n \rightarrow \infty, \text{ for some } a > 0 \text{ and all } s > 0;$$

1'. the limit df G is max-stable in the sense that

$$(3) \quad G^s(x) = G(L_s^{-1}(x)) \quad \forall s > 0, \quad L_s := \mathbf{U}_{\sqrt[s]{s}};$$

2'. the intensity function $\lambda(t)$ is continuous.

Thus, under conditions 1. and 2. and the regularity of the norming sequence, the limit extremal process $Y(t)$ is stochastically continuous with df $g(t, x) = G^{\lambda(t)}(x)$ and the process $Y \circ \lambda^{-1}(t)$ is max-stable in the sense of (3).

Let us come back to the point process \mathcal{N} with the random time points T_k . The Functional Transfer Theorem (FTT) in this framework gives conditions on \mathcal{N} for the weak convergence (1) and determines the explicit form of the limit df $f(t, x) := \mathbf{P}(\tilde{Y}(t) < x)$. In other words, the weak convergence (2) in the framework with non-random time points can be transfer to the framework of \mathcal{N} if some additional condition on the point process \mathcal{N} is met. In our case this is condition d) below.

Denote by $\mathcal{M}([0, \infty))$ the space of all strictly increasing, cadlac functions $y : [0, \infty) \rightarrow [0, \infty)$, $y(0) = 0$, $y(t) \rightarrow \infty$ as $t \rightarrow \infty$. We assume additionally to a)–c) the following condition

$$d) \quad \theta_n(s) := \tau_n^{-1}(T_{[sk_n]}) \Longrightarrow T(s)$$

where $T : [0, \infty) \rightarrow [0, \infty)$ is a random time change, i.e. stochastically continuous process with sample paths in $\mathcal{M}([0, \infty))$. Let us set $N_n(t) := N(\tau_n(t))$. In view of condition d) the sequence

$$\begin{aligned}
\Lambda_n(t) &:= \frac{N_n(t)}{k_n} = \frac{1}{k_n} \max\{k : T_k \leq \tau_n(t)\} \\
&= \sup\{s > 0 : \tau_n^{-1}(T_{[sk_n]}) \leq t\} \\
&= \sup\{s > 0 : \theta_n(s) \leq t\}
\end{aligned}$$

is weakly convergent to the inverse process of $T(s)$. Let us denote it by Λ and let $Q_t(s) = P(\Lambda(t) < s)$.

Now we are ready to state a general FTT for maxima of iid rv's on $(0, \infty)$.

Theorem (FTT): Let $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ be a Bpp described by conditions a)–c). Assume further that there is a regular norming sequence $\zeta_n(t, x) = (\tau_n(t), u_n(x))$ of time-space changes of \mathcal{S} such that for $n \rightarrow \infty$ and $t > 0$ conditions 1., 2. and d) hold. Then

- i) $\frac{N_n(t)}{k_n} \xrightarrow{d} \Lambda(t)$
- ii) $\mathbf{P}\left(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x\right) \xrightarrow{w} \mathbf{E}[G(x)]^{\Lambda(t)}$

Indeed, we have to show only ii). Observe that for $n \rightarrow \infty$

$$N_n(t) = k_n \cdot \frac{N_n(t)}{k_n} \sim k_n \cdot \Lambda(t) \sim k_n (\lambda^{-1} \circ \Lambda(t))$$

In the last asymptotic relation we have used condition 2). Then by convergence (2)

$$\tilde{Y}_n(t) = \bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) \implies Y(\lambda^{-1} \circ \Lambda(t))$$

and

$$\begin{aligned}
\mathbf{P}\left(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x\right) &\longrightarrow f(t, x) \\
\int_0^\infty G^s(x) dQ_t(s) &= \mathbf{E}[G(x)]^{\Lambda(t)}
\end{aligned}$$

Let us apply these results to a particular insurance risk model.

2. Application to ruin probability

The insurance model, we are dealing with here, can be described by a particular Bpp $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ where

a) the claim sizes $\{X_k\}$ are positive iid random variables which df F has a regularly varying tail, i.e. $1 - F \in RV_{-\alpha}$. We consider the “very heavy tail case” $0 < \alpha < 1$ when EX does not exist, briefly $EX = \infty$;

b) the claims occur at times $\{T_k\}$ where $0 < T_1 < T_2 < \dots < T_k \rightarrow \infty$ a.s. We denote the inter-arrival times by $J_k = T_k - T_{k-1}$, $k \geq 1$, $T_0 = 0$ and assume the random variables $\{J_k\}$ positive iid with df H . Suppose $1 - H \in RV_{-\beta}$, $0 < \beta < 1$;

c) both sequences $\{X_k\}$ and $\{T_k\}$ are independent and defined on the same probability space.

The point process \mathcal{N} generates the following random processes we are interested in.

i) The counting process $N(t) = \max\{k : T_k \leq t\}$. It is a renewal process with $\frac{N(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ for $EJ < \infty$. By the Stable CLT there

exists a normalizing sequence $\{b(n)\}$, $b(n) > 0$, such that $\sum_{k=1}^{[nt]} \frac{J_k}{b(n)}$ converges weakly to a β - stable Levy process $S_\beta(t)$. One can choose $b(n) \sim n^{1/\beta} L_J(n)$, where L_J denotes a slowly varying function. Let us determine $\tilde{b}(n)$ by the asymptotic relation $b(\tilde{b}(n)) \sim n$ as $n \rightarrow \infty$. Now the normalized counting process $\frac{N(nt)}{\tilde{b}(n)}$ is weakly convergent to the hitting time process $E(t) = \inf\{s : S_\beta(s) > t\}$ of S_β , see Meerschaert and Scheffler (2002). As inverse of S_β , $E(t)$ is β -selfsimilar.

ii) The extremal claim process $Y(t) = \{\vee_{k=1}^{N(t)} X_k : T_k \leq t\} = \vee_{k=1}^{N(t)} X_k$. In view of assumption a) there exist norming constants $B(n) \sim n^{1/\alpha} L_X(n)$ such that $\vee_{k=1}^{[nt]} \frac{X_k}{B(n)}$ converges weakly to an extremal process $Y_\alpha(t)$ with Frechet marginal df, i.e. $P(Y_\alpha(t) < x) \Phi_\alpha^t(x) = \exp -tx^{-\alpha}$. Consequently,

$$Y_n(t) := \vee_{k=1}^{N(nt)} \frac{X_k}{B(\tilde{b}(n))} \Longrightarrow Y_\alpha(E(t)).$$

Below we use the $\frac{\beta}{\alpha}$ -selfsimilarity of the compound extremal process $Y_\alpha(E(t))$ (see e.g. Pancheva et al. 2003).

iii) The accumulated claim process $S(t) = \sum_{k=1}^{N(t)} X_k$. Using the same norming sequence as above we observe that

$$S_n(t) := \sum_{k=1}^{N(nt)} \frac{X_k}{B(\tilde{b}(n))} \implies Z_\alpha(E(t)).$$

Here Z_α is an α -stable Levy process and the composition $Z_\alpha(E(t))$ is $\frac{\beta}{\alpha}$ -selfsimilar.

iv) The risk process $R(t) = c(t) - S(t)$. Here $u := c(0)$ is the initial capital and $c(t)$ denotes the premium income up to time t , hence it is an increasing curve. We assume $c(t)$ right-continuous.

Note, the extremal claim process $Y(t)$ and the accumulated claim process $S(t)$ need the same time-space changes $\zeta_n(t, x) = \left(nt, \frac{x}{B(\tilde{b}(n))} \right)$ to achieve weak convergence to a proper limiting process. In fact, $\{\zeta_n\}$ makes the claim sizes smaller and compensates this by increasing their number in the interval $[0, t]$. Both processes $Y_n(t)$ and $S_n(t)$ are generated by the point process $\mathcal{N}_n = \left\{ \left(\frac{T_k}{n}, \frac{X_k}{B(\tilde{b}(n))} \right) : k \geq 1 \right\}$. With the latter we also associate the sequence of risk processes $R_n(t) = \frac{c(nt)}{B(\tilde{b}(n))} - S_n(t)$. Let us assume additionally to a)–c) the condition

d) $\frac{c(nt)}{B(\tilde{b}(n))} \xrightarrow{w} c_0(t)$, c_0 increasing curve with $c_0(0) > 0$.

Under conditions a)–d) the sequence R_n converges weakly to the risk process (cf Furrer et al. 1997) $R_{\alpha,\beta}(t) = c_0(t) - Z_\alpha(E(t))$ with initial capital $u_0 = c_0(0)$. Using the $R_{\alpha,\beta}$ -approximation of the initial risk process $R(t)$, when time and initial capital increase with n , we next obtain upper ($\bar{\psi}$) and lower ($\underline{\psi}$) bound for the ruin probability $\Psi(c, t) : P(\inf_{0 \leq s \leq t} R(s) < 0)$. Let $Z_\alpha(1)$ and $E(1)$ have df's G_α and Q , resp. Then we have:

$$\begin{aligned}
\psi(c_0, t) &:= P\left(\inf_{0 \leq s \leq t} R_{\alpha, \beta}(s) < 0\right) \\
&\leq P\left(\sup_{0 \leq s \leq t} Z_{\alpha}(E(s)) > u_0\right) \\
&\leq P(Z_{\alpha}(E(t)) > u_0) \\
&= \int_0^{\infty} \bar{Q}\left(\frac{u_0}{xt^{\frac{\beta}{\alpha}}}\right)^{\alpha} dG_{\alpha}(x) =: \bar{\psi}(c_0, t)
\end{aligned}$$

Here $\bar{Q} = 1 - Q$. On the other hand

$$\begin{aligned}
\psi(c_0, t) &\geq P(Y_{\alpha}(E(t)) > c_0(t)) \\
&= \int_0^{\infty} \bar{Q}\left(\frac{c_0(t)}{xt^{\frac{\beta}{\alpha}}}\right)^{\alpha} d\Phi_{\alpha}(x) =: \underline{\psi}(c_0, t)
\end{aligned}$$

Here we have used the self-similarity of the processes Z_{α} , Y_{α} and E . Thus, finally we get

$$\underline{\psi}(c_0, t) \leq \psi(c_0, t) \leq \bar{\psi}(c_0, t)$$

Remember, our initial insurance model was described by the point process \mathcal{N} with the associated risk process $R(t)$. We have denoted the corresponding ruin probability by $\Psi(c, t)$ with $u = c(0)$. Then

$$\begin{aligned}
\Psi(c, t) &= P\left(\inf_{0 \leq s \leq t} \left\{c(s) - \sum_{k=1}^{N(s)} X_k\right\} < 0\right) \\
&= P\left(\inf_{0 \leq s \leq \frac{t}{n}} \left\{\frac{c(ns)}{B(\bar{b}(n))} - \sum_{k=1}^{N(ns)} \frac{X_k}{B(\bar{b}(n))}\right\} < 0\right)
\end{aligned}$$

Now let initial capital u and time t increase with $n \rightarrow \infty$ in such a way that $\frac{u}{B(\bar{b}(n))} = u_0$, $\frac{t}{n} = t_0$. We observe that under conditions a) – d) we may approximate

$$\Psi(c, t) \approx \psi(c_0, t_0)$$

and consequently for u and t “large enough”

$$(4) \quad \underline{\psi}(c_0, t_0) \leq \Psi(c, t) \leq \bar{\psi}(c_0, t_0)$$

3. Examples

Assume that our model is characterized by $\alpha = 0.5$, i.e. the df of $Z_\alpha(1)$ is the Levy df $G_\alpha(x) = 2 \left(1 - \Phi \left(\sqrt{\frac{1}{x}} \right) \right)$. Here Φ is the standard normal df.

We suppose also that the random variable $E(1)$ is Exp(1)-distributed, namely $Q(s) = 1 - e^{-s}$, $s \geq 0$. Further, let us take the income curve c_0 to be of the special form $c_0(t) = u_0 + t^{\frac{\beta}{\alpha}}c$, c positive constant, that agrees with the self-similarity of the process $Z_\alpha(E(t))$. Now the upper bound depends on (u_0, t_0, β) and the lower bound depends on (u_0, t_0, β, c) . We calculate the bounds $\underline{\psi}$ and $\bar{\psi}$ in two cases $\alpha > \beta = 0.25$ and $\alpha < \beta = 0.75$ by using MATLAB7. The results of the calculations show clearly that in case $\beta > \alpha$, when “large” claims arrive “often”, the bounds of the ruin probability are larger than in the case $\beta < \alpha$, even in small time interval.

Note, if we choose the income curve in the above special form, we may calculate the ruin probability $\psi(c_0, t_0)$ in the approximating model exactly, namely

$$\begin{aligned} \psi(c_0, t_0) &= P(\inf_{0 \leq s \leq t_0} \{u_0 + s^{\frac{\beta}{\alpha}}c - Z_\alpha(E(s))\} < 0) \\ &= P(\inf_{0 \leq s \leq t_0} \{s^{\frac{\beta}{\alpha}}(c - Z_\alpha(E(1)))\} < -u_0) \\ &= P(\inf_{0 \leq s \leq t_0} \{s^{\frac{\beta}{\alpha}}(c - Z_\alpha(E(1)))\} < -u_0, \quad c - Z_\alpha(E(1)) < 0) \\ &= P(t_0^{\frac{\beta}{\alpha}}(c - Z_\alpha(E(1))) < -u_0, \quad c - Z_\alpha(E(1)) < 0) \\ &= P(Z_\alpha(E(1)) > c + \frac{u_0}{t_0^{\frac{\beta}{\alpha}}}) \\ &= \int_0^\infty \bar{Q}\left(\left(\frac{c_0(t_0)}{xt_0^{\frac{\beta}{\alpha}}}\right)^\alpha\right) dG_\alpha(x) \end{aligned}$$

Below we give graphical results related to the computation of $\underline{\psi}(c_0, t_0)$, $\psi(c_0, t_0)$ and $\bar{\psi}(c_0, t_0)$ in the 6 cases: $c=0.1$, $c=1$, $c=10$ when $\alpha = 0.5$ and $\beta = 0.25$ $\beta = 0.75$.

4. Graphics of $\underline{\psi}(c_0, t_0)$, $\psi(c_0, t_0)$ and $\bar{\psi}(c_0, t_0)$

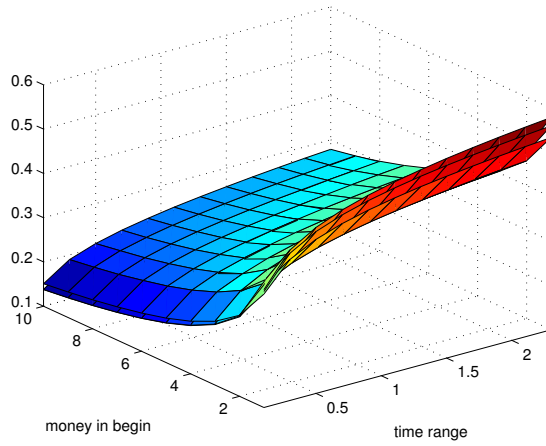


Figure 1: $\alpha = 0.5, \beta = 0.25, c = 0.1$

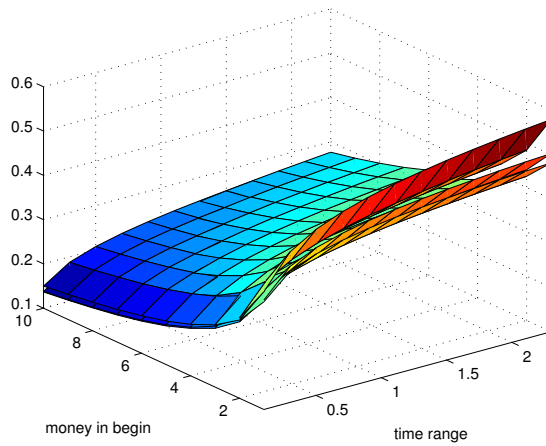


Figure 2: $\alpha = 0.5, \beta = 0.25, c = 0.5$

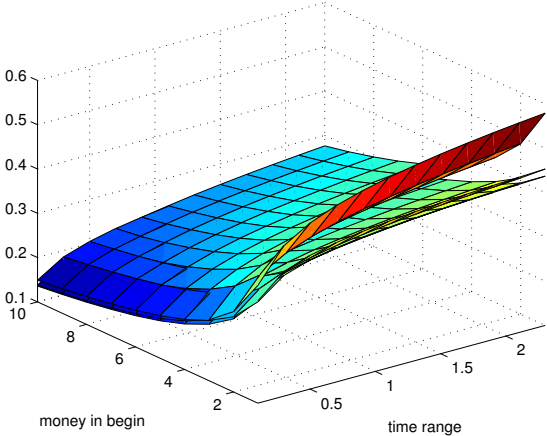


Figure 3: $\alpha = 0.5, \beta = 0.25, c = 1.0$

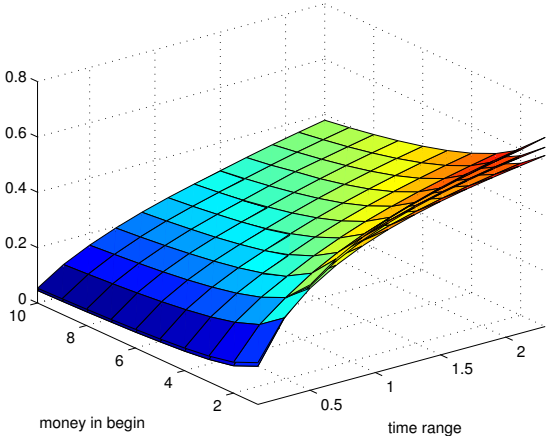


Figure 4: $\alpha = 0.5, \beta = 0.75, c = 0.1$

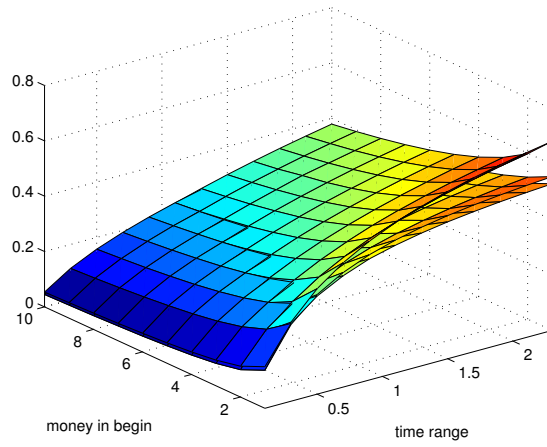


Figure 5: $\alpha = 0.5, \beta = 0.75, c = 0.5$

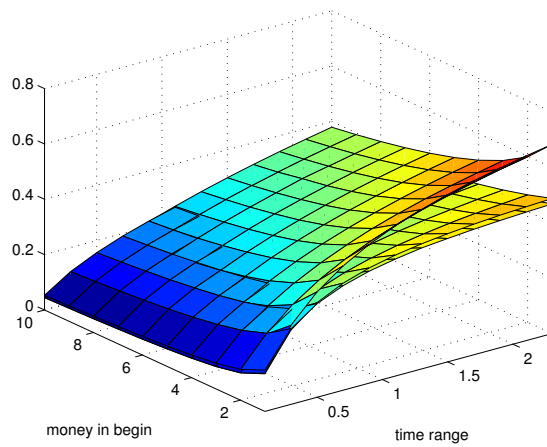


Figure 6: $\alpha = 0.5, \beta = 0.75, c = 1.0$

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